2.003J/1.053J Dynamics and Control I, Spring 2007
Professor Peacock
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Lecture 17

Lagrangian Dynamics: Examples and Equilibrium Analysis

Example: Cart with Pendulum and Spring (continued)

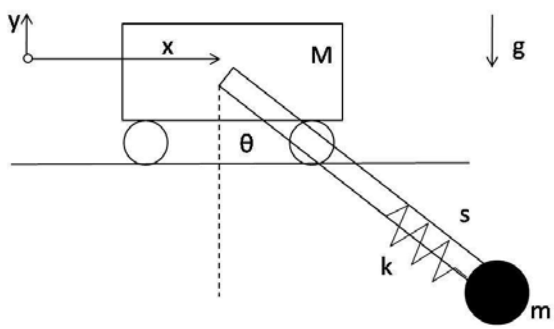


Figure 1: Cart with pendulum and spring. Figure by MIT OCW.

x, θ, s: Generalized Coordinates

Lagrangian

$$
\begin{aligned}
L &= T - V \\
&= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m[\dot{s}^2 + s^2\dot{\theta}^2 + 2\dot{x}(\dot{s}\sin\theta + s\dot{\theta}\cos\theta)] + mgs\cos\theta - \frac{1}{2}k(s-l)^2
\end{aligned}
$$

Equations of Motion

x:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \Xi_x$$

$$\boxed{(M+m)\ddot{x} + m\ddot{s}\sin\theta + 2m\dot{s}\dot{\theta}\cos\theta + ms\ddot{\theta}\cos\theta - ms\dot{\theta}^2\sin\theta = 0} \tag{1}$$

θ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \Xi_\theta$$

$$\boxed{s\ddot{\theta} + 2\dot{s}\dot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0} \tag{2}$$

s:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \left(\frac{\partial L}{\partial s}\right) = \Xi_s$$

$$\frac{\partial L}{\partial \dot{s}} = m\dot{s} + m\dot{x}\sin\theta$$

$$\frac{\partial L}{\partial s} = ms\dot{\theta}^2 + m\dot{x}\dot{\theta}\cos\theta + mg\cos\theta - k(s-l)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = m\ddot{s} + m\ddot{x}\sin\theta + m\dot{x}\dot{\theta}\cos\theta - ms\dot{\theta}^2 - m\dot{x}\dot{\theta}\cos\theta - mg\cos\theta - k(s-l) = 0$$

$$\boxed{m\ddot{s} + m\ddot{x}\sin\theta - ms\dot{\theta}^2 - mg\cos\theta + k(s-l) = 0} \tag{3}$$

These equations are highly non-linear.

In general, the equations of a real-world system will be highly nonlinear, what can we do analytically?

→ Identify equilibrium points and analyze their stability (2.003)
→ Nonlinear dynamics (2.05J)

Equilibrium Analysis

Let us look for equilibrium points.

Finding Equilibrium Points

Set velocities and acceleration to zero (Remaining generalized coordinates have values constant in time).

$$\dot{x} = \ddot{x} = \dot{\theta} = \ddot{\theta} = \dot{s} = \ddot{s} = 0$$

For (1), $0 = 0$
For (2), $g \sin\theta = 0$
For (3), $-mg\cos\theta + k(s - l) = 0$

In general, the equations can be much more complex: 3 nonlinear coupled equations to be solved simultaneously (can try fsolve in MATLAB)

Solve:

$$g \sin\theta = 0 \Rightarrow \theta_0 = 0, \pi$$

$$-mg\cos\theta + k(s - l) = 0 \Rightarrow s = l + \frac{mg}{k}\cos\theta$$

$$\theta_0 = 0 \Rightarrow s_0 = l + \frac{mg}{k}$$

$$\theta_0 = \pi \Rightarrow s_0 = l - \frac{mg}{k}$$

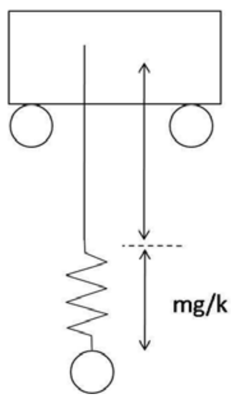
x is any value = ☺.

