2.003J/1.053J Dynamics and Control I, Spring 2007
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Lecture 11

2D Motion of Rigid Bodies: Finding Moments of Inertia, Rolling Cylinder with Hole Example

Finding Moments of Inertia

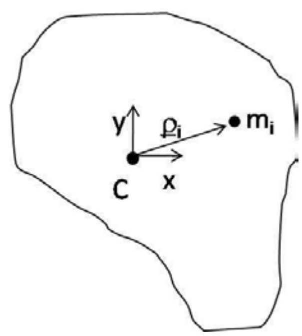

Figure 1: Rigid Body. Figure by MIT OCW.

$$I_C = \sum_i m_i |\rho_i|^2 = \sum_i m_i (x_i^2 + y_i^2)$$

I_C is the Moment of Inertia about C.

Example: Uniform Thin Rod of Length L and Mass M

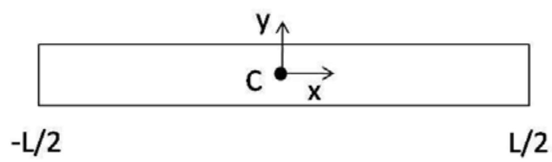

Figure 2: Uniform thin rod of length L and mass M. Figure by MIT OCW.

$$\begin{aligned} I_C &= \sum_i m_i(x_i^2 + y_i^2) \text{For very thin rod, } y_i \text{ is small enough to neglect.} \\ &\approx \sum_i m_i x_i^2 \end{aligned}$$

Rod has mass/length $= \rho$.
Convert to integral.

$$I_C = \int_{\text{rod}} x^2 dm$$
$$dm = \rho dx$$

$$\begin{aligned} I_C &= \int_{-L/2}^{L/2} x^2 \rho dx \\ &= \left[\rho \frac{x^3}{3}\right]_{-L/2}^{L/2} = \rho \frac{L^3}{12} \end{aligned}$$

We know that mass $M = \rho L$.
Therefore:

$$\boxed{I_C = \frac{ML^2}{12}}$$

Example: Uniform Thin Disc of Radius R

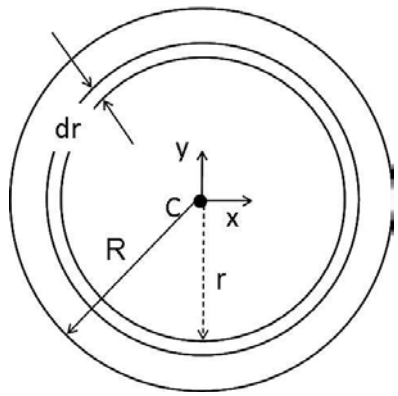

Figure 3: Uniform thin disc of radius R. Figure by MIT OCW.

Let ρ = mass/area. Consider a sliver that is a distance r from the center on this disc of radius R.

$$\begin{aligned} I_C &= \sum_i m_i(x_i^2 + y_i^2) \\ &= \int_{\text{disc}} (x^2 + y^2)dm \\ &= \int_0^R r^2 2\pi r\rho dr \\ &= \int_0^R \rho 2\pi r^3 dr = 2\pi\rho\frac{R^4}{4} \\ &= \pi R^2\rho\frac{R^2}{2} \end{aligned}$$

Mass of Disc: $M = \pi R^2\rho$. Thus,

$$\boxed{I_C = \frac{MR^2}{2}}$$

Example: Rolling Cylinder with a Hole

Find the equation of motion for a cylinder with a hole rolling without slip on a horizontal surface. In the hole with center A, $R_2 = R/2$.

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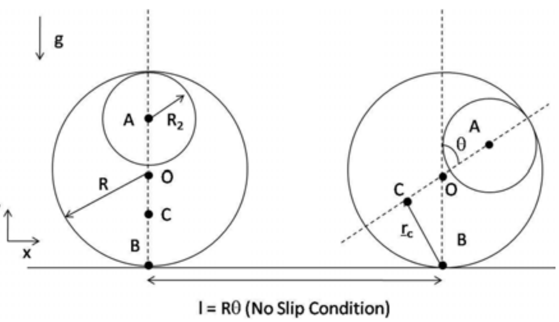

Figure 4: Rolling cylinder with hole shown at 2 distinct positions. Figure by MIT OCW.

Kinematics

2 Constraints: 1. Rolling on surface 2. No slip condition
Use 1 generalized coordinate θ to describe the motion
Only need 1 equation.
For this example, we will use the work-energy principle to obtain the equation.

1. Gravity is a potential force.
2. Normal force on object: At point of contact, velocity is zero so no work done.

No work done by external forces therefore $T + V =$ constant.

$T = \frac{1}{2}M|\underline{v}_C|^2 + \frac{1}{2}I_C|\underline{\omega}|^2$. Need center of mass. Where is the center of mass? Below O, because of hole.

Kinetics

Center of Mass Calculation

First find position of center of mass.

We know the center of mass of disc without hole: Point O. Can think of the hole to be "negative mass."

Consider moments about OX' at point O

$$\rho\pi R^2(0) = \rho(\pi R^2 - \frac{\pi R^2}{4})OC - \rho\frac{\pi R^2}{4}\frac{R}{2}$$

Distance from O to O is zero.
$\rho(\pi R^2 - \frac{\pi R^2}{4})OC$: Mass moment of cylinder with the hole.

