2.004 Dynamics and Control II
Spring 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

DEPARTMENT OF MECHANICAL ENGINEERING

2.004 *Dynamics and Control II*
Spring Term 2008

Lecture 31[1]

Reading:

- Nise: 10.1

- Class Handout: Frequency Response and Bode Plots

1   Sinusoidal Frequency Response (continued)

In Lecture 30 we saw that the steady-state response of a linear system with transfer function $H(s)$ to a sinusoidal input
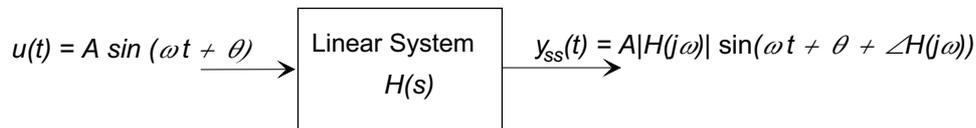
$$u(t) = A\sin(\omega t + \theta)$$

is

$$y_{ss}(t) = A\left|H(j\omega)\right|\sin\left(\omega t + \theta + \angle H(j\omega)\right)$$

where

$$
\begin{aligned}
|H(j\omega)| &= \sqrt{\Re^2\left\{H(j\omega)\right\} + \Im^2\left\{H(j\omega)\right\}}\\
\angle H(j\omega) &= \arctan\left(\frac{\Im\left\{H(j\omega)\right\}}{\Re\left\{H(j\omega)\right\}}\right)
\end{aligned}
$$

We note that

$$H(j\omega) = \left. H(s)\right|_{s=j\omega},$$

that is $H(j\omega)$ is $H(s)$ evaluated along the imaginary axis of the s-plane.

|H(jω)| is a "slice" of the |H(s)| surface along the imaginary axis of the s-plane.
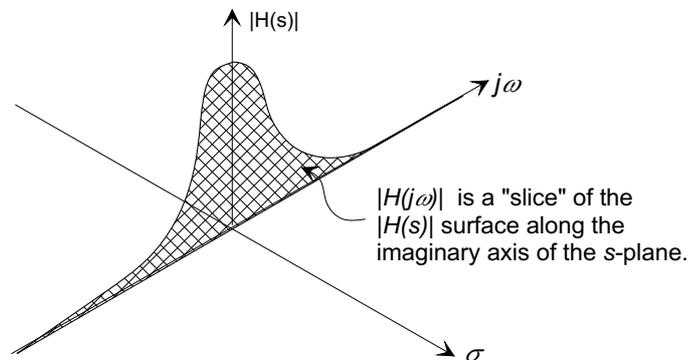
1.1 The Frequency Response of Systems with Zeros

If a system has a transfer function

$$H(s) = \frac{N(s)}{D(s)}$$

the frequency response function is

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)}.$$

For complex a and b, $|a/b| = |a| \, / \, |b|$ and $\angle(a/b) = \angle a - \angle b$, so that

$$\left| H(j\omega) \right| = \frac{|N(j\omega)|}{|D(j\omega)|} = \frac{\sqrt{\Re^2\{N(j\omega)\} + \Im^2\{N(j\omega)\}}}{\sqrt{\Re^2\{D(j\omega)\} + \Im^2\{D(j\omega)\}}} \tag{1}$$

$$\angle H(j\omega) = \angle N(j\omega) - \angle D(j\omega) = \arctan\left(\frac{\Im\{N(j\omega)\}}{\Re\{N(j\omega)\}}\right) - \arctan\left(\frac{\Im\{D(j\omega)\}}{\Re\{D(j\omega)\}}\right) \tag{2}$$

