2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology
Department of Mechanical Engineering

2.004 *Dynamics and Control II*
Spring Term 2008

Lecture 33[1]

Reading:

- Nise: 10.1
- Class Handout: Sinusoidal Frequency Response

1 Bode Plots (continued

1.1 Logarithmic Amplitude and Frequency Scales:

1.1.1 Logarithmic Amplitude Scale: The Decibel

Bode magnitude plots are frequently plotted using the decibel logarithmic scale to display the function $|H(j\omega)|$. The Bel, named after Alexander Graham Bell, is defined as the logarithm to base 10 of the *ratio* of two *power* levels. In practice the Bel is too large a unit, and the decibel (abbreviated dB), defined to be one tenth of a Bel, has become the standard unit of logarithmic power ratio. The power flow $\mathcal{P}$ into any element in a system, may be expressed in terms of a logarithmic ratio Q to a *reference* power level $\mathcal{P}_{ref}$:

$$Q = \log_{10}\left(\frac{\mathcal{P}}{\mathcal{P}_{ref}}\right) \text{ Bel} \quad \text{or} \quad Q = 10\log_{10}\left(\frac{\mathcal{P}}{\mathcal{P}_{ref}}\right) \text{ dB.} \tag{1}$$

Because the power dissipated in a D–type element is proportional to the square of the amplitude of a system variable applied to it, when the ratio of across or through variables is computed the definition becomes

$$Q = 10\log_{10}\left(\frac{A}{A_{ref}}\right)^2 = 20\log_{10}\left(\frac{A}{A_{ref}}\right) \text{ dB.} \tag{2}$$

where A and A_{ref} are amplitudes of variables.

> **Note:** This definition is only strictly correct when the two amplitude quantities are measured across a common D–type (dissipative) element. Through common usage, however, the decibel has been effectively redefined to be simply a convenient logarithmic measure of amplitude ratio of any two variables. This practice is widespread in texts and references on system dynamics and control system theory.

The table below expresses some commonly used decibel values in terms of the power and amplitude ratios.

Decibels	Power Ratio	Amplitude Ratio
-40	0.0001	0.01
-20	0.01	0.1
-10	0.1	0.3162
-6	0.25	0.5
-3	0.5	0.7071
0	1.0	1.0
3	2.0	1.414
6	4.0	2.0
10	10.0	3.162
20	100.0	10.0
40	10000.0	100.0

The magnitude of the frequency response function $|H(j\omega)|$ is defined as the *ratio* of the amplitude of a sinusoidal output variable to the amplitude of a sinusoidal input variable. This ratio is expressed in decibels, that is

$$20 \log_{10} |H(j\omega)| = 20 \log_{10} \frac{|Y(j\omega)|}{|U(j\omega)|} \text{ dB.}$$

As noted this usage is not strictly correct because the frequency response function does not define a power ratio, and the decibel is a dimensionless unit whereas $|H(j\omega)|$ may have physical units.

■ Example 1

An amplifier has a gain of 28. Express this gain in decibels.

We note that $28 = 10 \times 2 \times 1.4 \approx 10 \times 2 \times \sqrt{2}$. The gain in dB is therefore $20 \log_{10} 10 + 20 \log_{10} 2 + 20 \log_{10} \sqrt{2}$, or

$$\text{Gain(dB)} = 20 + 6 + 3 = 29 \text{ dB.}$$

The advantages of a logarithmic amplitude scale include:

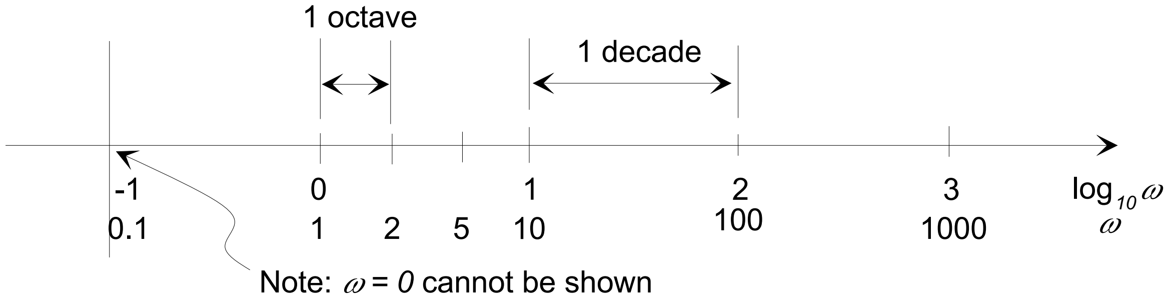
- Compression of a large dynamic range.
- Cascaded subsections may be handled by addition instead of multiplication, that is

