**2.25 Final Exam 2005 – SOLUTION – Question 1**

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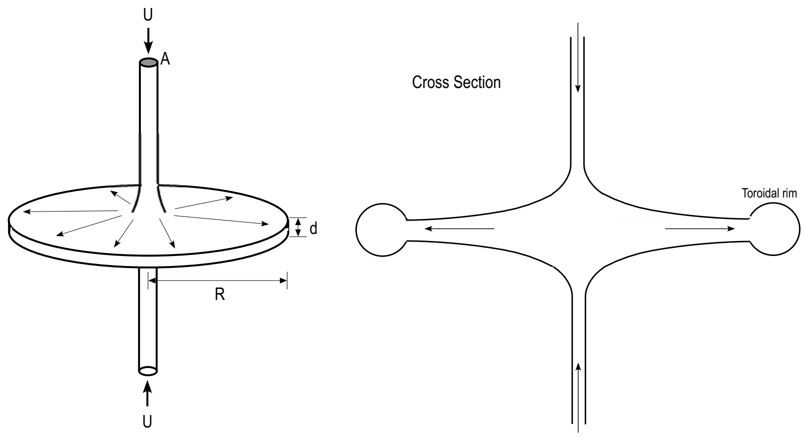
Figure 17l in Clanet, C. "Dynamics and stability of water bells." *J. Fluid Mech* 430 (2001): 111-147.




Two identical jets of **water** impact on one another forming a sheet of liquid as shown in the figure (and as shown in the surface tension movie). The sheet grows to a radius $R$ ending in a toroidal rim that ejects droplets of fluid. Consider incoming jet velocities, $U = 1\text{m/s}$, and cross-sectional jet areas, $A = \pi/4\ \text{cm}^2$.

1. Calculate the Reynolds number, Capillary number, and Froude number for this flow. Use these numbers to argue which effects (e.g. inertia, viscosity, gravity, etc.) are important in this system.

   *Solution:* $U = 100$ cm/s, $A = \pi/4\ \text{cm}^2 \Rightarrow D = 1$ cm, $\mu = 10^{-2}$ dyne-s/cm$^2$, $\rho = 1$ gm/cm$^3$, $\sigma = 70$ dynes/cm. Hence:

   $$Re = \frac{\rho U D}{\mu} \sim 10^4 \Rightarrow \text{ inertia} \gg \text{viscosity} \tag{1}$$

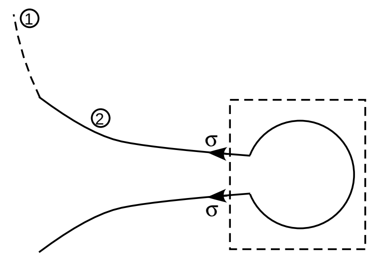
   $$Ca = \frac{\mu U}{\sigma} \sim 10^{-2} \Rightarrow \text{ surface tension} \gg \text{viscosity} \tag{2}$$

   $$Fr = \frac{U^2}{gD} \sim 10 \Rightarrow \text{ inertia} \gtrsim \text{gravity} \tag{3}$$

   If the sheet is flat, we can neglect gravity (if the sheet is curved, as in part 4, we must include gravity).

2. Derive an expression for the maximum radius $R$ of the liquid sheet and calculate $R$ using the values given above.

   *Solution:* The maximum radius occurs when the sheet is flat so we neglect gravity. Apply conservation of momentum on the following control volume:




