2.004 Dynamics and Control II
Spring 2008

Lecture 13[1]

<u>Reading:</u>

- Class Handout: *Modeling Part 2 – Summary of One-Port Primitive Elements*

1 Generalized System Representation

In the electrical and mechanical systems we have defined, and used, two *power variables* (variables whose product is power):

Electrical	Mechanical
$P = vi$, where	$P = Fv$, where
v – voltage drop	v – velocity drop
i – current	F – force

We have also developed a common graph notation to express the system topology, and have used extensions to Kirchoff's laws to draw *analogies* between pairs of system variables:

(1) voltage drop and velocity drop, and

(2) current and force.

In addition, we drew analogies between the primitive modeling elements in each domain:

(1) capacitors and masses

(2) inductors and springs

(3) resistors and dampers.

We now look at generalizing these associations.

Variables: In each energy domain we will class the two power variables as either:

(a) Those variables that are defined by measuring a difference, or drop, *across* an element, that is between nodes on a graph (across one or more branches). These variables sum to zero around any closed loop on the graph (they satisfy the *compatibility conditions*). These variables are defined to be <u>across-variables</u>.

The two across variables we have defined so far are (i) *velocity drop* in mechanical systems, and (ii) *voltage drop* in electrical systems.

Example: To measure voltage drop in an electrical circuit you must connect a voltmeter *across* an element:
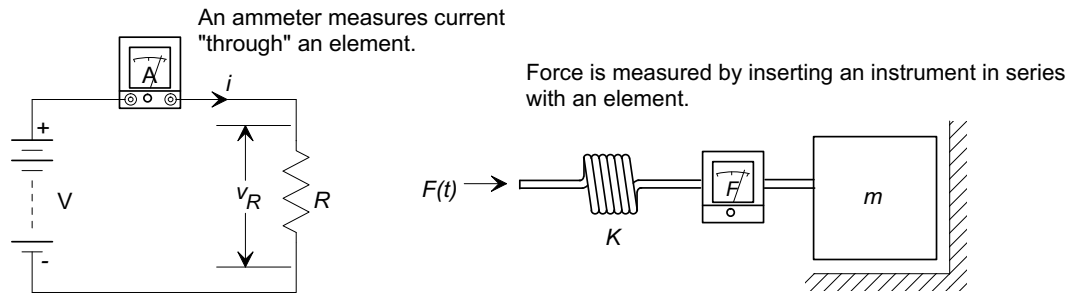
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A voltmeter measures voltage "across" an element
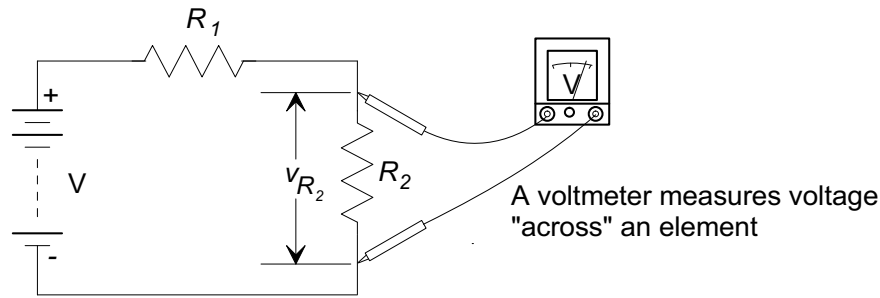
(b) Those variables that are measured *through* an element, that is are considered as being transmitted through an element unchanged. These variables sum to zero at the nodes on a graph, and are said to satisfy the *continuity condition*. They are defined to be *through variables*.

The through variables defined so far are (i) *current* in electrical systems, and (ii) *force* in mechanical systems.

Example: to measure force in a mechanical system, or current in an electrical system a sensor mus be inserted in series with an element.

An ammeter measures current "through" an element.

Force is measured by inserting an instrument in series with an element.

Generalized Variables: In dealing with generalized modeling power variables, without regard for a particular energy domain we now define

- A generalized across variable, designated v, and

- A generalized through variable, designated f.