Explaining their physical meaning

$\theta_0 = 0$: Spring and pendulum hanging vertically down
$0 = 0$ means cart can take any x value as long as the cart is at rest.
$\theta_0 = \pi$: Spring and pendulum standing vertically up.

$\theta_0 = 0$

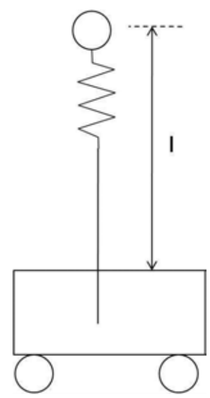
$s_0 = l + \dfrac{mg}{k}$



Stable. $x_0 =$ ☺

$\theta_0 = \pi$

$s_0 = l - \dfrac{mg}{k}$



Unstable. $x_0 =$ ☺

Analyzing Their Stability

Close to their equilibrium points, nonlinear systems behave like linear systems. Thus, if we linearize the equations of motion about those static equilibrum points and analyze the linearized system mathematically, we can draw conclusions about the nonlinear system's behavior and stability.

$$x = 0, \theta_0 = 0, s_0 = l + \frac{mg}{k}$$

If we make a small disturbance, what happens?

Imagine $x << 1$, $\theta_0 << 1$, $s = s_0 + \epsilon$ where $\epsilon << 1$. ϵ is a small perturbation away from s_0.

$$(\dot{s} = \dot{\epsilon}, \ddot{s} = \ddot{\epsilon})$$

Linearize by keeping only linear terms.

Approximations: For small angles: $\sin\theta \approx \theta$, $\cos\theta \approx 1$

1.

$$(M+m)\ddot{x} + m(\ddot{\epsilon}\theta + \dot{\epsilon}\dot{\theta} + \dot{\epsilon}\dot{\theta} + (s_0+\epsilon)\ddot{\theta} - (s_0+\epsilon)\dot{\theta}^2\theta)$$

$$\boxed{\approx (M+m)\ddot{x} + ms_0\ddot{\theta} = 0} \tag{1L}$$

2.

$$(s_0+\epsilon)\ddot{\theta} + 2\dot{\epsilon}\dot{\theta} + \ddot{x} + g\theta = 0$$

$$\boxed{s_0\ddot{\theta} + \ddot{x} + g\theta = 0} \tag{2L}$$

3:

$$m\ddot{\epsilon} + m\ddot{x}\theta - m(s_0+\epsilon)\dot{\theta}^2 - mg + k(s_0+\epsilon-l) = 0$$

$$m\ddot{\epsilon} - mg + k(l + \frac{mg}{k} + \epsilon - l) = 0 \Rightarrow \boxed{m\ddot{\epsilon} + k\epsilon = 0} \tag{3L}$$

(3L) is uncoupled from x, θ. Consequently, the spring oscillations are independent of x and θ in the linearized system.

(3L) is decoupled $\Rightarrow$ describes oscillatory motion with frequency $\sqrt{\frac{k}{m}}$.

(1L) and (2L) are coupled.

From equation (1L) we have:

$$(M+m)\ddot{x} = -ms_0\ddot{\theta}$$

Substitute in (2L)

$$s_0\ddot{\theta} - \frac{ms_0}{(M+m)}\ddot{\theta} + g\theta = 0$$

$$\frac{s_0 M}{(M+m)}\ddot{\theta} + g\theta = 0$$

Describes oscillations with frequency $\sqrt{\frac{g(M+m)}{M}}$.

To solve, assume a solution $A\sin\omega t + B\cos\omega t$.

If we perturb pendulum, an oscillation will result.

From (1L), $\ddot{x}(M+m) = -ms_0\ddot{\theta}$.