By Bernoulli, $u_1 = u_2 = U$. Conservation of momentum yields (note: $p = p_a$):

$$\frac{d}{dt}\int_{CV} \rho \mathrm{v}\, dV + \int_{CS} \rho \mathbf{v}(\mathbf{v}-\mathbf{v}_c)\cdot \mathbf{n}\, dA = -\int_{CS} p\mathbf{n}\, dA - \int_{CL} 2\sigma\, d\ell \tag{4}$$

$$-\rho U^2 2\pi r d = -2\sigma 2\pi r \tag{5}$$

where $d = d(r)$ is the thickness of the sheet. Thus inertia balances surface tension at steady state when

$$d = \frac{2\sigma}{\rho U^2}. \tag{6}$$

By conservation of mass (2 on the LHS from two incoming jets):

$$2UA = 2\pi r d U \Rightarrow r = \frac{A}{\pi d}. \tag{7}$$

Combining this with (6) we find

$$r = R_{rim} = \frac{\rho U^2 A}{2\pi\sigma} \approx 18 \text{ cm (for water)} \tag{8}$$

3. Using energy arguments, show that the toroidal rim is unstable and will break into droplets. Derive a lower bound for the size of the ejected droplets.

*Solution:*

$$E_{surf} = \sigma A_{surf} \tag{9}$$

Thus, if the torus has a larger surface area that the droplets (where the volume of fluid is identical in both configurations), surface tension will break the torus into drops.

$$V_{torus} = \pi R_0^2 (2\pi R_{rim}) = V_{drop} = N\frac{4}{3}\pi r_{drop}^3 \tag{10}$$

$$N = \frac{R_0^2 (2\pi R_{rim})^3}{4 r_{drop}^3} \tag{11}$$

The torus is unstable if

$$S_{torus} \approx 2\pi R_0 (2\pi R_{rim}) > S_{drop} = N 4\pi r_{drop}^2 \tag{12}$$

$$r_{drop} > \frac{3}{2} R_0 \tag{13}$$

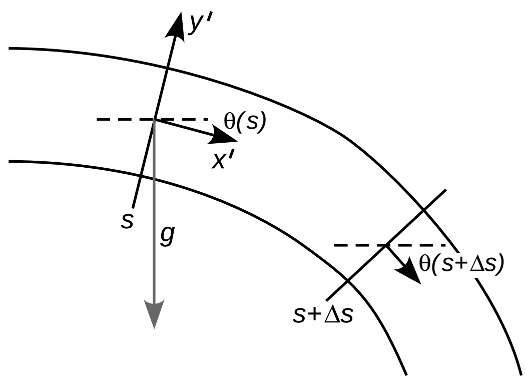
Thus the torus will break into droplets with radii $> \frac{3}{2}R_0$ to lower the surface energy. (Note, tiny droplets are no good since they create extra surface area and hence cost more energy.) To calculate the actual drop radius rather than a lower bound, we need to consider how to move fluid from the torus into the droplets. This calculation is a bit more involved and results in a critical wavelength of $9.02R_0$ (see Rayleigh-Plateau instability).

4. If gravity becomes important in the problem, the sheet will sag and become a "water bell" (instead of a flat sheet) as sketched below. Show that the shape of the "bell" is given by the solution to the following equation where $s$ is the coordinate along the sheet and other variables are defined in the sketch:

$$\rho g \cos\theta + \frac{2\pi\sigma r\sqrt{U^2+2gz}}{UA}\left(\frac{d\theta}{ds}+\frac{\sin\theta}{r}\right) = \frac{d\theta}{ds}\rho\left(U^2+2gz\right)$$

where $s$, $z$ and $\theta$ are related by $dz/ds = \sin\theta$. You do NOT need to solve this equation!

*Solution:*




Conservation of momentum:

$$\begin{aligned}\frac{d}{dt}\int_{CV}\rho\mathbf{v}\,dV \quad &+ \quad \int_{CS}\rho\mathbf{v}(\mathbf{v}-\mathbf{v}_c)\cdot\mathbf{n}\,dA = -\int p\mathbf{n}\,dA + \int_{CV}\rho\mathbf{g}\,dV \\ &- \int_{CL} 2\sigma\mathbf{t}\cdot\mathbf{y}'\,d\ell\bigg|_s + \int_{CL} 2\sigma\mathbf{t}\cdot\mathbf{y}'\,d\ell\bigg|_{s+\Delta s} + \underbrace{\int_{CS} 2\sigma\frac{\sin\theta}{r}\,dA}_{\text{out-of-plane curvature}}\end{aligned} \tag{14}$$

