Engineering Risk Benefit Analysis

1.155, 2.943, 3.577, 6.938, 10.816, 13.621, 16.862, 22.82, ESD.72

CBA 4. Including Uncertainty

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Uncertainty

- **Practically any CBA requires consideration of uncertainty.**
- **Most methodologies in use are *ad hoc*, due to the intrinsic difficulty of the generalized problem.**

Methods

1. **Scenario analysis**
2. **Adjustments of interest rates**
3. **Decision Theory**
4. **Simplified probabilistic models**

Scenario Analysis

- **Preparation and analysis of scenarios:**

  -- "Optimistic" or "most favorable estimate"

  -- "Most likely" or "best estimate" or "fair estimate"

  -- "Pessimistic" or "least favorable estimate"

- **Interpretation is difficult without assignment of probabilities to scenarios.**

- **Benefit: Brings additional information into the process.**

Example

- A new machine is to be purchased for producing units in a new manner.

	Pessim.	Fair	Optim.
Annual number of units:	900	1,000	1,100
Savings per unit:	$50	55	60
Operating costs:	$2,000	1,600	1,200
etc.			
PW:	-$49,000	22,000	120,000

- What are we to do with such information?

Developing Scenarios

- For each element in the problem, e.g., interest rate and costs, define the three values.
- We really don't know how conservative (pessimistic) the final answer is.
- People are bad processors of information.
- Point estimates tend to cluster around the median value. Possibility of displacement bias.
- Extremes greater than the 75^{th} or smaller than the 25^{th} percentile are difficult to imagine.
- Overconfidence.

Forecasting Oil Prices

D. Draper, "Assessment and propagation of model uncertainty," Journal of the Royal Statist. Soc., B (1995) 45-97.

- In 1980, 43 economists and energy experts forecast the price of oil from 1981 to 2020 to aid in policy planning.
- They used 10 leading econometric models under each of 12 scenarios embodying a variety of assumptions about inputs, such as supply, demand, and growth rates.

The Plausible Scenario

- **One scenario was termed as the "plausible median case." It represented "the general trends to be expected."**

- **The 10 models were applied to the plausible scenario.**

- **Results for 1986:**
  - **Actual price: $13**
  - **Range of predictions: $27 to $ 51.**

High Interest Rates

- **Justify choice of alternatives using a high interest rate, e.g., 30%.**
- **Example**

Total annual income: $55,000

Capital cost: $80,000

Annual capital recovery with return:

$80,000 (A/P, 30%, 6yrs) = $30,272

Annual operating cost $28,600

Net annual profit: 55,000 – (30,272 + 28,600) =

= -$3,872

Example (cont'd)

- The high rate of 30% is intended to cover uncertainty.
- If the annual income were $60,000, then the net annual profit would be 60,000 - (30,272 + 28,600) = = $1,128 and the venture would be accepted.
- A high interest rate does not guarantee that all uncertainties are accounted for. Its choice is arbitrary.

Decision Theory: Manufacturing Example

- **<u>Decision:</u>** **To continue producing old product (O) or convert to a new product (N).**

  The payoffs depend on the market conditions:

  s: strong market for the new product

  w: weak market for the new product

Manufacturing Example Payoffs

- **Earnings (payoffs):**

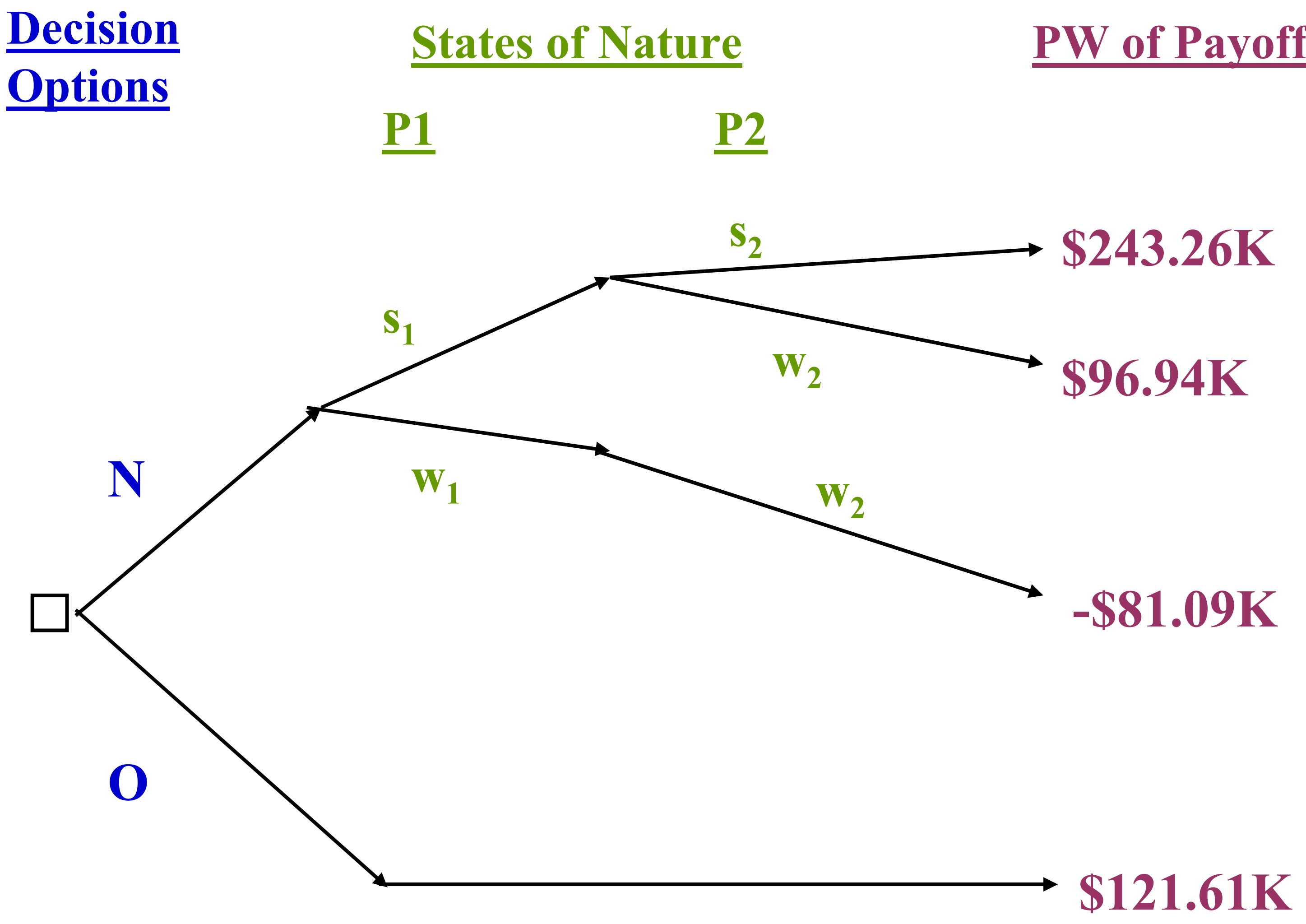
L_1: $15,000/yr, old product,
L_2: $30,000/yr, new product and the market is strong,
L_3: -$10,000/yr, new product and the market is weak

- **Demand and Probabilities:**

Period 1 (5 yrs)	Period 2 (5 yrs)	Probabilities
s_1	s_2	$0.4 = P(s_1 s_2)$
s_1	w_2	$0.4 = P(s_1 w_2)$
w_1	w_2	$0.2 = P(w_1 w_2)$

$P(s_1) = P(s_1 s_2) + P(s_1 w_2) = 0.8$; $P(w_1) = 0.2$; $P(s_2/s_1) = 0.5$; $P(w_2/w_1) = 1.0$