  $$\log(|H_1(j\omega)H_2(j\omega)H_3(j\omega)|) = \log(|H_1(j\omega)|) + \log(|H_2(j\omega)|) + \log(|H_3(j\omega)|)$$

  which is the basis for the sketching rules.
- High and low frequency asymptotes become straight lines when $\log(|H(j\omega)|)$ is plotted against $\log(\omega)$.

1.1.2 Logarithmic Frequency Scales

In the Bode plots the frequency axis is plotted on a logarithmic scale. Two logarithmic units of frequency ratio are commonly used: the *octave* which is defined to be a frequency ratio of 2:1, and the *decade* which is a ratio of 10:1.



Given two frequencies ω_1 and ω_2 the frequency ratio $W = (\omega_1/\omega_2)$ between them may be expressed logarithmically in units of decades or octaves by the relationships

$$\begin{aligned} W &= \log_2(\omega_1/\omega_2) \text{ octaves} \\ &= \log_{10}(\omega_1/\omega_2) \text{ decades.} \end{aligned}$$

The terms "above" and "below" are commonly used to express the positive and negative values of logarithmic values of W. A frequency of 100 rad/s is said to be two octaves (a factor of 2^2) above 25 rad/s, while it is three decades (a factor of 10^{-3}) below 100,000 rad/s.

1.2 Asymptotic Bode Plots of Low-Order Transfer Functions

The Bode plots consist of (1) a plot of the logarithmic magnitude (gain) function, and (2) a separate linear plot of the phase shift, both plotted on a logarithmic frequency scale. In this section we develop the plots for first and second-order terms in the transfer function. The approximate sketching methods described here are based on the fact that an approximate log–log magnitude plot can be derived from a set of simple straight line asymptotic plots that can be easily combined graphically.

The system transfer function in terms of factored numerator and denominator polynomials is

$$H(s) = K\frac{(s-z_1)(s-z_2)\dots(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_{n-1})(s-p_n)}, \tag{3}$$

where the z_i, for $i = 1, \dots, m$, are the system zeros, and the p_i, for $i = 1, \dots, n$, are the system poles.

In general a system may have complex conjugate pole and zero pairs, real poles and zeros, and possibly poles or zeros at the origin of the s-plane. Bode plots are constructed from a rearranged form of Eq. (**??**), in which complex conjugate poles and zeros are combined into second-order terms with real coefficients. For example a pair of complex conjugate poles $s_i, s_{i+1} = \sigma_i \pm j\omega_i$ is written

$$\left.\frac{1}{\left(s-(\sigma_i+j\omega_i)\right)\left(s-(\sigma_i-j\omega_i)\right)}\right|_{s=j\omega} = \left(\frac{1}{\omega_n^2}\right)\frac{1}{(1-(\omega/\omega_n)^2)+j2\zeta\omega/\omega_n} \tag{4}$$

and described by parameters ω_n and ζ. The constant terms $1/\omega_n^2$ is absorbed into a redefinition of the gain constant K.

In the following sections Bode plots are developed for the first and second-order numerator and denominator terms:

1.2.1 Constant Gain Term:

The simplest transfer function is a constant gain, that is $H(s) = K$.

$$|H(j\omega)| = K \quad \text{and} \quad \angle H(j\omega) = 0,$$

and converting to the logarithmic decibel scale

$$\boxed{20\log_{10}|H(j\omega)| = 20\log_{10} K \quad \text{and} \quad \angle H(j\omega) = 0 \text{ dB.}}$$

The Bode magnitude plot is a horizontal line at the appropriate gain and the phase plot is identically zero for all frequencies.