In the $\mathbf{y}'$ direction at steady state:

$$\rho u^2 2\pi r d\sin(\Delta\theta) = \rho g 2\pi r d\Delta s\cos\theta + 2\sigma 2\pi r\sin(\Delta\theta) + 2\sigma\frac{\sin\theta}{r}2\pi r\Delta s \tag{15}$$

In the limit $\Delta s \to 0$, $\sin(\Delta\theta) \to \Delta\theta$ and $\frac{\Delta\theta}{\Delta s} \to \frac{\partial\theta}{\partial s}$. Hence

$$\rho u^2\frac{\partial\theta}{\partial s} = \rho g\cos\theta + \frac{2\sigma}{d}\left(\frac{\partial\theta}{\partial s}+\frac{\sin\theta}{r}\right) \tag{16}$$

From Bernoulli:

$$u^2 = U^2 + 2gz \tag{17}$$

From conservation of mass:

$$2UA = 2\pi r dU \Rightarrow d = \frac{A}{\pi r} \tag{18}$$

Combining (16), (17) and (18) we find

$$\rho g \cos\theta + \frac{2\pi\sigma r\sqrt{U^2+2gz}}{UA}\left(\frac{d\theta}{ds}+\frac{\sin\theta}{r}\right) = \frac{d\theta}{ds}\rho\left(U^2+2gz\right) \tag{19}$$

## Solutions to Question 2: Flow Focusing in Microfluidics

### A) Dimensional Analysis [an easy 10 points!]

i) We seek a solution which describes

$$R_j = f\left(H, \mu_i, \mu_o, Q_i, Q_o, \sigma, \rho, \alpha\right)$$

Se we have $n = 9$, $r = 3$ (MLT) and hence $n - r = 6$. Most of these can be found by inspection. Picking $\mu_i$ $Q_i$ and $H$ to characterize fluid, flow and geometry respectively we find:

$$\frac{R_j}{H} = \phi\left(\frac{Q_0}{Q_i}, \frac{\mu_0}{\mu_i}, \frac{\sigma H^2}{\mu_i Q_i}, \alpha, \frac{\rho Q_i}{\mu_i H}\right)$$

where $Ca_i \equiv \mu\left(Q_i / H^2\right) / \sigma$ is a capillary number and $\mathrm{Re}_i \equiv \rho Q_i / \mu_i H$ is the Reynolds number for the inner fluid. Note that $\alpha$ (angle in radians) is already dimensionless and can be varied independently of *H*, so it is also a dimensionless group!!

ii) We need to find a product group involving surface tension, viscosity and density that has units of length:

i.e. $l_c \equiv \mu^a \rho^b \sigma^c$ or equivalently a dimensionless group: $\Pi = H \mu_i^{-a} \rho^{-b} \sigma^{-c}$

Since we have 3 unknowns (*a, b, c*) and 3 equations there is one unique solution:

$$l_c \equiv \mu^2 / \rho\sigma$$

This is often referred to now as the *Eggers length* (J. Eggers, Phys. Rev. Lett.,71 (1993). The corresponding dimensionless group is the *Ohnesorge* number

$$Oh^2 = \frac{\mu_i^2}{(\rho\sigma \mathrm{H})} = \frac{l_c}{H}.$$

and it measures relative importance of viscous effects to inertial and capillary effects in jet breakup

For water: $l_c = \frac{\left(10^{-3}\right)^2}{\left(10^3\right)\left(72 \times 10^{-3}\right)} =$ 13.9 nm; For glycerol: $l_c = \frac{1^2}{\left(10^3\right)\left(62 \times 10^{-3}\right)} =$ 16mm

Note that even in a microfluidic geometry (with say $H = 10$ μm) the breakup of a water jet is not affected by viscosity until scales of 0(10-20 nm)!! In contrast for glycerol we should expect viscosity to dominate all aspects of the process.

