2.003J/1.053J Dynamics and Control I, Spring 2007
Professor Peacock
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Lecture 24

Vibrations: Forced Response of Multi-Degree-of-Freedom Systems

Forced Response of Multi-Degree-of-Freedom Systems

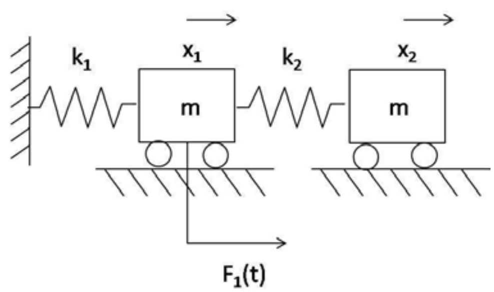

Figure 1: Two carts connected by two springs. Motion only in the x direction. Generalized coordinates x_1 and x_2. Figure by MIT OCW.

Equations of Motion

$$m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) = F_1(t)$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$$

Rewrite in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix} \quad (1)$$

Let:

$$F_1(t) = F_0 \cos \omega t$$

What is the response?

General Form of Solution

2 natural frequencies. 2 modes of oscillation.

$$\left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} = \underbrace{A_1 \left\{ \; c \; \right\}_1 \cos(\omega_1 t - \phi_1) + A_2 \left\{ \; c \; \right\}_2 \cos(\omega_2 t - \phi_2)}_{\text{Free Response}} + \underbrace{\left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\}_p \cos(\omega t - \phi)}_{\text{Forced Response}}$$

$\{c_1\}$ and $\{c_2\}$ are vectors describing the modes shapes for the free response.
ω_1 and ω_2 are the two natural frequencies of oscillation.
Constants A_1, A_2, ϕ_1, and ϕ_2 are set by the initial conditions.
For more information on free response, see lectures 21, 22, and 23.

For the particular solution, we will restrict the guess further.

$$\boxed{\left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} = \left\{ \begin{array}{c} \mathbb{X}_1 \\ \mathbb{X}_2 \end{array} \right\} \cos \omega t} \tag{2}$$

This guess assumes phase change of 0 or π. Acceptable, because the system has no damping. Justify with a 1-D example.

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

We use:

$$x = \text{Re} \left\{ \mathbb{X} e^{(i\omega t)} \right\} = |\mathbb{X}| \cos(\omega t - \phi)$$

This allows $0 < \phi < \pi$ because $c\dot{x}$ term however, in the case $m\ddot{x} + kx = F_0 \cos \omega t$, one is restricted to guessing $x = \mathbb{X} \cos \omega t$.

Derivations of Equations for Particular Solution

Substitute Equation (2) into Equation (1).

$$\left(-\omega^2 \left[\begin{array}{cc} m_1 & 0 \\ 0 & m_2 \end{array} \right] + \left[\begin{array}{cc} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{array} \right] \right) \left\{ \begin{array}{c} \mathbb{X}_1 \\ \mathbb{X}_2 \end{array} \right\} = \left[\begin{array}{c} F_0 \\ 0 \end{array} \right]$$

$$\left\{ \begin{array}{c} \mathbb{X}_1 \\ \mathbb{X}_2 \end{array} \right\} = \left[\begin{array}{cc} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{array} \right]^{-1} \left\{ \begin{array}{c} F_0 \\ 0 \end{array} \right\}$$

For:

$$\underline{\underline{A}} = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right]$$

$$\underline{\underline{A}}^{-1} = \frac{1}{|\underline{\underline{A}}|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\begin{Bmatrix} \mathbb{X}_1 \\ \mathbb{X}_2 \end{Bmatrix} = \frac{1}{\begin{vmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{vmatrix}} \begin{bmatrix} k_2-m_2\omega^2 & k_2 \\ k_2 & k_1+k_2-m_2\omega^2 \end{bmatrix} \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

Note that the determinant in the denominator is the free response determinant. There will be resonance at the system's natural frequencies.

Particular Solution

Look at case $m_1 = m$, $m_2 = \frac{2}{3}m$, $k_1 = k_2 = k$

$$\begin{Bmatrix} \mathbb{X}_1 \\ \mathbb{X}_2 \end{Bmatrix} = \frac{1}{\begin{vmatrix} 2k-m\omega^2 & -k \\ -k & k-\frac{2}{3}\omega^2 \end{vmatrix}} \begin{bmatrix} k-\frac{2}{3}m\omega^2 & k \\ k & 2k-m\omega^2 \end{bmatrix} \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \mathbb{X}_1 \\ \mathbb{X}_2 \end{Bmatrix} = \frac{1}{(2k-m\omega^2)(k-\frac{2}{3}m\omega^2)-k^2} \begin{Bmatrix} (k-\frac{2}{3}m\omega^2)F_0 \\ kF_0 \end{Bmatrix}$$