1.2.2 A Pole at the Origin of the s-plane:

A single pole at the origin of the s-plane, that is $H(s) = 1/s$, has a frequency response

$$|H(j\omega)| = \frac{1}{\omega} \quad \text{and} \quad \angle H(j\omega) = -\pi/2.$$

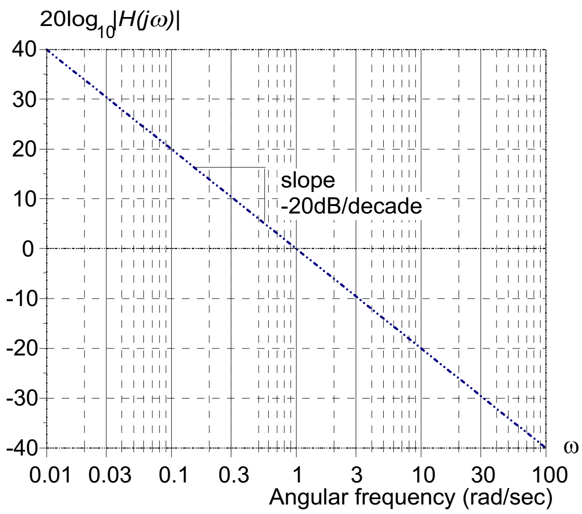
The value of the magnitude function in logarithmic units is

$$\log|H(j\omega)| = -\log(\omega)$$

or using the decibel scale

$$\boxed{20\log_{10}|H(j\omega)| = -20\log_{10}(\omega) \text{ dB.}}$$

The decibel based Bode magnitude plot is therefore a straight line with a slope of -20 dB/decade and passing through the 0 dB line ($|H(j\omega)| = 1$) at a frequency of 1 rad/s. The phase plot is a constant value of $-\pi/2$ rad, or $-90°$, at all frequencies. The magnitude Bode plot for this system is shown below.



1.2.3 A Single Zero at the Origin:

A single zero at the origin of the s-plane, that is $H(s) = s$, has a frequency response $H(j\omega)$ with magnitude and phase

$$|H(j\omega)| = \omega \quad \text{and} \quad \angle H(j\omega) = \pi/2.$$

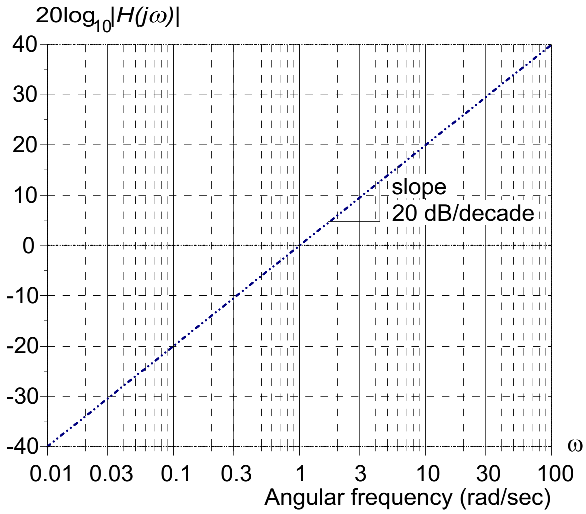
The logarithmic magnitude function is therefore

$$\log |H(j\omega)| = \log(\omega)$$

or in decibels

$$\boxed{20 \log_{10} |H(j\omega)| = 20 \log_{10}(\omega) \text{ dB.}}$$

The Bode magnitude plot is a straight line with a slope of +20 dB/decade. This curve also has a gain of 0 dB (unity gain) at a frequency of 1 rad/s. The phase plot is a constant of $\pi/2$ radians, or +90°, at all frequencies. The magnitude plot is shown in below.



1.2.4 A Single Real Pole

The frequency response of a unity-gain single real pole factor is

$$H(s) = \frac{1}{\tau s + 1},$$

and the frequency response is:

$$\boxed{|H(j\omega)| = \frac{1}{\sqrt{(\omega\tau)^2 + 1}} \quad \text{and} \quad \angle H(j\omega) = \tan^{-1}(-\omega\tau).}$$

The logarithmic magnitude function is

$$\log |H(j\omega)| = -0.5 \log \left((\omega\tau)^2 + 1 \right),$$

or as a decibel function

$$\boxed{20 \log_{10} |H(j\omega)| = -10 \log_{10} \left((\omega\tau)^2 + 1 \right) \text{ dB.}}$$

- When $\omega\tau \ll 1$, the first term may be ignored and the magnitude may be approximated by a *low-frequency asymptote*

$$\lim_{\omega\tau\to 0} 20\log_{10}|H(j\omega)| = -10\log_{10}(1) = 0 \text{ dB}$$

  which is a horizontal line on the plot at 0dB (unity) gain.