### B. Converging Planar Channel

i) the continuity equation in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0$$

Substituting for $v_r = -f(\sigma)/r$ we find $\frac{\partial}{\partial r}\left(r\times\frac{-f(\theta)}{r}\right)+0+0=0$

The volumetric flow rate (per unit depth) through the device at any distance *r* is:

$$Q_i' = \int_{-\alpha}^{+\alpha} v_\theta(-\mathbf{e}_r)\cdot\mathbf{e}_r r d\theta = \int_{-\alpha}^{+\alpha} f(\theta)d\theta = \text{ constant!} \tag{B1}$$

Note that the function $f(\theta)$ has units of $\left[m^2/s\right]$ and $Q_i'$ is *positive* here (i.e. it is the magnitude of the flow rate into the throat).

ii) If we substitute $\boldsymbol{v}=\left[-f(\theta)/r,0,o\right]^T$ into the Navier-Stokes equations we obtain

$$\frac{\partial p}{\partial r}=\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}\right]-\rho v_r\frac{\partial v_r}{\partial r} \quad\text{and}\quad \frac{\partial p}{\partial\theta}=\frac{2\mu}{r}\frac{\partial v_r}{\partial\theta}$$

Substituting for $rv_r = f(\theta)$, we see that the first term on the right hand side is zero and hence we obtain:

$$\frac{\partial p}{\partial r}=-\frac{\mu f''}{r^3}+\rho\frac{f^2}{r^3} \tag{B2}$$

Similarly for the $\theta$ component we obtain: $\frac{\partial p}{\partial\theta}=-\frac{2\mu}{r^2}f'$ (B3)

where a prime indicates a derivative with respect to theta.

iii) If we take cross-derivatives of B2 and B3 we obtain:

$$\frac{\partial}{\partial\theta}\left(\frac{\partial p}{\partial r}\right)=-\frac{\mu_i}{r^3}f'''+\frac{\rho}{r^3}2ff' \quad\text{and}\quad \frac{\partial}{\partial r}\left(\frac{\partial p}{\partial\theta}\right)=+\frac{4\mu_i}{r^3}f'$$

Equating and then rearranging by multiplying by $\left(-r^3/\mu\right)$ gives:

$$f'''-\frac{2\rho}{\mu}ff'+4f'=0$$

or equivalently if we define a dimensionless function $f^*$ or $F == f/Q_i'$ then we obtain:

$$\frac{d^3F}{d\theta^3}-\left(\frac{2\rho Q'}{\mu_i}\right)F\frac{dF}{d\theta}+4F'=0 \tag{B4}$$

The dimensionless group $\Re = 2\rho Q_i'/\mu_i$ is the Reynolds number for this problem (as we found in part A). The appropriate boundary conditions for this problem are

a) no slip at the walls: at $\theta=\pm\alpha$; $v_r=0$ so hence $f(\pm\alpha)=0$

b) this gives two boundary conditions…but (B4) is a third order ODE so we need another boundary condition. A statement of symmetry at the centerline is NOT sufficient as it is tantamount to the same thing as (a). However we can also note that the volumetric flow rate through the device is constant or: $\int_{-\alpha}^{+\alpha} F(\theta)\,d\theta = -1$.

iv) In the limit $\Re \to 0$, the nonlinear 3rd order ODE (B4) becomes a linear ODE. You should recognize that this is easy to solve in terms of trig. Functions (hint; you can directly integrate it one to find a 2nd order ODE and find the complementary function and particular integral). Regardless, we are given $F(\theta) = A + B\sin 2\theta + C\cos 2\theta$ and hence we immediately find:

$$\begin{aligned} F' &= 2B\cos 2\theta - 2C\sin 2\theta \\ F'' &= -4B\sin 2\theta - 4C\cos 2\theta \qquad \Rightarrow \qquad F''' + 4F' = 0 \\ F''' &= -8B\cos 2\theta + 8C\sin 2\theta \end{aligned}$$