Decision Tree



Calculation of the Payoffs

$$PW_{s1} = 30x(P/A,0.04,5) = 30x\frac{(1+0.04)^5 - 1}{0.04x(1+0.04)^5} = 133.52K$$

$$PW_{s2} = 30x(P/A,0.04,5)x(P/F,0.04,5) =$$

$$= 30x\frac{(1+0.04)^5 - 1}{0.04x(1+0.04)^5}x\frac{1}{(1+0.04)^5} = 109.74K$$

$$PW_{s1s2} = 133.52 + 109.74 = 243.26K$$

$$PW_{w1} = -133.52x\frac{10}{30} = -44.51K$$

$$PW_{w2} = -109.74x\frac{10}{30} = -36.58K$$

$$PW_{s1w2} = 133.52 - 36.58 = 96.94K$$

$$PW_{w1w2} = -44.51 - 36.58 = -81.09K$$

Calculation of the EMV

Old Product

$$PW_O = 15x(P/A,0.04,10) = 15x\frac{(1+0.04)^{10}-1}{0.04x(1+0.04)^{10}} = 121.61K$$

New Product

$EMV_N = 243.26x0.4 + 96.94x0.4 - 81.09x0.2 = 119.86K$

Decision

Stay with the old product?

Calculation using Utilities

Let the utility of payoffs be $U(x) = 1.18 \ln(x+5) - 1.29$

$-2 \leq x \leq 2$ (x in $M) $[U(2) = 1, U(-2)= 0]$

$U(243.26) = 0.665$; $U(121.61) = 0.637$; $U(96.94) = 0.632$;

$U(-81.09) = 0.590$

Old Product

$EU(O) = 0.637$

New Product

$EU(N) = 0.665 \times 0.4 + 0.632 \times 0.4 + 0.590 \times 0.2 = 0.6368$

Decision

Stay with the old product?

Probabilistic Models

- **We have:**

$$PW[X(T)] = X_0 + \frac{X_1}{(1+i)} + \frac{X_2}{(1+i)^2} + ... + \frac{X_T}{(1+i)^T} \quad (1)$$

where $X_j \equiv B_j - C_j$ are the net benefits in year j.

- **All X_j are r.v.'s, $\Rightarrow$ PW[X(T)] is a r.v.**
- **Note that (1) is of the form:**

$$Y = a_0 X_0 + a_1 X_1 + a_2 X_2 + ... + a_T X_T$$

Analysis

- **Computing the probability density function (pdf) of PW is usually difficult in practice.**

- **Try to compute the quantities E[PW], the expected value of PW,**

and σ^2_{PW} **i.e., the variance of PW.**

Fundamental Relationships from Probability Theory (1)

- **Let $Z = aW + b$**

(Z and W are r.v.'s, a and b constants)

$\Rightarrow$ $$E[Z] = a\,E[W] + b$$

$\Rightarrow$ $$\sigma_Z^2 = a^2 \sigma_W^2$$

Fundamental Relationships from Probability Theory (2)

- Let

$$Z = W_1 + W_2 \qquad \text{(}W_1\text{, }W_2\text{ : independent r.v.'s)}$$

$$\Rightarrow \qquad E[Z] = E[W_1] + E[W_2]$$

$$\Rightarrow \qquad \sigma_Z^2 = \sigma_{W_1}^2 + \sigma_{W_2}^2$$

Note: Extends to any number of mutually independent r.v.'s.

Fundamental Relationships from Probability Theory (3)

- Let $$Z = aW_1 + bW_2 + c$$

$\Rightarrow$ $$E[Z] = aE[W_1] + bE[W_2] + c$$

$$\sigma_Z^2 = a^2\sigma_{W_1}^2 + b^2\sigma_{W_2}^2$$

- If W_1 and W_2 are normal, then Z is also normal.
- We often assume that Z is normal even if W_1 and W_2 are not.

Example: Reliability Physics

- **(RPRA 3, slide 30)** ***A capacitor is placed across a power source. Assume that surge voltages occur on the line at a rate of one per month and they are normally distributed with a mean value of 100 volts and a standard deviation of 15 volts. The breakdown voltage of the capacitor is 130 volts.***

- ***Suppose that the breakdown voltage is also normally distributed with standard deviation of 15 volts.***

Example (2)

