2.003J/1.053J Dynamics and Control I, Spring 2007
Professor Peacock
5/9/2007

Lecture 22

Vibrations: Free Response of Multi-Degree-of-Freedom Systems

Motivation

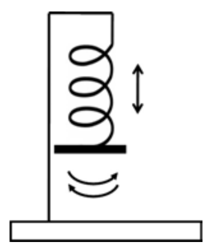

Figure 1: 2 degrees of freedom. Spring with mass on end. The mass rotates in both counterclockwise and clockwise directions. The spring extends and retracts by varying amounts. Sometimes only the spring moves with the mass relatively still; sometimes only the mass rotates. What is happening? Figure by MIT OCW.

Two Degrees of Freedom: System of Three Springs and Two Masses

Consider the following:

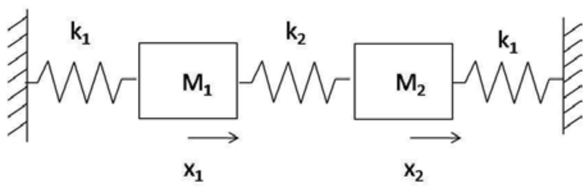

Figure 2: System of three springs and two masses. Figure by MIT OCW.

2 degrees of freedom
2 generalized coordinates
$M_1 = M_2 = m$
x_1 and x_2 are the coordinates that describe displacement from equilibrium

Lagrangian

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$
$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_1x_2^2 + \frac{1}{2}k_2(x_2 - x_1)^2$$
$$L = T - V$$

Equations of Motion

$$m\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) = 0$$
$$m\ddot{x}_2 + k_2x_2 + k_2(x_2 - x_1) = 0$$

Can solve by elimination of x_2, for example

$$x_1^{''''} + 2\frac{k_1 + k_2}{m}\ddot{x}_1 + k_1\frac{2k_1 + 2k_2}{m^2}x_1 = 0$$

System is equivalent to a single 4th order equation.

Rather than solving this, we prefer to keep a system of equations. Use matrix notation as this then allows us to use to use tools from linear algebra.

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1}$$

Mass Matrix ($\underline{\underline{M}}$):

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

Stiffness Matrix ($\underline{\underline{K}}$):

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix}$$

$$\underline{\underline{M}}\ddot{x} + \underline{\underline{K}}x = 0, \text{ where } x = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

(Often referred to as 'state variables' form)

Solution

By analogy with $m\ddot{x} + kx = 0$, look for solutions of the form:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \cos(\omega t - \phi) \tag{2}$$

This solution assumes both masses are in phase and both masses are oscillating at same frequency.

Substitute Equation (2) into Equation (1).

$$\begin{bmatrix} -m\omega^2 & 0 \\ 0 & -m\omega^2 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \cos(\omega t - \phi) + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \cos(\omega t - \phi) = 0$$

Divide by $\cos(\omega t - \phi)$

Finding c_1 and c_2

$$\begin{bmatrix} -m\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m\omega^2 + k_1 + k_2 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3}$$

Trivial Solution $c_1 = c_2 = 0$
Determinant of matrix $= 0$ will give c_1 and c_2 nonzero.

Expand matrix to see why:

$$(k_1 + k_2 - m\omega^2)c_1 - k_2 c_2 = 0 \tag{4}$$

$$-k_2 c_1 + (k_1 + k_2 - m\omega^2)c_2 = 0 \tag{5}$$

To make the system nontrivial, we make the equations linearly dependent (they say the same thing).

If:

$$\frac{(k_1 + k_2 - m\omega^2)}{-k_2} = \frac{-k_2}{(k_1 + k_2 - m\omega^2)},$$

Then the two equations are just multiples of each other.

This sets a ratio of $\frac{c_1}{c_2}$ and not absolute values.

$$(k_1 + k_2 - m\omega^2)^2 = k_2^2 \Rightarrow (k_1 + k_2 - m\omega^2)^2 - k_2^2 = 0$$

which is just the determinant of the matrix.

Solving for ω

$$k_1 + k_2 - m\omega^2 = \pm k_2$$

Two possibilities:

$$m\omega_1^2 = k_1$$
$$m\omega_2^2 = k_1 + 2k_2$$

$$\omega_1 = \sqrt{\frac{k_1}{m}}$$

$$\omega_2 = \sqrt{\frac{k_1 + 2k_2}{m}}$$

Both are natural frequencies.

We will not prove this, but number of degrees of freedom equals the number of natural frequencies.

