MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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Computational Geometry
Spring Term, 2003
Problem Set 1 on Differential Geometry

Issued: Day 2
Due: Day 6
Weight: 15% of total grade

Individual Effort

1. Show that the curvature of a planar curve is independent of the parametrization. Namely, if

$$\mathbf{r}(t) = [x(t), y(t)] \tag{1}$$

is the curve then a change of variables

$$t = w(u) \;\; with \;\; w'(u) \neq 0 \tag{2}$$

does not affect the curvature. (Problem 18 in the textbook)

2. Let a curve $\mathbf{X}$ be defined by

$$\mathbf{X}(t) = a \int \mathbf{g}(t) \times \mathbf{g}'(t) dt, \quad a = const \neq 0, \tag{3}$$

where $\mathbf{g}(t)$ is a vector function satisfying $|\mathbf{g}(t)| = 1$ and $[\mathbf{g}\mathbf{g}'\mathbf{g}''] \neq 0$. Show that the curvature and the torsion of the curve are $\kappa \neq 0$ and $\tau = 1/a$, respectively.

3. Find the parametric equation of a curve whose curvature κ and torsion τ are respectively

$$\kappa = \frac{a}{a^2 + b^2}, \quad \tau = \frac{b}{a^2 + b^2}, \tag{4}$$

where $a > 0$ and b are constants.

4. A curve $\mathbf{C}_1$ is called an *involute* of a given curve $\mathbf{C}$ if tangents of $\mathbf{C}$ are normal to $\mathbf{C}_1$. The curve $\mathbf{C}$ is called an *evolute* of $\mathbf{C}_1$. Show that the curvature κ_1 of $\mathbf{C}_1$ is given by

$$\kappa_1^2 = \frac{\kappa^2 + \tau^2}{\kappa^2 (c - s)^2}, \tag{5}$$

where c is a constant, s is the arc length of C measured from a fixed point on C, and κ and τ are the curvature and torsion of C.

5. Let E, F, G be the coefficients of the first fundamental form of a regular surface $\mathbf{R} = \mathbf{R}(u, v)$. Let $f(u, v) = c$ and $g(u, v) = d$ be two families of regular curves defined in the parameter space $u - v$ of the surface with images in 3D space obtained for various constants c and d. Prove that the 3D images of these two families of curves are orthogonal (i.e. whenever two curves of distinct families meet, their tangents are orthogonal) if and only if

$$E f_v g_v - F(f_u g_v + f_v g_u) + G f_u g_u = 0 \tag{6}$$

where $E = \mathbf{R}_u \cdot \mathbf{R}_u, F = \mathbf{R}_u \cdot \mathbf{R}_v, G = \mathbf{R}_v \cdot \mathbf{R}_v$, and subscripts u, v denote partial derivatives.

6. Consider a torus parametrized as follows:

$$\mathbf{r}(u, v) = [(R + a\ cos\ u)cos\ v, (R + a\ cos\ u)sin\ v, a\ sin\ u] \tag{7}$$

where $0 \le u \le 2\pi, 0 \le v \le 2\pi$, and R and a are constants such taht $R > a$. Derive formulae for the Gauss, mean and principal curvatures. Sketch the torus and subdivide it into hyperbolic, parabolic and elliptic regions. In a follow-up sketch illustrate the lines of curvature of the torus. Explain the above subdivision and sketches. (Problem 17 in the textbook)

7. Show that the surface area on a Monge patch $\mathbf{X}(u, v) = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u, v)\mathbf{e}_3$ is given by the integral

$$A = \int\int_W \sqrt{1 + f_u^2 + f_v^2} du dv, \tag{8}$$

where W is the parameter domain, and $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ the unit coordinate vectors.

8. Show that the second fundamental form on a Monge patch $\mathbf{X}(u, v) = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u, v)\mathbf{e}_3$ is

$$II = (f_u^2 + f_v^2 + 1)^{-\frac{1}{2}} \left[f_{uu} du^2 + 2 f_{uv} du dv + f_{vv} dv^2 \right], \tag{9}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the unit coordinate vectors.

9. Show that the principal curvatures of the surface $f(x, y, z) = x \sin(z) - y \cos(z) = 0$ are $\pm(x^2 + y^2 + 1)^{-1}$.

10. Consider the parametrized surface

$$\mathbf{r}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2 \right). \tag{10}$$

Show that

(a) The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, F = 0. \tag{11}$$

(b) The coefficients of the second fundamental form are

$$L = 2, M = -2, N = 0. \tag{12}$$

(c) The principal curvatures are

$$\kappa_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad \kappa_2 = -\frac{2}{(1 + u^2 + v^2)^2}. \tag{13}$$