Computational Ocean Acoustics

- Ray Tracing
- Wavenumber Integration
- Normal Modes
- Parabolic Equation

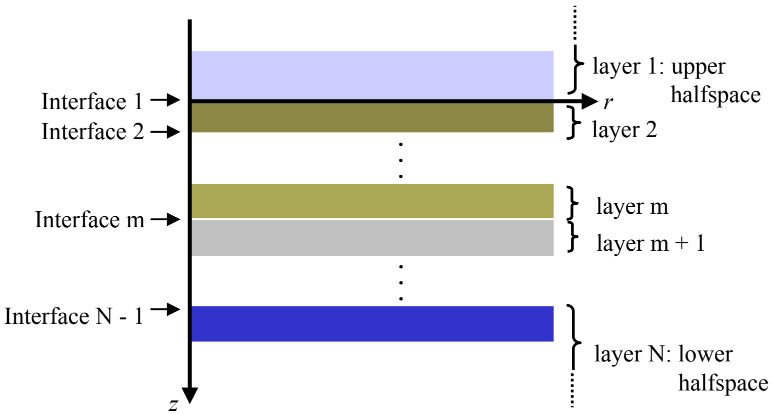
Wavenumber Integration

- Range-independent – Integral Transform solution
- Exact depth-dependent solution
  - Global Matrix Approach
  - Propagator Matrix Approach
  - Invariant Embedding
- Numerical issues:
  - Numerical stability of depth solution
  - Evaluation of inverse transforms

Global Equations and Unknowns

Wavefield Unknowns		Boundary Conditions
0	Vacuum	
		2
4	Elastic Ice Cover	
		3
2	Fluid Water Column	
		2
2	Fluid Sediment Layer	
		3
4	Elastic Sediment Layer	
		4
2	Elastic Halfspace	
14 unknowns		14 equations

Propagator Matrix Approach



- Until 1980's the Direct Global Matrix (DGM) approach was assumed to be unstable.
- memory was an issue in computing

Layer m - field at interface m

$$\mathbf{v}_m(k_r, z_m) = \left\{ \begin{array}{c} p(k_r, z_m) \\ w(k_r, z_m) \end{array} \right\}$$

$$= \begin{bmatrix} \rho_m \omega^2 e^{ik_{z;m}(z_m - z_{m-1})} & \rho_m \omega^2 \\ ik_{z;m} e^{ik_{z;m}(z_m - z_{m-1})} & -ik_{z;m} \end{bmatrix} \left\{ \begin{array}{c} A_m^+ \\ A_m^- \end{array} \right\} = \mathbf{c}_m(k_r, z_m)\, \mathbf{a}_m(k_r)$$

Layer m - field at interface $m-1$

$$\mathbf{v}_m(k_r, z_{m-1}) = \left\{ \begin{array}{c} p(k_r, z_{m-1}) \\ w(k_r, z_{m-1}) \end{array} \right\}$$

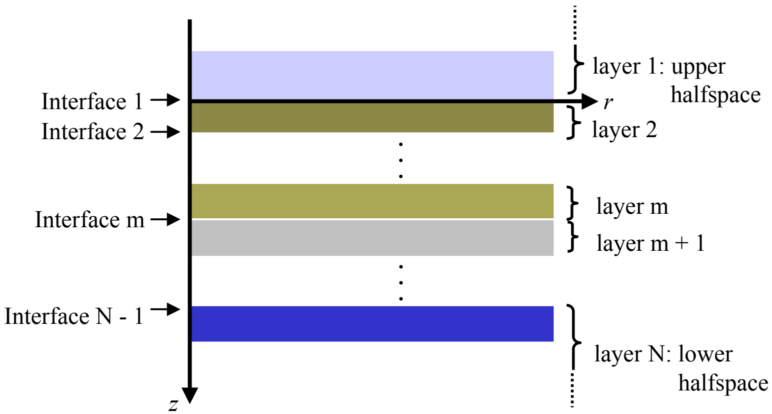
$$= \begin{bmatrix} \rho_m \omega^2 & \rho_m \omega^2 e^{-ik_{z;m}(z_{m-1} - z_m)} \\ ik_{z;m} & -ik_{z;m} e^{ik_{z;m}(z_m - z_{m-1})} \end{bmatrix} \left\{ \begin{array}{c} A_m^+ \\ A_m^- \end{array} \right\} = \mathbf{c}_m(k_r, z_{m-1})\, \mathbf{a}_m(k_r)$$