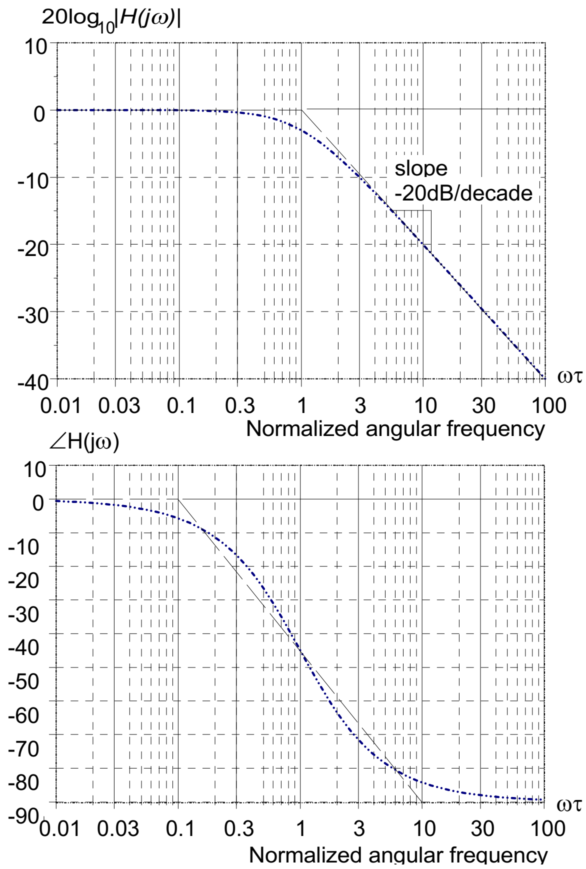
- At high frequencies, for which $\omega\tau \gg 1$, the unity term in the magnitude expression may be ignored and the magnitude function is approximated by a *high-frequency asymptote*

$$20\log_{10}|H(j\omega)| \approx -10\log_{10}((\omega\tau)^2) = -20\log_{10}(\omega) - 20\log_{10}(\tau) \quad \text{dB}.$$

  which is a straight line when plotted against $\log(\omega)$, with a slope of -20 dB/decade.

- The high and low frequency asymptotes intersect on the plot on the 0 dB line at a *corner* or *break* frequency of $\omega = 1/\tau$. We note that when $\omega = 1/\tau$ the magnitude is $|H(j\omega)| = 1/\sqrt{2}$ or -3 dB.

The complete asymptotic Bode magnitude plot as defined by these two line segments is shown in (a) below using a normalized frequency axis. The exact response is also shown in the figure; at the break frequency $\omega = 1/\tau$ the actual response is $20\log_{10}|H(j\omega)| = -10\log_{10}(2) = -3$ dB.



The phase characteristic is also plotted against a normalized frequency scale in (a). At low frequencies the phase shift approaches 0 radians. It passes through a phase shift of $-\pi/4$ radians at the break frequency $\omega = 1/\tau$, and asymptotically approaches a maximum phase lag of $-\pi/2$ radians as the frequency becomes very large. A piece-wise linear approximation may be made by assuming that the curve has a phase shift of 0 radians at frequencies more than one decade below the break frequency, a phase shift of $-\pi/2$ radians at frequencies more than a decade above the break frequency, and a linear transition in phase between these two frequencies on the logarithmic frequency scale. This approximation is within 0.1 radians of the exact value at all frequencies.

1.2.5 A Single Real Zero

A numerator term, corresponding to a single real zero, written in the form $H(s) = \tau s + 1$ (where τ is not strictly a time constant), is handled in a manner similar to a real pole. In this case

$$H(j\omega) = j\omega\tau + 1$$

and the magnitude and phase responses are

$$\boxed{|H(j\omega)| = \sqrt{1 + (\omega\tau)^2} \quad \text{and} \quad \angle H(j\omega) = \tan^{-1}(\omega\tau)}$$

respectively. In decibels the magnitude expression is

$$\boxed{20\log_{10}|H(j\omega)| = 10\log_{10}(1 + (\omega\tau)^2) \text{ dB.}}$$

- The low frequency asymptote is found by assuming that $\omega\tau \ll 1$ in which case

$$\lim_{\omega\tau \to 0} 20\log_{10}|H(j\omega)| = 10\log_{10}(1) = 0 \text{ dB},$$