∎ Example 1

Find and plot the frequency response of

$$H(s) = \frac{s+5}{s+10}$$

The frequency response function is
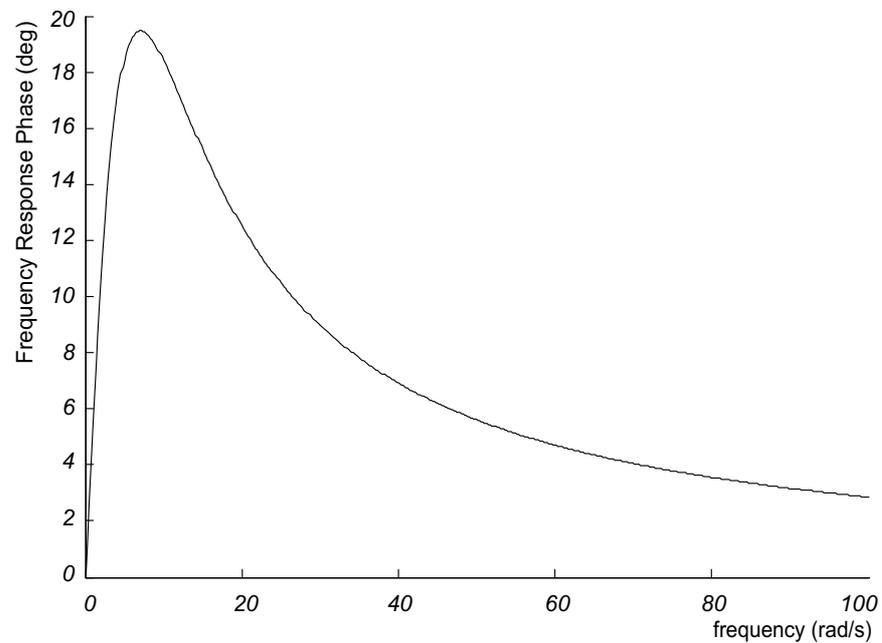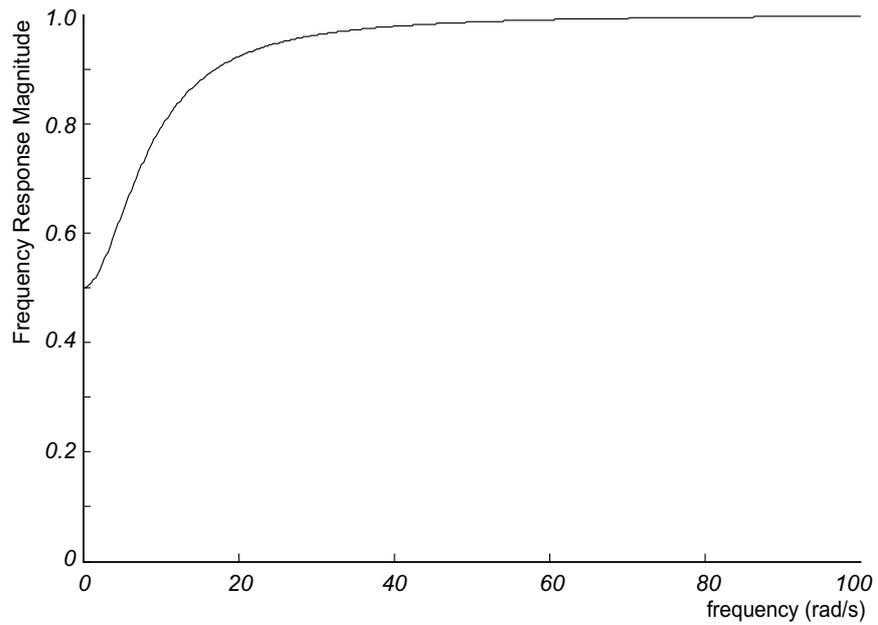
$$H(j\omega) = \frac{j\omega + 5}{j\omega + 10}$$

and

$$\left| H(j\omega) \right| = \frac{|N(j\omega)|}{|D(j\omega)|} = \frac{\sqrt{\omega^2 + 25}}{\sqrt{\omega^2 + 100}}$$

$$\angle H(j\omega) = \angle N(j\omega) - \angle D(j\omega) = \arctan\left(\frac{\omega}{5}\right) - \arctan\left(\frac{\omega}{10}\right)$$

The following MATLAB commands were used to plot the frequency response:

```
w=0:.2:100;
sys=zpk(-5,-10,1)
y=freqresp(sys,w);
plot(w,squeeze(abs(y)))
plot(w,squeeze(angle(y)))
```

which produced the following plots:

We note that

- When $\omega = 0$, $|H(j\omega)| = 0.5$ and $\angle H(j\omega) = 0°$.
- When $\omega \to \infty$, $|H(j\omega)| \to 1$ and $\angle H(j\omega) \to 0°$.

■ Example 2

Find the frequency response functions for (i) a differentiator, and (ii) an integrator.