$\rho \frac{\pi R^2}{4} \frac{R}{2}$: Mass moment of the hole.

$$\frac{3\pi}{4}(OC) = \frac{\pi R}{8} \Rightarrow OC = \frac{R}{6}$$

Calculation of $\frac{1}{2}M|v_C|^2$

We know what $\underline{r}_C$ is. $\underline{r}_C$ is from point B to point C.

$$x_C = R\theta - \frac{R}{6}\sin\theta$$
$$y_C = R - \frac{R}{6}\cos\theta$$

Differentiate:

$$\dot{x}_C = R\dot{\theta} - \frac{R}{6}\dot{\theta}\cos\theta$$
$$\dot{y}_C = \frac{R}{6}\dot{\theta}\sin\theta$$

$$\frac{1}{2}Mv_C^2 = \frac{1}{2}\left(\pi R^2 \rho - \pi \frac{R^2}{4}\rho\right)(\dot{x}_C^2 + \dot{y}_C^2)$$
$$= \frac{1}{2}\frac{3}{4}\pi R^2 \rho \left(R^2\dot{\theta}^2 + \frac{R^2}{36}\dot{\theta}^2\cos^2\theta - \frac{R^2\dot{\theta}^2}{3}\cos\theta + \frac{R^2\dot{\theta}^2}{36}\sin^2\theta\right)$$

$$\boxed{\frac{1}{2}Mv_c^2 = \frac{1}{2}\frac{3}{4}\pi R^4 \dot{\theta}^2 \rho \left(1 - \frac{1}{3}\cos\theta + \frac{1}{36}\right)}$$

Calculation of $\frac{1}{2}I_C|\underline{\omega}|^2$

2nd term of kinetic energy is $\frac{1}{2}I_C|\underline{\omega}|^2$.

What is I_C?

We want to find I_C^{cwh}.
cwh: cylinder with hole
mc: missing cylinder
cyl: cylinder

First find I_O^{cwh} around O. Then shift to C with the Parallel-Axis Theorem.

$$I_O^{cyl} = I_O^{cwh} + I_O^{mc}$$

$$I_O^{cwh} = I_O^{cyl} - I_O^{mc}$$
$$I_O^{cyl} = \frac{1}{2}\rho\pi R^2 R^2$$

$$\begin{aligned} I_O^{mc} &= I_A^{mc} + M^{mc}(OA)^2 \leftarrow \text{ Parallel Axis Theorem} \\ &= \frac{1}{2}\left(\rho\pi\frac{R^2}{4}\right)\frac{R^2}{4} + \left(\rho\pi\frac{R^2}{4}\right)\frac{R^2}{4} \\ &= \frac{3}{32}\rho\pi R^4 \end{aligned}$$

$$I_O^{cwh} = \frac{1}{2}\rho\pi R^4 - \frac{3}{32}\rho\pi R^4 = \frac{13}{32}\rho\pi R^4$$

Now:

$$I_O^{cwh} = I_C^{cwh} + M^{cwh}(OC)^2 \leftarrow \text{ Parallel Axis Theorem}$$
$$I_C^{cwh} = I_O^{cwh} - M^{cwh}(OC)^2$$
$$I_C^{cwh} = \frac{13}{32}\rho\pi R^4 - \left(\rho\pi R^2 - \rho\frac{\pi R^2}{4}\right)\frac{R^2}{36} = \frac{37}{96}\rho\pi R^4$$

So we have that:

$$\boxed{\frac{1}{2}I_C|\underline{\omega}|^2 = \frac{1}{2}\frac{37}{96}\rho\pi R^4\dot{\theta}^2}$$

$$T+V = \frac{3}{8}\rho\pi R^4\dot{\theta}^2\left(\frac{37}{36}-\frac{1}{3}\cos\theta\right)+\frac{37}{192}\rho\pi R^4\dot{\theta}^2+V = \rho\pi R^4\dot{\theta}^2\left(\frac{37}{64}-\frac{1}{8}\cos\theta\right)+V$$

Calculation of Potential Energy (V)

What is V?

$$V = mgh = \frac{3}{4}\pi R^2\rho\left(R - \frac{R}{6}\cos\theta\right)$$

Finding Equation of Motion

So we have $T + V$ = Constant

$$\frac{d}{dt}(T+V) = 0$$

Differentiate:

$$\boxed{2\rho\pi R^4\dot{\theta}\ddot{\theta}\left(\frac{37}{64}-\frac{1}{8}\cos\theta\right)+\frac{1}{8}\rho\pi R^4\dot{\theta}^2\sin\theta\dot{\theta}+\frac{1}{8}\pi R^3\rho g\sin\theta\dot{\theta}=0}$$

Equation of motion: motion is complicated.

Alternative Approach: Using Angular Momentum (Sketch)

Angular momentum about *moving point B*.

B is not on cylinder. B is not on ground. B is contact point between ground and cylinder.

$$\underline{\tau}_B = \frac{d}{dt}\underline{H}_B + \underline{v}_B \times \underline{P}$$

$\underline{\tau}_B$: Torque due to gravity
$\underline{H}_B = \underline{H}_C + \underline{r}'_C \times \underline{P}$
$\underline{v}_B$: Moving Point ($R\dot{\theta}$)
$\underline{P} = M\underline{v}_C$

1. Still have to find velocity and location of center of mass.
2. Still have to find I_C.
3. But, even more work because need to take torques.