To find c_1 and c_2 for each frequency, we use equations (4) and (5).

$$\begin{bmatrix} -m\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m\omega^2 + k_1 + k_2 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For $\omega_1^2 = k_1/m$:

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$k_2 c_1 - k_2 c_2 = 0$$
$$-k_2 c_1 + k_2 c_2 = 0$$

$c_1 = c_2$

Notice the two equations are linearly related by a factor -1.
For $\omega_2^2 = \frac{k_1 + 2k_2}{m}$

$$\begin{bmatrix} -k_2 & -k_2 \\ -k_2 & -k_2 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$-k_2 c_1 - k_2 c_2 = 0$$
$$c_1 = -c_2$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \cos(\omega t - \phi)$$

$$\omega_1^2 = \frac{k_1}{m}$$

$$\left\{ \begin{array}{c} c_1 \\ c_2 \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} A$$

$$\omega_2^2 = \frac{k_1 + 2k_2}{m}$$

$$\left\{ \begin{array}{c} c_1 \\ c_2 \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\} B$$

General Solution

$$\left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} = A \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} \cos(\omega_1 t - \phi_1) + B \left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\} \cos(\omega_2 t - \phi_2)$$

A, B, ϕ_1, and ϕ_2 are determined by initial conditions.

Discussion

$$\omega_1^2 = \frac{k_1}{m}$$

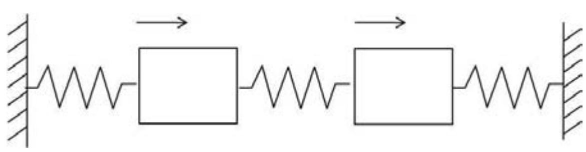

Figure 3: When x_1 moves 1 unit, x_2 moves 1 unit in the same direction, both masses are moving the same distance while oscillating. Figure by MIT OCW.

$$\omega_2^2 = \frac{k_1 + 2k_2}{m}$$

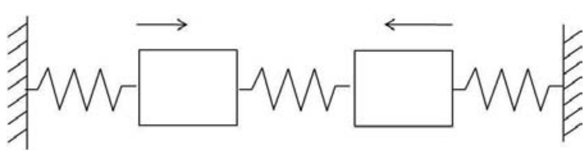

Figure 4: Masses move in opposite directions. Figure by MIT OCW.

$\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ means antiphase. Masses move in opposite directions.

ω_1 and ω_2 are the system's natural frequencies. $\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}_1$ and $\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}_2$ are the mode shapes in which the system naturally oscillates at the natural frequencies.

Examples of Determining Mode Shape and Natural Frequencies by Inspection

One can use forces and system symmetry to deduce ω and the mode shapes by inspection.

System of Two Masses and Three Springs

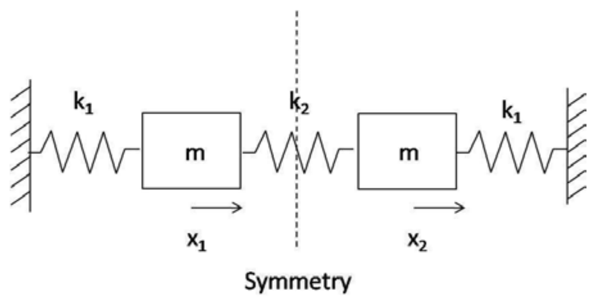

Figure 5: System of Two Masses and Three Springs. Due to the symmetry of the system, we can look at just one half. Figure by MIT OCW.

1. If both masses move together, spring k_2 will never stretch.

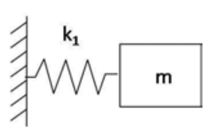

Figure 6: Left hand mass only experiences 1 spring. Figure by MIT OCW.

Mode shape: $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$\omega_1^2 = \frac{k_1}{m}$

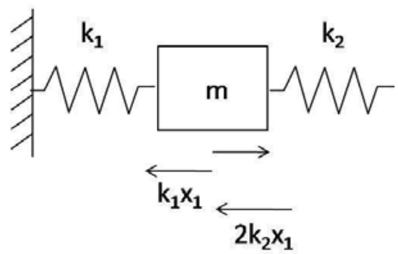

Figure 7: Because of symmetry, both masses could move in opposite directions.

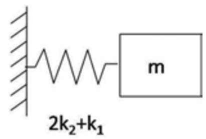

Figure 8: There is twice as much of k_2, because mass 2 also moves a distance x_1 when mass 1 moves a distance x_1. Figure by MIT OCW.

Assume mode shape $\left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\}$.

$\omega_2^2 = \frac{2k_2+k_1}{m}$

System of 2 carts and 1 spring

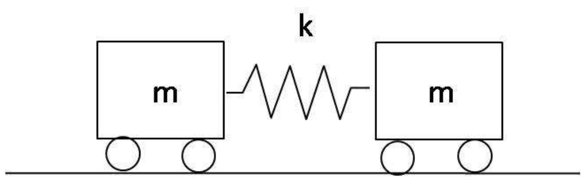

Figure 9: System of two carts and one spring. Figure by MIT OCW.

What are the frequencies of oscillations?
$\omega_1^2 = 0$, $\left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}$ mode shape. Translation in one direction is one of the natural frequencies.

$\omega_2^2 = \frac{2k}{m} \left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\}$

Symmetry helps when analyzing systems by inspection.