$$\begin{Bmatrix} \mathbb{X}_1 \\ \mathbb{X}_2 \end{Bmatrix} = \frac{1}{k^2-\frac{7}{3}m\omega^2+\frac{2}{3}m^2\omega^4} \begin{Bmatrix} (k-\frac{2}{3}m\omega^2)F_0 \\ kF_0 \end{Bmatrix}$$

$$\begin{Bmatrix} \mathbb{X}_1 \\ \mathbb{X}_2 \end{Bmatrix} = \frac{1/m}{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)} \begin{Bmatrix} (\frac{3}{2}\frac{k}{m}-\omega^2)F_0 \\ \frac{3}{2}\frac{k}{m}F_0 \end{Bmatrix}$$

$$\omega_1^2 = \frac{k}{2m}, \quad \omega_2^2 = \frac{3k}{m}$$

Response Diagrams

(Forcing $\rightarrow F_0$). As limit $\omega \rightarrow 0$, steady state response: $\left\{ \begin{array}{c} F_0/k \\ F_0/k \end{array} \right\}$

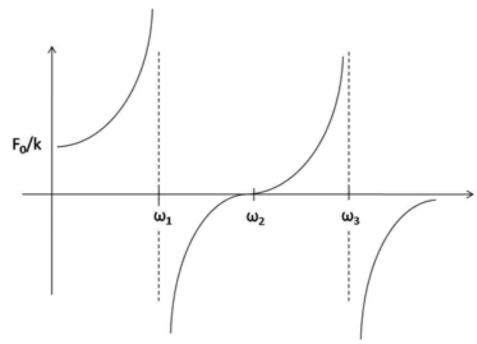

Figure 2: Forced response diagram for $\mathbb{X}_1$, which represents Cart 1. For ω approaching 0, the response approaches steady state F_0/k. For ω equal to ω_3, Cart 1 is stationary and only Cart 2 moves. Figure by MIT OCW.

$\mathbb{X}_1$ (Figure 2):

As $\omega \rightarrow \omega_1$, $\mathbb{X}_1 \rightarrow \infty$.
As ω is just over $\omega_1 \rightarrow \omega_2$, $\omega_3 = \sqrt{\frac{3k}{m}}$, sign change.
As ω goes over ω_2, starts at $-\infty$, then decays to 0.

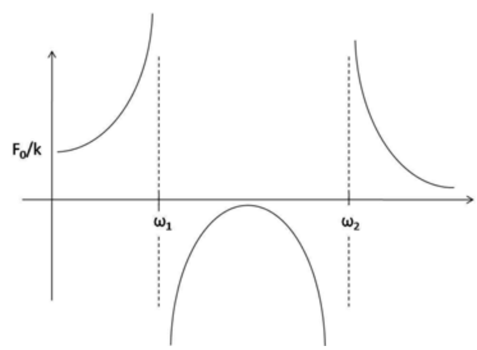

Figure 3: Forced response diagram for $\mathbb{X}_2$, which represents Cart 2. For ω approaching 0, the response approaches steady state F_0/k. For ω equal to ω_3, Cart 2 still exhibits oscillatory behavior. Figure by MIT OCW.

$\mathbb{X}_2$ (Figure 3):

As $\omega \to \omega_1$, increase without bound.
As $\omega \to \omega_2$, but just above ω_1, negative. As $\omega \to \omega_2$, approach $-\infty$.
As $\omega \to \omega_2$, comes from ∞ down to 0.
At ω_3, first cart is stationary. Anti-resonance (mass 2 sees mass 1 as a rigid wall).

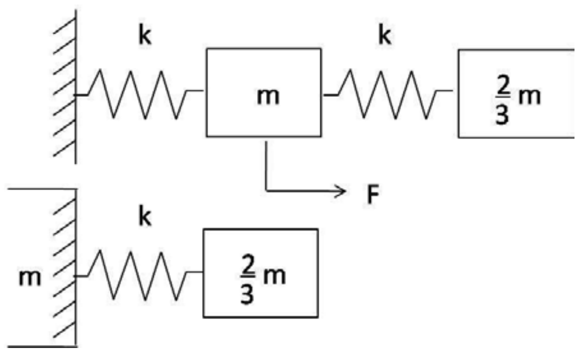

Figure 4: m is not moving so mass 2 sees a "wall" at mass 1. How does this occur? The forcing is equal and opposite to the forces provided by the strings. Figure by MIT OCW.

Adding damping will make the matrix inversion more complicated.

Real life examples of dynamics and vibrations include the Tacoma Narrows bridge collapse, nonlinear deep sea ocean wave formation and breaking, and dynamic pattern formation in nature (for example, shifting formations of starlings in flight).