Applying the boundary condition (a) we obtain: $B = 0,\ A = -C\cos 2\alpha$. To determine $A$ and $C$ we now use the final boundary condition:

$$\int_{-\alpha}^{+\alpha} F(\theta)d\theta = \int_{-\alpha}^{+\alpha} \{C\cos 2\theta - C\cos 2\alpha\}d\theta = 1 \qquad \Rightarrow \qquad C = \frac{1}{[\sin 2\alpha - 2\alpha\cos 2\alpha]}$$

and the final velocity field is:

$$v_r = -\frac{f(\theta)}{r} = -\frac{Q_i'}{r}\left\{\frac{\cos 2\theta - \cos 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}\right\} = -\frac{Q_i'}{r[\tan 2\alpha - 2\alpha]}\left\{1 - \frac{\cos 2\theta}{\cos 2\alpha}\right\}$$

A quick sign check shows that this is inward everywhere, has greatest magnitude at the centerline ($\theta = 0$) and goes smoothly to zero at the wall $\theta = \pm\alpha$.

v) The radial pressure gradient is (from B2): $\dfrac{\partial p}{\partial r} = -\mu\dfrac{f''}{r^3} + \rho\dfrac{f^2}{r^3}$. The second term is obviously positive. Also from the form of the velocity field we can see that the second derivative of the function $f(\theta) \sim Q_i'\cos 2\theta$ is $f'' \sim -4Q_i'\cos 2\theta$ and thus the first term also increases with *r*. So the conclusion is the pressure always increases away from the apex (*r* = 0)…both viscous effects and inertial effects (Bernoulli) lead to a pressure drop as the fluid accelerates into the contraction throat.

**C. Axisymmetric Potential Flow Into A Capillary Tube**

(i) We are given the velocity potential $\Phi = -\dfrac{m}{r^2}\cos\theta$ and thus the velocity field is given by

$$v_r = \frac{\partial \Phi}{\partial r} = \frac{-2m\cos\theta}{r^3} \quad \text{and} \quad v_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = \frac{m\sin\theta}{r^3}.$$

Note that there are TWO components so the velocity vector is **not** purely radial in this case. Also note that it decays away as $r^{-3}$ (unlike the viscous flow solution above). Finally note that $m$ has units [m$^4$/s]. The streamtubes can be found from either expression for $v_r$ or $v_\theta$ to be:

$$\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = v_r = \frac{-2m\cos\theta}{r^3} \Rightarrow \quad \psi = \frac{-m\sin^2\theta}{r} \tag{C1}$$

Note that the streamfunction is only defined to within a constant…also note it has units of [m$^3$/s]. To draw this in the $\{R,z\}$ plane we can substitute for $z = r\cos\theta$, $R = r\sin\theta$ and by Pythagoras $r = \sqrt{R^2+z^2}$, $\sin^2\theta = R^2/(R^2+z^2)$ to find

$$\psi = \frac{-mR^2}{\left(R^2+z^2\right)^{3/2}} \quad \text{or equivalently } (-\psi/2m)^2\left(z^2+R^2\right)^3 = R^4$$

Given a doublet of strength $m$ this is an implicit equation for a streamline $R_i(z)$ for any value of $\psi$ and could be plotted in Matlab. Alternately, we can see that the flow is *focused* down and gets progressively thinner so we can rearrange to give:

$$(-\psi/2m)^2 z^6\left(1+(R_i/z)^2\right)^3 = R_i{}^4$$

or simplifying it for small values of $R/z \ll 1$: $\quad R_i \approx \pm(\psi/2m)z^{3/2} = Kz^{3/2}$ (C2)