(i) A differentiator. The transfer function is $H(s) = s$, so that $H(j\omega) = j\omega$.
Then

$$|H(j\omega)| = \omega, \qquad \angle H(j\omega) = \frac{\pi}{2} \text{ (or } +90°)$$

(ii) An integrator. The transfer function is $H(s) = 1/s$, so that $H(j\omega) = 1/j\omega$.
Then

$$|H(j\omega)| = \frac{1}{\omega}, \qquad \angle H(j\omega) = -\frac{\pi}{2} \text{ (or -90°)}$$

1.2 The Frequency Response of a Second-Order System

Consider the unity-gain second-order system:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The frequency response is

$$H(j\omega) = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2}$$

so that

$$|H(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \tag{3}$$

$$\angle H(j\omega) = -\tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} \tag{4}$$

Equations (3) and (4) show the following:

$$\text{When } \omega = 0, \quad |H(j\omega)| = 1$$
$$\angle H(j\omega) = 0$$

$$\text{When } \omega \to \infty, \quad |H(j\omega)| \to 0$$
$$\angle H(j\omega) \to -\pi \text{ (or -180°)}$$

$$\text{When } \omega = \omega_n, \quad |H(j\omega)| = \frac{1}{2\zeta}$$
$$\angle H(j\omega) = -\frac{\pi}{2} \text{ (or -90°)}$$

The response of the system to frequencies close to the undamped natural frequency clearly depends on the damping ratio ζ. For a lightly damped system $\zeta < 0.5$, $|H(j\omega_n)| > 1$ and the system demonstrates *amplification* due to *resonance*.
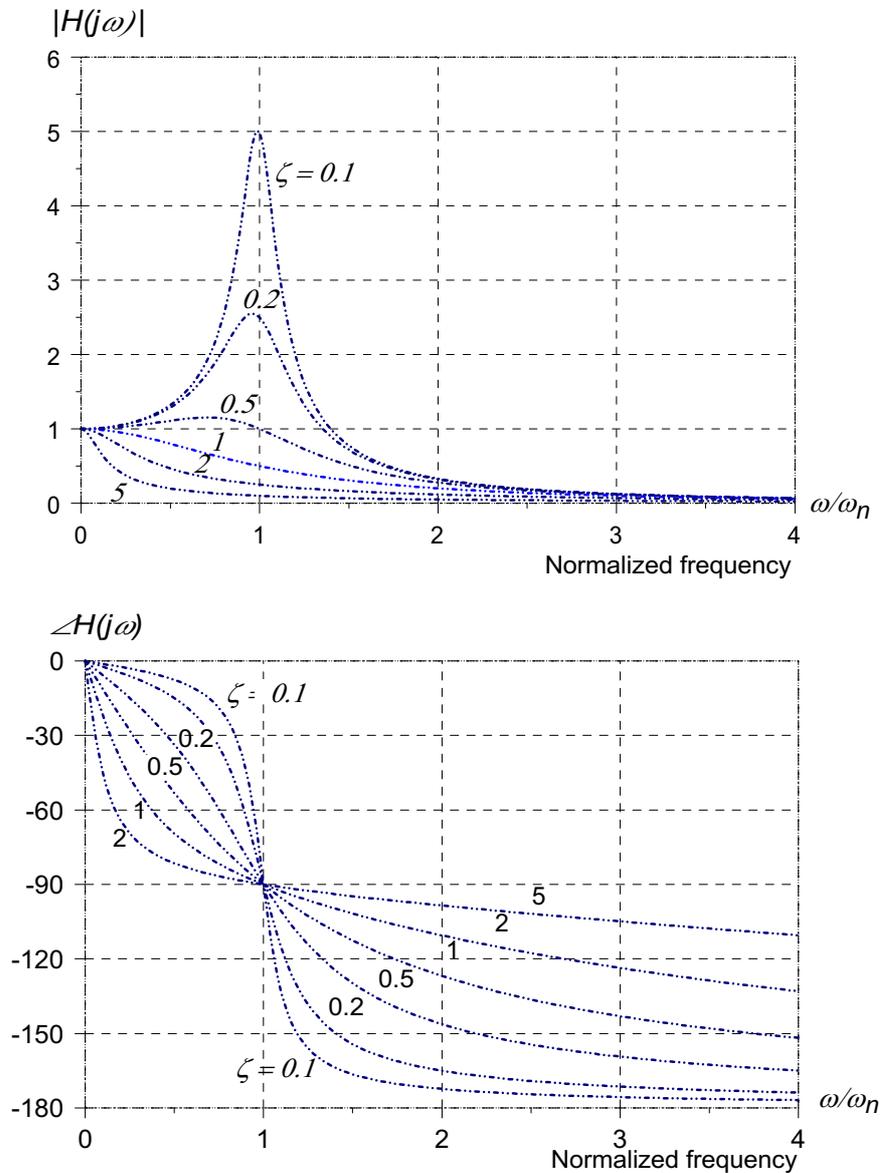
Differentiation of Eq. (4) shows the ω_{peak}, the frequency of the peak response is not ω_n but is in fact