- **The capacitor fails when the surge voltage, S, is greater than the capacity, C.**
- **S: rv with E[S] = 100, σ_S = 15 volts**
- **C: rv with E[C] = 130, σ_C = 15 volts**
- **Define a new rv D $\equiv$ C – S = aC + bS**
- **Then, E[D] = 130 – 100 = 30 volts**

and

$$\sigma_D = \sqrt{\sigma_C^2 + \sigma_S^2} = 21.21 \quad \text{volts}$$

Example (3)

- D is also normally distributed, therefore
- $P_{d/sv}(D < 0) = P(Z < -(30/21.21)) = P(Z < - 1.41) =$ $= P(Z > 1.41) = 0.5 – 0.42 = 0.08$
- RPRA 3, page 31, shows that $P_{d/sv}$ = conditional probability of damage given a surge voltage = P(surge voltage>130 volts/surge voltage)

$$= P(Z > \frac{130 - 100}{15}) = P(Z > 2) =$$

$$= 1 - P(Z < 2) = 1 - 0.9772 = 0.0228 < 0.08$$

The uncertainty in the breakdown voltage increased the failure probability.

Assuming Independence of X_j

$$PW_{Ind}[X(T)] = X_0 + \frac{X_1}{(1+i)} + \frac{X_2}{(1+i)^2} + ... + \frac{X_T}{(1+i)^T}$$

- X_j (j = 0, 1,........,T) are mutually independent r.v.'s with known $E[X_j]$ and $\sigma^2_{X_j} (= \sigma^2_j)$

$$E[PW_{Ind}\{X(T)\}] = E[Y_{Ind}] = \sum_{j=0}^{T} \frac{E[X_j]}{(1+i)^j}$$

$$\sigma^2_{PW,Ind} = \sigma^2_{Y,Ind} = \sum_{j=0}^{T} \frac{\sigma^2_j}{(1+i)^{2j}}$$

- <u>Note:</u> Gaussian approximation for pdf of *Y* may work well in this case.

Example: Project A

T = 3 yrs.; i = 8%; Initial Cost = $10K

	Net Benefits		
Probability, p	t=1	t=2	t=3
0.10	$3K	$3K	$3K
0.25	$4K	$4K	$4K
0.30	$5K	$5K	$5K
0.25	$6K	$6K	$6K
0.10	$7K	$7K	$7K
1.00			

Example (2)

- Denote with X1, X2, and X3, the net benefits of A in years 1, 2 and 3, respectively.
- <u>Note:</u> Net benefits, X1, X2, and X3, do not have to be identically distributed or symmetric or discrete; these choices are made just to keep the example simple.
- Denote with Y the present worth of A, PW(A). Then:

$$PW_{Ind}(A) = Y_{Ind} = -10K + X1/(1.08) + X2/(1.08)^2 + X3/(1.08)^3$$

Observations

- **Y is a *random variable* (takes more than one value with different probabilities for any given implementation of project A)**
- **Value of Y will be determined by the values of the combination of X1, X2, and X3 that will actually materialize**
- **Corresponding *a priori* probability of any value of Y is equal to probability of that particular combination of X1, X2 and X3.**

Expectation and Variance of Annual Net Benefits

- It is easy to determine the expected value and variance of each of X1, X2, and X3, separately:

E[X1] = 0.1x3+0.25x4+0.3x5 +0.25x6+0.1x7 = $5,000

(Similarly, we have E[X2] = $5K and E[X3] = $5K.)

$$\sigma^2_{X_1} = 0.1x(3\text{-}5)^2 + 0.25x(4\text{-}5)^2 + 0.3x(5\text{-}5)^2 + 0.25x(6\text{-}5)^2 + \\ + 0.1x(7\text{-}5)^2 = 1{,}300{,}000 \approx (1{,}140)^2$$

or, $\sigma_{X_1} \approx \$1{,}140 = \sigma_{X_2} = \sigma_{X_3}$

Independence: Calculations

- **Assume that the net benefits obtained from Project A in years 1, 2 and 3 are determined independently of one another.**
- **This means the probability of the combination {X1 = 3, X2 = 6, X3 = 4} is equal to**