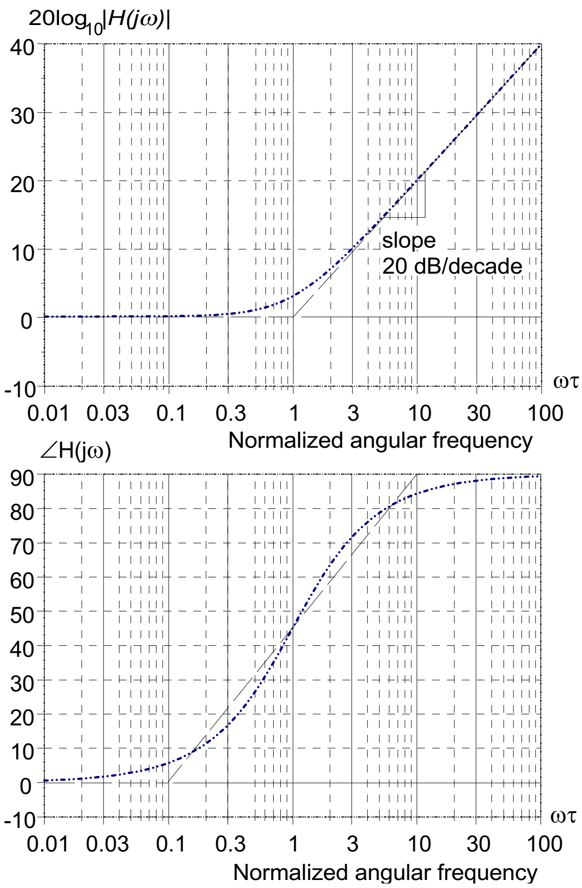
- The high frequency asymptote is found by assuming that $\omega\tau \gg 1$,

$$20\log_{10}|H(j\omega)| \approx 20\log_{10}(\omega\tau) = 20\log_{10}(\omega) - 20\log_{10}(\tau) \text{ dB} \quad \text{when } \omega \gg 1/\tau$$

  which is a straight line on the log-log plot, with a slope of +20 dB/decade.

- The break frequency, defined by the intersection of these two asymptotes is at a frequency $\omega = 1/\tau$, and at this frequency the exact value of $|H(j\omega)|$ is $\sqrt{2}$ or +3 dB. The complete asymptotic Bode magnitude plot using a normalized frequency scale is shown below.

The phase characteristic asymptotically approaches 0 radians at low frequencies and approaches a maximum phase lead of $\pi/2$ radians at frequencies much greater than the break frequency. At the break frequency the phase shift is $\pi/4$ radians. A piece-wise linear approximation, similar to that described for a real pole, is also shown below.



1.2.6 Complex Conjugate Pole Pair:

The classical second-order system,

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

has a frequency response

$$\boxed{\begin{aligned} |H(j\omega)| &= \frac{1}{\sqrt{\left(1-(\omega/\omega_n)^2\right)^2 + \left(2\zeta(\omega/\omega_n)\right)^2}} \\ \text{and} \quad \angle H(j\omega) &= \tan^{-1}\frac{-2\zeta\,(\omega/\omega_n)}{\left(1-(\omega/\omega_n)^2\right)}. \end{aligned}}$$

In logarithmic units the magnitude response is

$$\boxed{20\log_{10}|H(j\omega)| = -10\log_{10}\left[\left(1-(\omega/\omega_n)^2\right)^2 + \left(2\zeta(\omega/\omega_n)\right)^2\right]}$$

The Bode forms of the magnitude and phase responses are plotted in below, with the damping ratio ζ as a parameter.

- The low-frequency asymptote is found by assuming that $\omega/\omega_n \ll 1$ so that