(ii) The pressure field is given by the Bernoulli equation. Far away from the tube the velocity decays to zero as $r^{-3}$ and the pressure is $p_\infty$. Hence we find

$$p_\infty + 0 = p(r,\theta) + \tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} = p(r,\theta) + \tfrac{1}{2}\left\{v_r{}^2 + v_\theta{}^2\right\}$$

$$\text{or} \quad p(r,\theta) = p_\infty - \frac{m^2}{r^6}\left\{\sin^2\theta + 4\cos^2\theta\right\} = p_\infty - \frac{m^2}{r^6}\left\{1+3\cos^2\theta\right\} \tag{C3}$$

(iii) At the interface, the shear stresses balance and the normal stresses only differ by the additional pressure associated with curvature. Formally (if $n$ is a normal vector and $t$ is a tangent vector):

$\tau_{tn}{}^{(i)} = \tau_{tn}{}^{(o)}$ and $-p^{(i)} + \tau_{nn}{}^{(i)} = -p^{(o)} + \tau_{nn}{}^{(o)} + 2\mathcal{H}\sigma$

where the mean curvature for this axisymmetric jet is $2H \approx \dfrac{1}{R_i} - R_i''$. Also note that we require the velocity <u>vectors</u> to match; $\boldsymbol{v_i} = \boldsymbol{v_o}$. It thus looks like we have FOUR boundary conditions at the interface (which we do…we have to solve a 2nd order ODE for each fluid!)

(iv) As we have shown in C2, the dividing streamline is a curve of the form $R_i = Kz^{3/2}$ (in cylindrical polars). If we write down the lubrication equations, we obtain

$$0 = -\frac{\partial p}{\partial z} + \mu_i \frac{\partial^2 v_z}{\partial R^2} \text{ with solution } v_z = \frac{1}{2\mu}\frac{\partial p}{\partial z}R^2 + c_1 R + c_2 \text{ where } \frac{\partial p}{\partial z} = \frac{24m^2}{r^7}$$

However when we try and match the velocity at the interface there is no slope since $\tau_{Rz}^{(i)} = 0 = \mu_i \frac{\partial v_z}{\partial r}$. Similarly there is no slope at the centerline $R = 0$ and the solution is thus a simple plug flow with

$$v_r = \frac{\partial \Phi}{\partial r} = \frac{-2m\cos\theta}{r^3} \approx \frac{-2m}{z^3} \quad \text{and} \quad v_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = \frac{m\sin\theta}{r^3} \approx 0.$$

Note that the angle $\tan\theta = R/z = Kz^{1/2} \to 0$, so the flow is increasingly close to a lubrication flow. Although not asked to check this, you can see that this solution also satisfies conservation of mass because:

$$Q_i = \pi R_j^2 v_r = \pi\left(Kz^{3/2}\right)^2\left(-2m/z^3\right) = \text{const.}$$

(v) Consider a coordinate system *(x,y)* that is locally normal to the interface and in the flow direction $R = R_i(z)$; i.e. $x \approx -z$ and $y = R - R_i(z)$. The boundary layer equations are then (neglecting *g)*:

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p^{(o)}}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y^2} \quad \text{and} \quad 0 = \frac{\partial p^{(o)}}{\partial y}$$

where $p^{(o)}$ is the pressure field for inviscid flow which is 'imposed' on the boundary layer and which we found in part (ii) – see eq. C3. The solution to this equation would be different than the boundary layer solution for two reasons:

1. There is a favorable pressure gradient; the pressure decreases as we focus the flow towards the inlet; with the gradient given by:

$$\frac{\partial p^{(o)}}{\partial x} = -\frac{dp^{(o)}}{dz} = -\frac{24m^2}{z^7}$$

2. The appropriate boundary condition to apply at the boundary is NOT $v_x(y=0) = 0$ but rather the no-tangential stress condition: $\tau_w = \mu \left.\frac{\partial v_x}{\partial y}\right|_{y=0} = 0$.

Both of these effects are likely to make the boundary layer grow much more slowly than the Blausius solution.