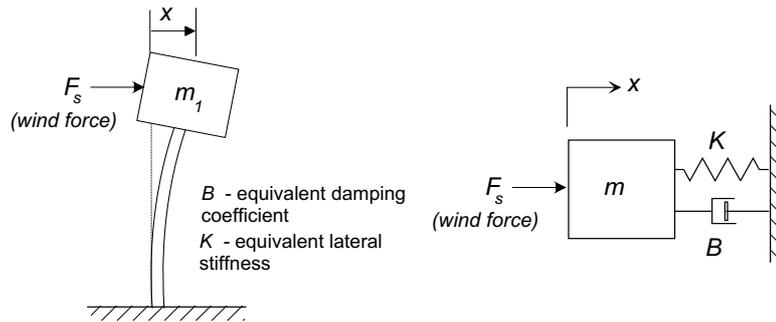$$\omega_p = \omega_n\sqrt{1 - 2\zeta^2}$$

for $\zeta < 1/\sqrt{2}$, and

$$|H(j\omega_p)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}.$$

Frequency response plots for several values of ζ are shown below:

■ Example 3

A tall slender structure, excited by wind forces, is modeled as a mass-spring-damper system. Find the frequency response of the displacement x of the building to a sinusoidal wind loading.

The transfer function is

$$H(s) = \frac{X(s)}{F_s(s)} = \frac{1}{ms^2 + Bs + K} \quad \rightarrow \quad H(j\omega) = \frac{1}{(K - m\omega^2) + jB\omega}$$

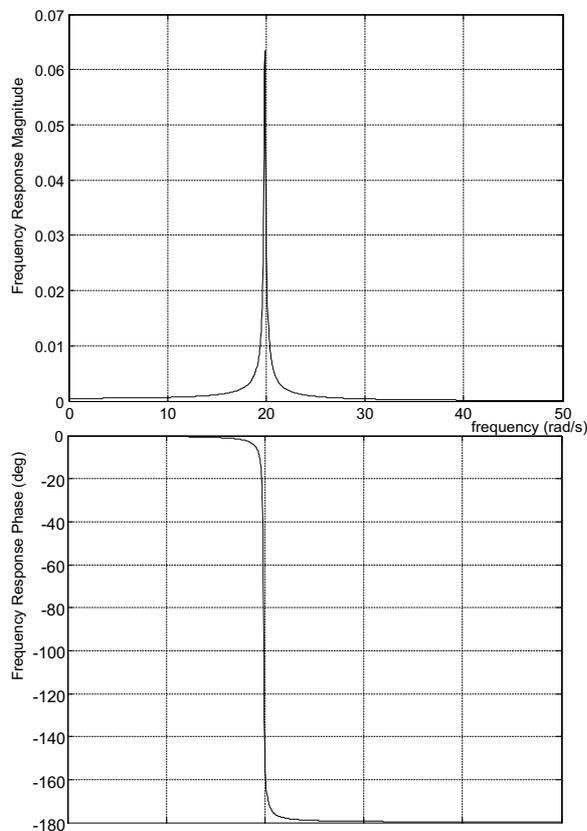$$|H(j\omega)| = \frac{1}{\sqrt{(K - m\omega^2)^2 + (B\omega)^2}}$$

$$\angle H(j\omega) = -\tan^{-1}\frac{B\omega}{K - m\omega^2}$$

With values $m = 5.11$ kg, $B = 0.77$ N-s/m, and $K = 2020$ N/m,

$$\omega_n = \sqrt{K/m} = 19.9 \text{ rad/s}$$
$$\zeta = B/(2m\omega_n) = 0.0038 \text{ N-s/m}$$

and the system is very lightly damped. The frequency response is plotted below:

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Note the extremely sharp and high resonant peak in the magnitude plot, and the rapid phase transition about resonance in the phase plot.