We also generalize the constraints on across and through variables imposed by the structure of a system's graph:

- The compatibility condition

$$\sum_{i=1}^{n} v_i = 0 \qquad \text{around the } n \text{ elements in any loop on a graph}$$

which is clearly a generalization of Kirchoff's voltage law, and
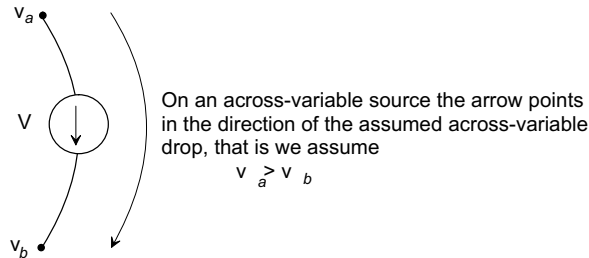
- The continuity condition

$$\sum_{i=1}^{n} f_i = 0 \qquad \text{in the } n \text{ elements connected to any node on a graph}$$

which is clearly a generalization of Kirchoff's current law.

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Generalized Sources: Along with the generalized variables we define a pair of ideal sources

(1) The Across Variable Source: An *across variable source* maintains the across variable between the nodes it is connected to, regardless of the magnitude of the through variable supplied to the system.

On an across-variable source the arrow points in the direction of the assumed across-variable drop, that is we assume
$$v_a > v_b$$

The examples we have seen so far are the *voltage source* in electrical systems, and the *velocity source* in mechanical systems.

(2) The Through Variable Source: An *through variable source* maintains the through variable into the nodes it is connected to, regardless of the magnitude of the across variable it must generate to maintain its through variable.


On an through-variable source the arrow points in the direction of the assumed through-variable "flow".
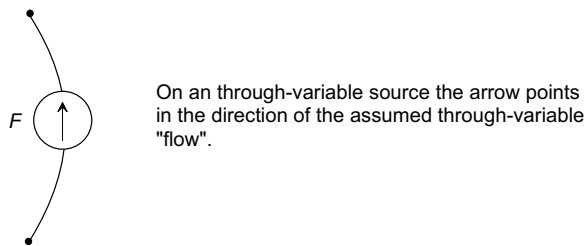
The examples we have seen so far are the *current source* in electrical systems, and the *force source* in mechanical systems.

Generalized Elements: We have seen that in each of the energy domains studied so far we have defined three primitive modeling elements: two energy storage elements and a dissipative element. We now classify these elements according to (i) the variable that accounts for the stored energy in energy storage elements, and (ii) we group together the dissipative elements.

A-Type Elements These are the energy storage elements in which the *stored energy is a function of the across-variable.*
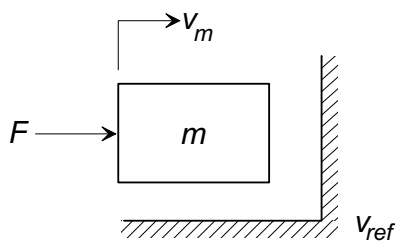**Electrical:** For a capacitor, let $v = v_a - v_b$ be the across variable:
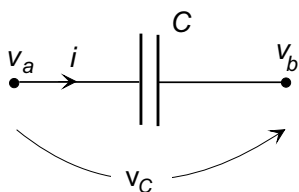
$$i = C\frac{dv}{dt}$$
$$E = \int_{-\infty}^{t} vi \, dt = \int_{0}^{t} Cv \, dv$$
$$= \frac{1}{2}Cv^2$$

which defines the capacitor as an A-type element.

Mechanical: For a mass element

$$F = m\frac{dv}{dt}$$

$$E = \int_{-\infty}^{t} vF\,dt = \int_{0}^{t} mv\,dv$$

$$= \frac{1}{2}mv^2$$

which defines the mass element as an A-type element.

> Define a generalized capacitance C that represents A-type elements independent of the energy domain, and write its elemental equation in terms of generalized across and through variables:
>
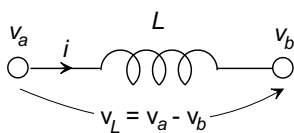> $$\text{f} = \text{C}\frac{d\text{v}}{dt}$$
>
> generalized through variable · generalized across variable · generalized capacitance

A-Type elements may be summarized as in the following table:

Element	Elemental equation	Energy
Generalized A-type	$\text{f} = \text{C}\dfrac{d\text{v}}{dt}$	$\mathcal{E} = \dfrac{1}{2}\text{Cv}^2$
Translational mass	$F = m\dfrac{dv}{dt}$	$\mathcal{E} = \dfrac{1}{2}mv^2$
Electrical capacitance	$i = C\dfrac{dv}{dt}$	$\mathcal{E} = \dfrac{1}{2}Cv^2$

T-Type Elements These are the energy storage elements in which the *stored energy is a function of the through-variable.*

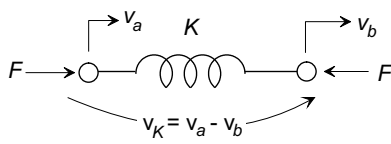**Electrical:** For a inductor, let $v = v_a - v_b$ be the across variable:

$$v = L\frac{di}{dt}$$

$$E = \int_{-\infty}^{t} vi\,dt = \int_{0}^{t} Li\,di$$

$$= \frac{1}{2}Li^2$$

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which defines the inductor as an T-type element.