Eliminate $\mathbf{a}_m$

$$\mathbf{v}_m(k_r, z_{m-1}) = \mathbf{c}_m(k_r, z_{m-1}) \left[\mathbf{c}_m(k_r, z_m)\right]^{-1} \mathbf{v}_m(k_r, z_m) = \mathbf{P}_m(k_r) \mathbf{v}_m(k_r, z_m),$$

Propagator Matrix

$$\mathbf{P}_m(k_r) = \mathbf{c}_m(k_r, z_{m-1}) \; \left[\mathbf{c}_m(k_r, z_m)\right]^{-1} .$$



Recursion - Layer $m-n$ below source

$$\begin{aligned}
\mathbf{v}_m(k_r, z_m) &= \mathbf{v}_{m+1}(k_r, z_m) \\
&= \mathbf{P}_{m+1}(k_r)\mathbf{v}_{m+1}(k_r, z_{m+1}) \\
&= \mathbf{P}_{m+1}(k_r)\mathbf{P}_{m+2}(k_r)\mathbf{v}_{m+2}(k_r, z_{m+2}) \\
&= \mathbf{P}_{m+1}(k_r)\mathbf{P}_{m+2}(k_r)\cdots\mathbf{P}_n(k_r)\mathbf{v}_n(k_r, z_n) \\
&= \prod_{\ell=m+1}^{n} \mathbf{P}_\ell(k_r)\mathbf{v}_n(k_r, z_n) \\
&= \mathbf{R}_n^m(k_r)\, \mathbf{v}_n(k_r, z_n) \\
&= \mathbf{R}_n^m(k_r)\, \mathbf{v}_{n+1}(k_r, z_n)
\end{aligned}$$

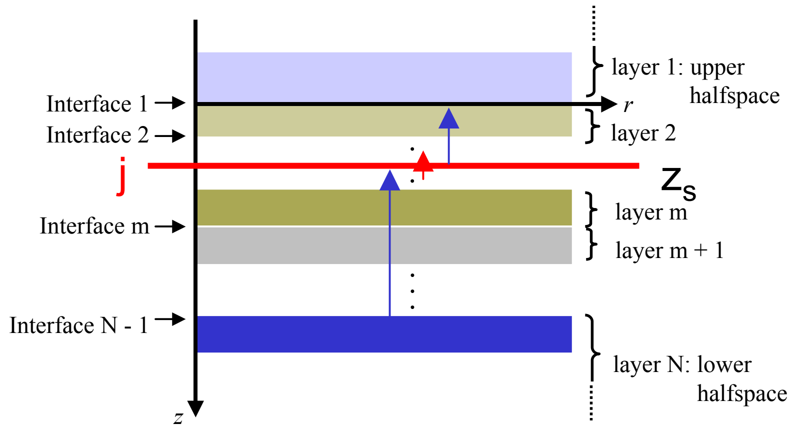
Recursive Propagator Matrix

$$\mathbf{R}_n^m(k_r) = \prod_{\ell=m+1}^{n} \mathbf{P}_\ell(k_r)\,.$$

Layer N - Lower halfspace radiation condition

$$\left.\begin{aligned}
p(k_r, z_{N-1}) &= A_N^+ \rho_N \omega^2 \\
w(k_r, z_{N-1}) &= A_N^+ i k_{z;N}
\end{aligned}\right\} \Rightarrow p(k_r, z_{N-1}) = \frac{\rho_N \omega^2}{i k_{z;N}} w(k_r, z_N - 1)$$

$$\mathbf{v}_N(k_r, z_{N-1}) = \begin{Bmatrix} \frac{\rho_N \omega^2}{i k_{z;N}} \\ 1 \end{Bmatrix} w(k_r, z_{N-1})$$