1.3   Frequency Response and the Pole-Zero Plot

The frequency response may be written in terms of the system poles and zeros by substituting directly into the factored form of the transfer function:

$$H(j\omega) = K\frac{(j\omega - z_1)(j\omega - z_2)\dots(j\omega - z_{m-1})(j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2)\dots(j\omega - p_{n-1})(j\omega - p_n)}. \tag{5}$$

Because the frequency response is the transfer function *evaluated on the imaginary axis* of the s-plane, that is when $H(s)_{j\omega}$, the graphical method for evaluating the transfer function may be applied directly to the frequency response. Each of the vectors from the n system poles to a test point $s = j\omega$ has a magnitude and an angle:

$$|j\omega - p_i| = \sqrt{\sigma_i^2 + (\omega - \omega_i)^2},$$

$$\angle(s - p_i) = \tan^{-1}\left(\frac{\omega - \omega_i}{-\sigma_i}\right),$$
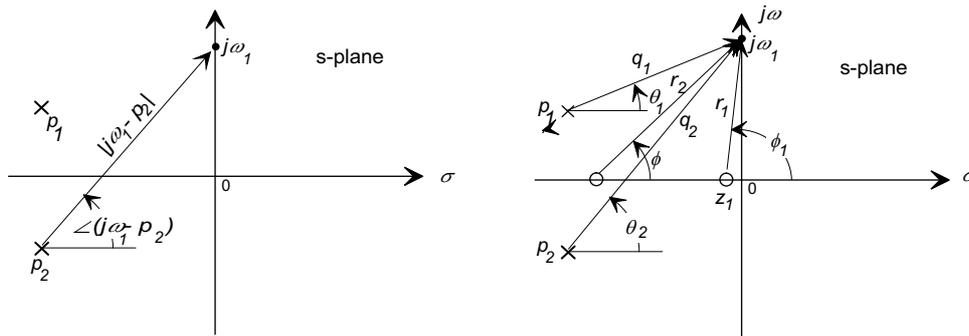
as shown above, with similar expressions for the vectors from the m zeros. The magnitude and phase angle of the complete frequency response may then be written in terms of the magnitudes and angles of these component vectors

$$|H(j\omega)| = K\frac{\prod_{i=1}^{m}|(j\omega - z_i)|}{\prod_{i=1}^{n}|(j\omega - p_i)|}$$

$$\angle H(j\omega) = \sum_{i=1}^{m}\angle(j\omega - z_i) - \sum_{i=1}^{n}\angle(j\omega - p_i).$$

If the vector from the pole p_i to the point $s = j\omega$ has length q_i and an angle θ_i from the horizontal, and the vector from the zero z_i to the point $j\omega$ has a length r_i and an angle ϕ_i, the value of the frequency response at the point $j\omega$ is

$$|H(j\omega)| = K\frac{r_1 \dots r_m}{q_1 \dots q_n}$$

$$\angle H(j\omega) = (\phi_1 + \dots + \phi_m) - (\theta_1 + \dots + \theta_n)$$

■ Example 4

Explain the nature of the sinusoidal response of a first-order system with a pole on the real axis at $s = -1/\tau$ as shown below, in terms of the pole-zero plot.

(a)   (b)
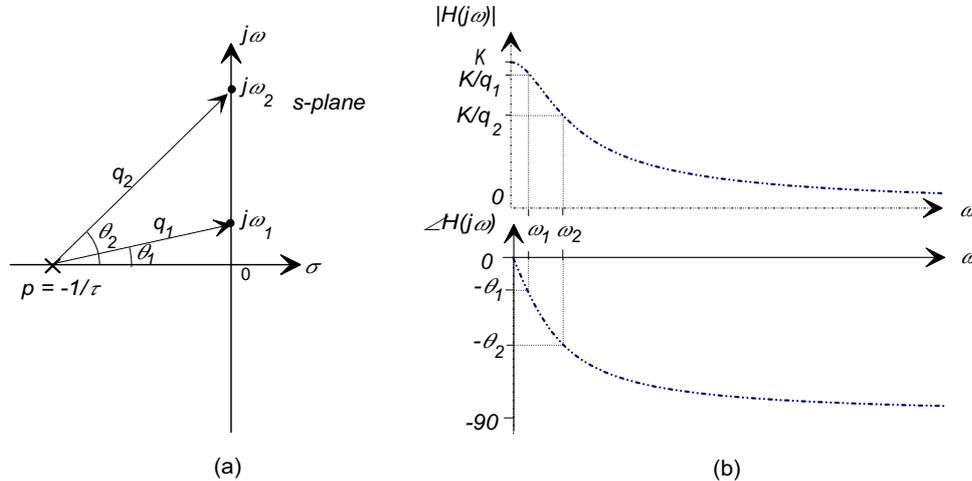
Even though the gain constant K cannot be determined from the pole-zero plot, the following observations may be made directly by noting the behavior of the magnitude and angle of the vector from the pole to the imaginary axis as the input frequency is varied:

- At low frequencies the gain approaches a finite value, and the phase angle has a small but finite lag.

- As the input frequency is increased the gain decreases (because the length of the vector increases), and the phase lag also increases (the angle of the vector becomes larger).

- At very high input frequencies the gain approaches zero, and the phase angle approaches $\pi/2$.

1.3.1 High Frequency Response

As $\omega \to \infty$ we note the following
Magnitude Response: The magnitude response for the s-plane is

$$|H(j\omega)| = K\frac{r_1 \dots r_m}{q_1 \dots q_n}$$

and at high frequencies all vectors have approximately the same length, that is

$$r_i \approx q_j \approx \omega \qquad \text{for } i = 1 \dots m, \ j = 1 \dots n$$

Then

$$\lim_{\omega \to \infty} |H(j\omega)| = K\frac{1}{\omega^{n-m}}$$

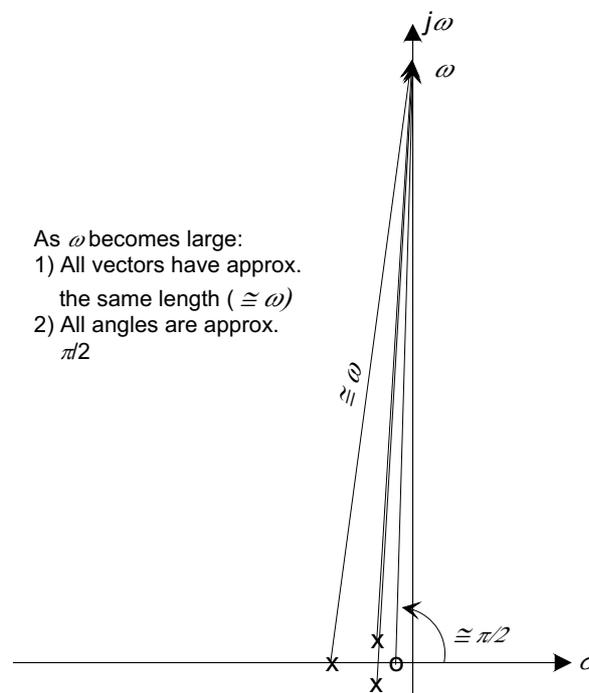**Phase Response**:   From the pole-zero plot

$$\angle H(j\omega) = (\phi_1 + \ldots + \phi_m) - (\theta_1 + \ldots + \theta_n)$$

As ω becomes large all of the angles of the vectors approach $\pi/2$,

$$\phi_i \approx \theta_j \approx \frac{\pi}{2} \qquad \text{for } i = 1\ldots m, \, j = 1\ldots n$$

and

$$\lim_{\omega \to \infty} \angle H(j\omega) = -(n-m)\frac{\pi}{2}$$

As ω becomes large:
1) All vectors have approx.
   the same length ($\cong \omega$)
2) All angles are approx.
   $\pi/2$

Then

	$n > m$	$n = m$	$n < m$
$\lim_{\omega \to \infty} |H(j\omega)|$	0	K	∞
$\lim_{\omega \to \infty} \angle H(j\omega)$	$-(n-m)\pi/2$	0	$(m-n)\pi/2$

If a system has an excess of poles over the number of zeros ($n > m$) the magnitude of the frequency response tends to zero as the frequency becomes large. Similarly, if a system has an excess of zeros the gain increases without bound as the frequency of the input increases. (This cannot happen in physical energetic systems because it implies an infinite power gain through the system.)