Mechanical: For a spring element

$$v = \frac{1}{K}\frac{dF}{dt}$$

$$E = \int_{-\infty}^{t} vF\,dt = \frac{1}{K}\int_{0}^{t} F\,dF$$

$$= \frac{1}{2K}F^2$$

which defines the spring element as an T-type element.

> Define a generalized inductance L that represents T-type elements independent of the energy domain, and write its elemental equation in terms of generalized across and through variables:
>
>
> $$\mathsf{v} = \mathsf{L}\frac{d\mathsf{f}}{dt}$$

T-Type elements may be summarized as in the following table:

Element	Elemental equation	Energy
Generalized T-type	$\mathsf{v} = \mathsf{L}d\mathsf{f}/dt$	$\mathcal{E} = \frac{1}{2}\mathsf{L}\mathsf{f}^2$
Translational spring	$v = \frac{1}{K}\frac{dF}{dt}$	$\mathcal{E} = \frac{1}{2K}F^2$
Electrical inductance	$v = L\frac{di}{dt}$	$\mathcal{E} = \frac{1}{2}Li^2$

D-Type Elements   These are the dissipative elements (non-energy storage, the power flow is always into the element).

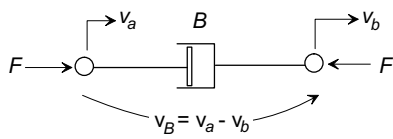**Electrical:** For a resistor, let $v = v_a - v_b$ be the across variable:
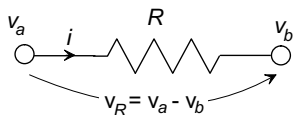
$$v = iR$$

$$P = vi = i^2R = v^2/R$$

$$\geq 0$$

which defines the resistor as an D-type element.

Mechanical: For a damper element

$$
\begin{aligned}
F &= Bv \\
P &= vF = Bv^2 = F^2/B \\
&\geq 0
\end{aligned}
$$

which defines the damper element as an D-type element.

> Define a generalized resistance R that represents D-type elements independent of the energy domain, and write its elemental equation in terms of generalized across and through variables:
>
> generalized through variable
> $$\mathsf{v} = R\mathsf{f}$$
> generalized across variable   generalized resistance

D-Type elements may be summarized as in the following table:

Element	Elemental equations		Power dissipated
Generalized D-type	$\mathsf{f} = \dfrac{1}{R}\mathsf{v}$	$\mathsf{v} = R\mathsf{f}$	$\mathcal{P} = \dfrac{1}{R}\mathsf{v}^2 = R\mathsf{f}^2$
Translational damper	$F = Bv$	$v = \dfrac{1}{B}F$	$\mathcal{P} = Bv^2 = \dfrac{1}{B}F^2$
Electrical resistance	$i = \dfrac{1}{R}v$	$v = Ri$	$\mathcal{P} = \dfrac{1}{R}v^2 = Ri^2$

Generalized Impedances: The generalized impedance of an is
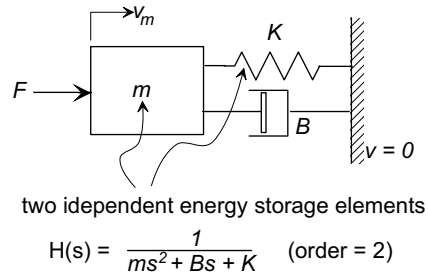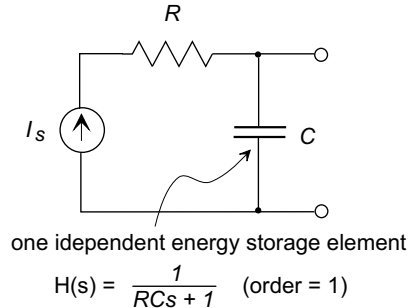
$$
\boxed{Z = \frac{V(s)}{F(s)}}
$$

where $V(s)$ and $F(s)$ are the Laplace domain representation of the generalized across and through variables. The impedances are summarized in the following table:

|	A-Type	T-Type	D-Type
Generalized	$\dfrac{1}{Cs}$	sL	R
Translational	$\dfrac{1}{sm}$	$\dfrac{1}{K}s$	$\dfrac{1}{B}$
Electrical	$\dfrac{1}{Cs}$	sL	R