Propagator Matrix Solution Procedure

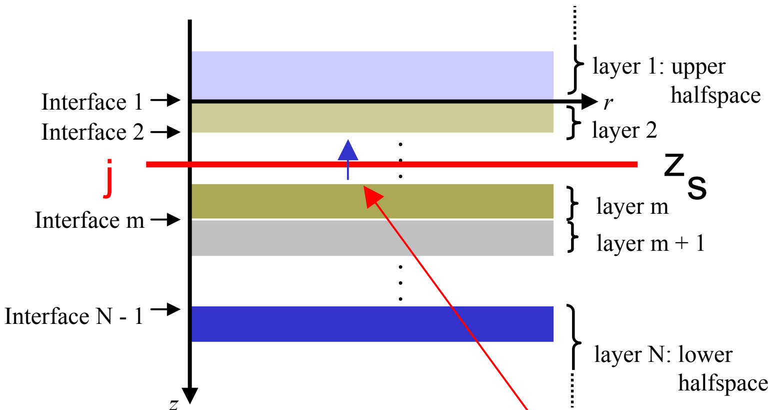
- Introduce interface number j at source depth $z_j = z_s$
- Compute recursive propagator R_N^m from lower halfspace to source depth $z_j = z_s$.
- Introduce source as discontinuity in field vector
- Propagate using recursive operator to sea surface
- Solve for unknown fields at lowermost and uppermost intefaces
- Propagate field from borttom interface to receiver depth

Source Treatment

Source in infinite medium

$$\hat{\psi}(k_r, z) = S_\omega g_\omega(k_r, z, z_s) = \frac{S_\omega}{4\pi} \frac{e^{ik_{z;s}|z-z_s|}}{ik_{z;s}}$$

Field discontinuity at source depth



$$\left.\begin{array}{l} \hat{p}(k_r, z_s^+) = \frac{S_\omega \rho_s \omega^2}{4\pi i k_{z;s}} \\ \hat{p}(k_r, z_s^-) = \frac{S_\omega \rho_s \omega^2}{4\pi i k_{z;s}} \end{array}\right\} \Rightarrow \hat{p}(k_r, z_s^-) = \hat{p}(k_r, z_s^+)$$

$$\left.\begin{array}{l} \widehat{w}(k_r, z_s^+) = \frac{S_\omega}{4\pi} \\ \widehat{w}(k_r, z_s^-) = -\frac{S_\omega}{4\pi} \end{array}\right\} \Rightarrow \widehat{w}(k_r, z_s^-) = \widehat{w}(k_r, z_s^+) - \frac{S_\omega}{2\pi}$$

Field Vector Discontinuity at Source Depth

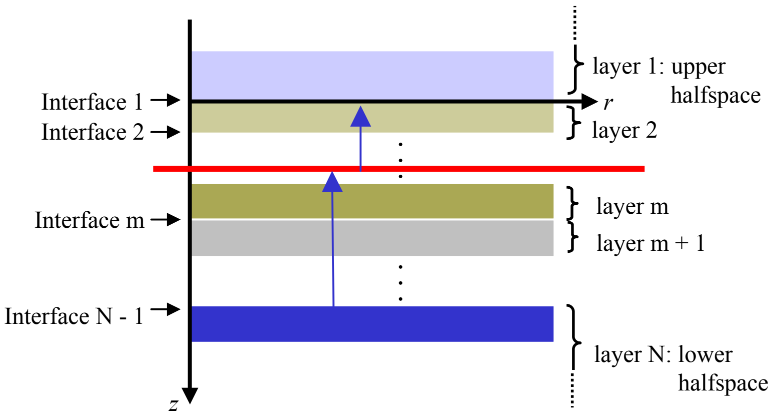
$$\mathbf{v}_{j+1}(k_r, z_j) = \left\{ \begin{array}{c} \hat{p}(k_r, z_s^+) \\ \widehat{w}(k_r, z_s^+) \end{array} \right\}$$

$$\mathbf{v}_{j}(k_r, z_j) = \left\{ \begin{array}{c} \hat{p}(k_r, z_s^-) \\ \widehat{w}(k_r, z_s^-) \end{array} \right\}$$

Propagator across Source Depth

$$\begin{aligned} \mathbf{v}_j(k_r, z_j) &= \mathbf{v}_{j+1}(k_r, z_j) + \left\{ \begin{array}{c} 0 \\ -\frac{S_\omega}{2\pi} \end{array} \right\} \\ &= \mathbf{v}_{j+1}(k_r, z_j) + \hat{\mathbf{v}}(k_r, z_j) \end{aligned}$$