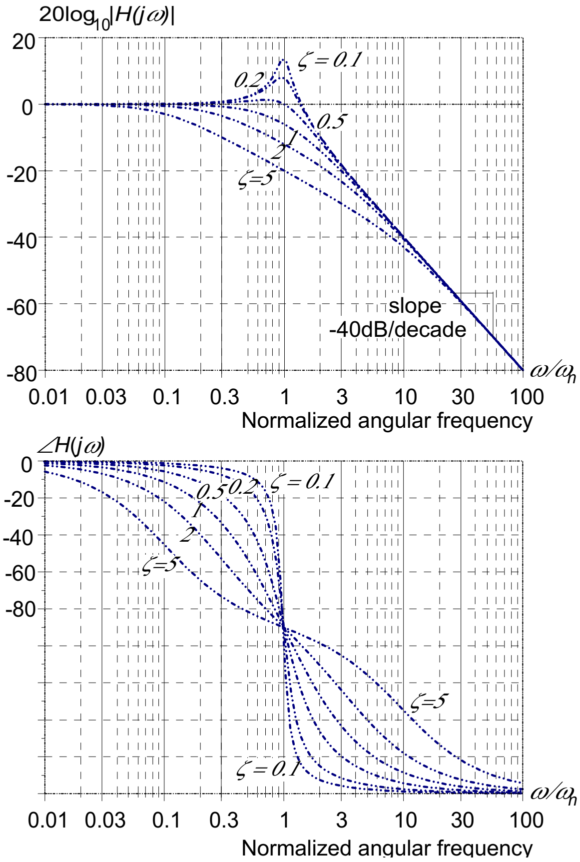
$$\lim_{(\omega/\omega_n)\to 0} \left(20\log_{10}|H(j\omega)|\right) = -10\log_{10}(1) = 0 \text{ dB}.$$

- The high frequency response can be found by retaining only the dominant term when $\omega/\omega_n \gg 1$:

$$\begin{aligned} 20\log_{10}|H(j\omega)| &\approx -10\log_{10}\left[(\omega/\omega_n)^4\right] \\ &= -40\log_{10}(\omega) + 40\log_{10}(\omega_n) \text{ dB} \quad \text{when } \omega \gg \omega_n, \end{aligned}$$

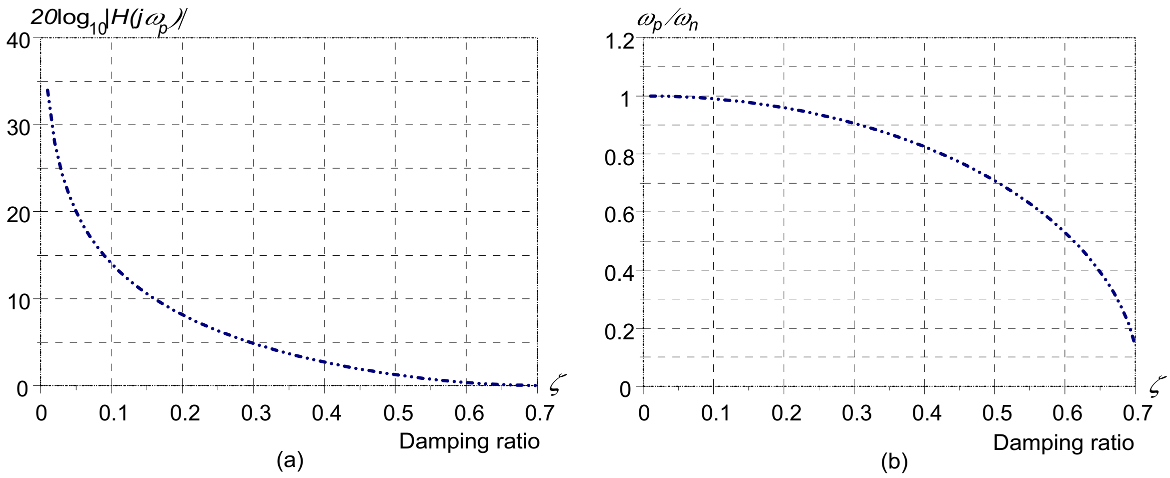
  which is a linear function of $\log_{10}\omega$ with a slope of -40 dB/decade.

- The two asymptotes intersect at a break frequency of $\omega = \omega_n$ as shown below. The straight line asymptotic form does not account in any way for the damping ratio.



The phase characteristic asymptotically approaches 0 radians at low frequencies, has a phase lag of $-\pi/2$ at the break frequency ω_n, and approaches $-\pi$ radians at high frequencies. The steepness of the transition is a function of the damping ratio ζ and so must be sketched using the information contained above.

The resonance peak (for values of $\zeta < 0.707$) must be sketched in after the asymptotes have been drawn. The figure below plots the logarithmic magnitude correction and frequency of the resonant peak as a function of ζ; it is a simple matter to sketch in the resonant peak from these values.