2   Notes on Transfer Function Generation

The system order (highest order derivative on the l.h.s. of the differential equation, or the highest power in s in the denominator of the transfer function) is determined by the number of *independent energy storage elements* in the system.

one idependent energy storage element

$$H(s) = \frac{1}{RCs + 1} \quad \text{(order = 1)}$$

two idependent energy storage elements

$$H(s) = \frac{1}{ms^2 + Bs + K} \quad \text{(order = 2)}$$

An independent energy storage element is one whose stored energy is not directly proportional to the energy stored in other elements or completely defined by the system sources. For example, in the following figure the two springs are not independent energy storage elements, because their stored energies are directly related. Therefore this system will generate a second-order differential equation even though there are three energy storage elements.
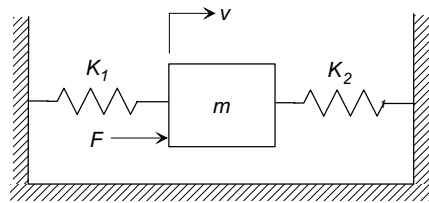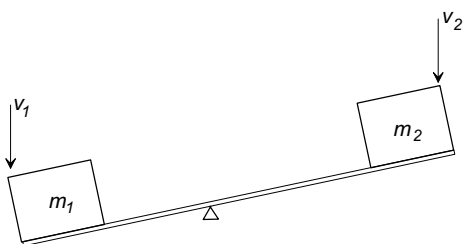

K_1 and K_2 are not independent energy storage elements since $E_{K_1} \propto E_{K_2}$
This is a second-order system.
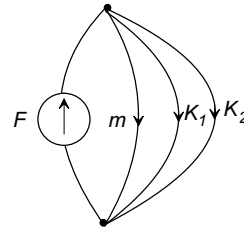
K_1 and K_2 are effectively in parallel and may be replaced by an equivalent spring.

similarly, in the following figure the velocities of the two masses are related by the lever, and they are not independent energy storage elements.

m_1 and m_2 are not independent energy storage elements because $v_1 \propto v_2$ and therefore their energies are algebraically related.

Another situation where an energy storage element does not increase the order of a system is when the stored energy is completely defined by a source.

(a)   Any element connected directly in parallel with an across variable source will not affect the system dynamics, and in fact will not appear in the differential equation (or

transfer function). The reason is that the source completely defines the across variable at its two nodes, and therefore the element can be removed without affecting the rest of the system. For example in the following system the output is the velocity of the mass m_2. The system input is a velocity source V applied to the mass m_1. The velocity at the left-hand node is completely defined by the source, and would not be affected if m_1 was removed from the system. The transfer function relating v_{m_2} to V is

$$H(s) = \frac{V_{m_2}(s)}{V(s)} = \frac{K/m_2}{s^2 + K/m_2}$$

and does not involve m_1.

The velocity of m_1 is completely defined by the source
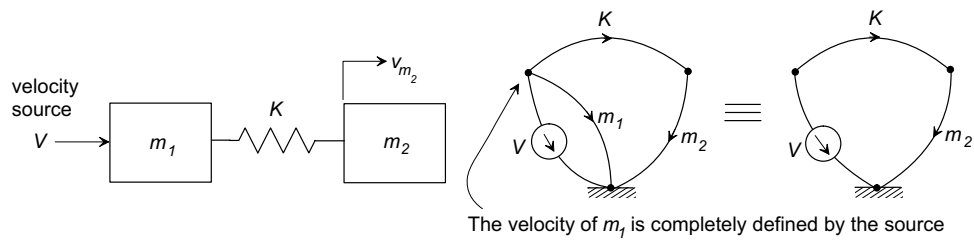
(b) Any element connected directly in series with a through variable source will not affect the system dynamics. The through variable transmitted from the source to the rest of the system will be unchanged by the presence of the series element(s). In the electrical system below the resistance R_1 and the inductor L are in series with the current source I. Since $i_L = I$, the current supplied to the right-hand node is unaffected by the presence if either L or R_1, and this is effectively a first-order system.

R_1 and L do not affect the system dynamics since the source defines the current through them.