Propagator Matrix Solution



Propagator to above source depth

$$\mathbf{v}_j(k_r, z_j^-) = \mathbf{R}^j_{N-1}(k_r)\mathbf{v}_N(k_r, z_{N-1}) + \hat{\mathbf{v}}(k_r, z_j)$$

Propagation to first interface

$$\mathbf{v}_1(k_r, z_1) = \mathbf{R}^1_j(k_r)\left[\mathbf{R}^j_{N-1}(k_r)\,\mathbf{v}_N(k_r, z_{N-1}) + \hat{\mathbf{v}}(k_r, z_j)\right] \;.$$

Surface Pressure Release Condition

$$p(k_r, z_1) = 0\,,$$

- 2 equations with 2 unknowns: $w(k_r, z_1)$ and $w(k_r, z_{N-1}$

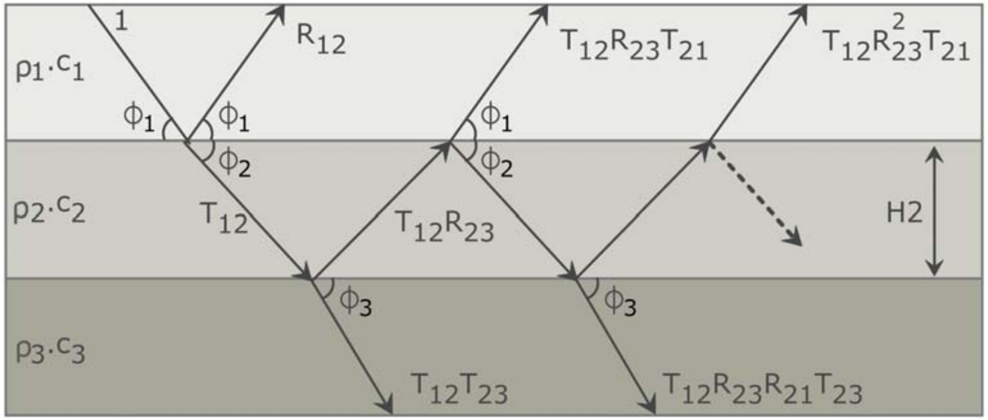
Numerical Stability

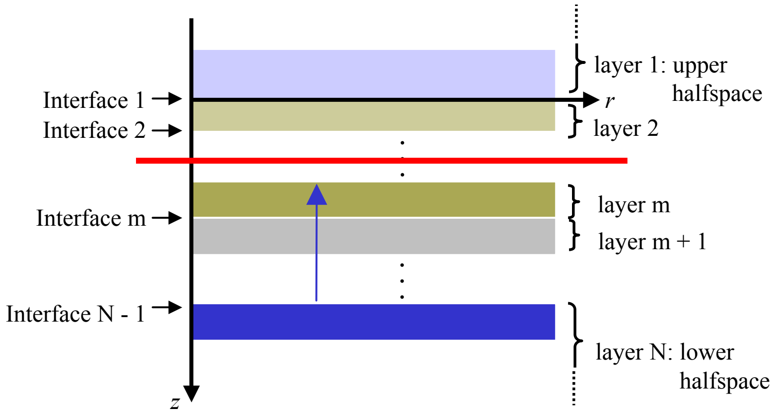
Layer Propagator Matrix

$$\mathbf{P}_m(k_r) = \mathbf{d}_m(k_r)\,\mathbf{e}_m(k_r, z_{m-1})\,[\mathbf{e}_m(k_r, z_m)]^{-1}\,[\mathbf{d}_m(k_r)]^{-1}\,,$$

$$\mathbf{e}_m(k_r, z_{m-1})\,[\mathbf{e}_m(k_r, z_m)]^{-1} = \begin{bmatrix} e^{-ik_{z,m}h_m} & 0 \\ 0 & e^{ik_{z,m}h_m} \end{bmatrix}\,.$$

Numerically unstable in evanescent regime





Invariant Embedding Approach

Reflection Coefficient Recursion

$$R_m = \frac{R_{m-1,m} + R_{m,m+1}\exp(2ik_{z,m}h_m)}{1 + R_{m-1,m}\,R_{m,m+1}\exp(2ik_{z,m}h_m)} ,$$

Numerical Stability

Recursion in Strongly Evanescent Layer

$$R_m \simeq R_{m-1,m} ,$$

Unconditionally Numerically Stable