(a) (b)

1.2.7 Complex Conjugate Zero Pair

Bode plots for a pair of complex conjugate zeros can be derived in a manner similar to the conjugate pole pair described above. In this case the block is assumed to have a transfer function

$$H(s) = \frac{1}{\omega_n^2}\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)$$

and a frequency response

$$\boxed{\begin{aligned} |H(j\omega)| &= \sqrt{\left(1-(\omega/\omega_n)^2\right)^2 + \left(2\zeta(\omega/\omega_n)\right)^2} \\ \text{and} \quad \angle H(j\omega) &= \tan^{-1}\frac{2\zeta\left(\omega/\omega_n\right)}{\left(1-(\omega/\omega_n)^2\right)}. \end{aligned}}$$

The logarithmic magnitude response is

$$20\log_{10}|H(j\omega)| = 10\log_{10}\left[\left(1-(\omega/\omega_n)^2\right)^2 + \left(2\zeta(\omega/\omega_n)\right)^2\right] \text{ dB}$$

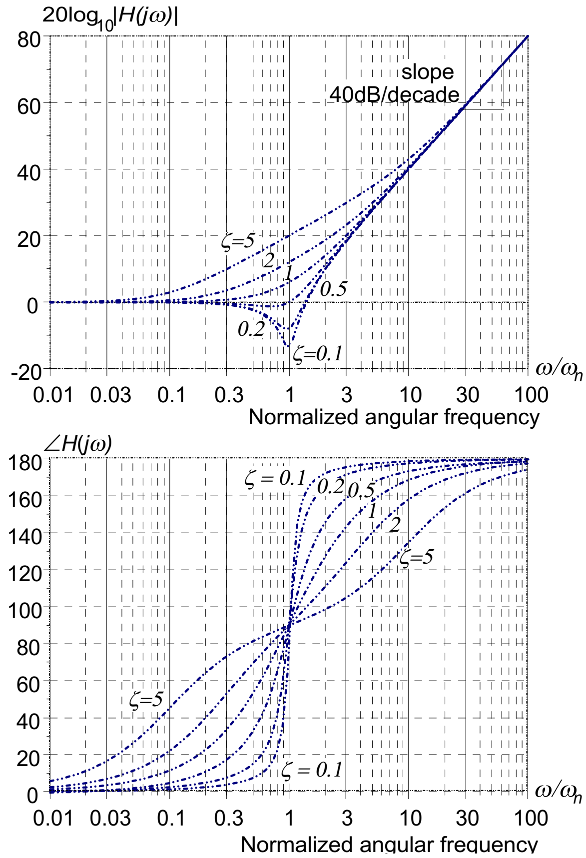
The asymptotic responses are derived in a similar manner to the complex pole pair; the low frequency asymptote is

$$\lim_{(\omega/\omega_n)\to 0}\left(20\log_{10}|H(j\omega)|\right) = 10\log_{10}(1) = 0 \text{ dB},$$

and the high frequency asymptote is

$$\begin{aligned} 20\log_{10}|H(j\omega)| &\approx 10\log_{10}\left[(\omega/\omega_n)^4\right] \\ &= 40\log_{10}(\omega) - 40\log_{10}(\omega_n) \text{ dB} \quad \text{for } \omega \gg \omega_n. \end{aligned}$$

The exact form of the magnitude response is plotted below. This is effectively an inverse of the characteristic of complex-conjugate pole pair described above. There is a “notch” in the response in the region of the frequency ω_n, and the depth is a function of the parameter ζ. The plot has a low frequency asymptote of 0 dB, a break frequency of $\omega = \omega_n$, and a high-frequency asymptote is a straight line with a slope of +40 dB/decade. The phase characteristic is also a flipped version of that of a pair of complex conjugate poles; it approaches 0 radians at low frequencies, passes through $-\pi/2$ at the break frequency, and shows a maximum phase lead of π radians at high frequencies. As above, the slope of the curve in the transition region is dependent on the value of ζ.



1.2.8 Summary

The essential features of the asymptotic forms of the seven components of the magnitude plot are summarized below.

Description	Transfer Function	Break Frequency (radians/sec.)	High Frequency Slope (dB/decade)
Constant gain	K	-	0
Pole at the origin	$\frac{1}{s}$	-	-20
Zero at the origin	s	-	+20
Real pole	$\frac{1}{\tau s+1}$	$1/\tau$	-20
Real zero	$(\tau s+1)$	$1/\tau$	+20
Conjugate poles	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	ω_n	-40
Conjugate zeros	$\frac{1}{\omega_n^2}(s^2+2\zeta\omega_n s+\omega_n^2)$	ω_n	+40