Integrate twice with respect to time to get x and θ relationship.

$$\ddot{x}\frac{(M+m)}{-ms_0} = \ddot{\theta} \Rightarrow \text{ We know that } x \text{ must be oscillatory.}$$

We know that:

$$\boxed{x_0 = \theta_0 = 0, s = l + \frac{mg}{k} \text{ is stable.}}$$

Small disturbances give oscillations.

Let us do the same for $x_0 = 0$, $\theta_0 = \pi$, $s_0 = l - \frac{mg}{k}$

Approximations:

Note $\cos\theta \approx -1$ for $\theta \approx \pi$

$$\begin{aligned}
\sin(\theta_0 + \delta\theta) &= \sin\theta_0 + \frac{d}{d\theta}\sin\theta|_{\theta_0}\delta\theta && (4)\\
&= \sin\theta_0 + \cos\theta_0\delta\theta && (5)\\
&= \sin\pi + \cos\pi\delta\theta && (6)\\
&= 0 - \delta\theta \text{ so } \sin(\theta+\delta\theta)|_{\theta_0} \approx -\delta\theta = -\phi && (7)
\end{aligned}$$

$$x << 1, \theta = \pi + \phi, s = s_0 + \epsilon(\phi, \epsilon << 1)$$

$$\begin{aligned}
\dot{\theta} &= \dot{\phi}, \dot{s} = \dot{\epsilon}\\
\ddot{\theta} &= \ddot{\phi}, \ddot{s} = \ddot{\epsilon}
\end{aligned}$$

$$\begin{aligned}
\phi &= \delta\theta\\
-\phi &= -\delta\theta
\end{aligned}$$

Linearize equations about this equilibrium point

1:

$$(M+m)\ddot{x} + m\ddot{s}\sin\theta + 2m\dot{s}\dot{\theta}\cos\theta + ms\ddot{\theta}\cos\theta - ms\dot{\theta}^2\sin\theta = 0$$

$$(M+m)\ddot{x} + m\ddot{\epsilon}\sin(\pi+\phi) + 2m\dot{\epsilon}\dot{\phi}(-1) + m(s_0+\epsilon)\ddot{\phi}(-1) - m(s_0+\epsilon)\dot{\phi}^2\sin(\pi+\phi) = 0$$

$$\boxed{(M+m)\ddot{x} - ms_0\ddot{\phi} = 0} \tag{1L$_\phi$}$$

2.

$$s\ddot{\theta} + 2\dot{s}\dot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0$$
$$(s_0+\epsilon)\ddot{\phi} + 2\dot{\epsilon}\dot{\phi} + \ddot{x}(-1) + g(-\phi) = 0$$

$$\boxed{s_0\ddot{\phi} - \ddot{x} - g\phi = 0} \tag{2L$_\phi$}$$

3.

$$m\ddot{s} + m\ddot{x}\sin\theta - ms\dot{\theta}^2 - mg\cos\theta + k(s-l) = 0$$
$$m\ddot{\epsilon} + m\ddot{x}(-\phi) - m(s_0+\epsilon)\dot{\phi}^2 - mg(-1) + k(s_0+\epsilon-l)$$

$$\boxed{m\ddot{\epsilon} + k\epsilon = 0} \tag{3L$_\phi$}$$

Note (3L$_\phi$) is the same as (3L).

Use (1L$_\phi$) to obtain $\ddot{x} = \frac{ms_0}{(M+m)}\ddot{\phi}_0$. Substitute in (2L$_\phi$) to get:

$$s_0\ddot{\phi} - \frac{ms_0}{(M+m)}\ddot{\phi} - g\phi \Rightarrow \boxed{\left(\frac{Ms_0}{M+m}\right)\ddot{\phi} - g\phi = 0} \tag{8}$$

Equation (8) has solutions of the form $\phi = Ae^{\lambda t} + Be^{-\lambda t}$, where $\lambda = \sqrt{\frac{g(M+m)}{s_0 M}}$. If we perturb by an angle ϕ, the angle will grow. Thus, the point is unstable.