P(X1 = 3, X2 = 6, X3 = 4) = P(X1 =3)·P(X2 = 6)·P(X3 =4) =

= (0.1)(0.25)(0.25) = 0.00625

that is, with probability 0.00625, the r.v. Y, i.e., the PW of Project A, will take on the value

$$PW_{Ind} = Y_{Ind} = -10K + 3K/(1.08) + 6K/(1.08)^2 + 4K/(1.08)^3 \approx \$1{,}097.14$$

Independence: Calculations (2)

- Note that Y can take on a total of 125 (= 5·5·5) different values of independent outcomes in years 1, 2 and 3.
- Using the expressions on slide 21 we get

$E[Y_{Ind}] = -10K + E[X1]/(1.08) + E[X2]/(1.08)^2 + E[X3]/(1.08)^3 =$

$= -10K + (5K)\cdot[1/1.08 + 1/(1.08)^2 + 1/(1.08)^3]$

$= -10K + (5K)\cdot(P/A, 0.08, 3) \approx -10K + (5K)(2.5771) \approx \$2,885$

$\sigma^2_{Y,Ind} = \sigma^2_{X_1}[1/1.08]^2 + \sigma^2_{X_2}[1/(1.08)^2]^2 + \sigma^2_{X_3}[1/(1.08)^3]^2$

$= (1,140)^2\cdot[1/(1.08)^2 + 1/(1.08)^4 + + 1/(1.08)^6] =$

$= (1,140)^2\cdot(2.2225)$, or, $\sigma_{Y,Ind} = (1,140)\cdot(2.2225)^{1/2} \approx \$1,700$

Conclusion

- **Project A will, "on average," have a net present value equal to about $2,885 and a standard deviation of approximately $1,700 around that average.**

What can we do with this information?

Alternative	E[Y] = E[PW]	σ_Y
1	20	15
2	5	7
3	15	12
4	17	16

Possible Decision Criteria

- Choose the alternative with the highest mean value (A1).
- Note how close A4 is and how large the standard deviations are. Depending on the uncertainties, A1 and A4 may be indistinguishable.

- Minimize the probability of loss.
- Assume normal distributions and find P(PW < 0).

Probability of Loss

- **A1: $P(PW < 0) = P(Z < -(20/15)) = P(Z < -1.33) =$**
  $= 0.09$ "best" alternative
- **A2: $P(PW < 0) = P(Z < -(5/7)) = P(Z < -0.71) =$**
  $= 0.24$
- **A3: $P(PW < 0) = P(Z < -(15/12)) = P(Z < -1.25) =$**
  $= 0.10$
- **A4: $P(PW < 0) = P(Z < -(17/16)) = P(Z < -1.06) =$**
  $= 0.14$

Probability of Loss for Project A

- $P_{Ind}(PW_{Ind} < 0) = P(Z < -(2885/1700)) = P(Z < -1.7) = 0.04$

- The fundamental assumption is that of independence of the annual benefits.

Complete Dependence of X_j

- **Once the net benefits, X1, for year 1 are known, we shall also know exactly the net benefits for years 2 and 3.**

	Net Benefits		
Probability, p	t=1	t=2	t=3
0.10	$3K	$3K	$3K
0.25	$4K	$4K	$4K
0.30	$5K	$5K	$5K
0.25	$6K	$6K	$6K
0.10	$7K	$7K	$7K
1.00			

Mean and Variance (Dependence)

$$PW_{Dep}[X(T)] = X_0 + X\left[\frac{1}{(1+i)} + \frac{1}{(1+i)^2} + \dots + \frac{1}{(1+i)^T}\right]$$

- PW_{Dep} (A) = Y_{Dep} = -10K + X[1/(1.08) + 1/(1.08)^2 +1/(1.08)^3] = -10K + X(P/A, 8, 3)
- From slide 15 we get
- E[Y_{Dep}] = -10K + E[X]·(P/A, 8, 3) ≈ \$2,885, as before

$$\sigma^2_{Y,Dep} = \sigma^2_X [(P/A, 8, 3)]^2 = (1{,}140)^2 \cdot [2.5771]^2$$

or

$$\sigma_{Y,Dep} = (1{,}140) \cdot (2.5771) \approx \$2{,}938$$

Comparison

$$PW_{Ind}[X(T)] = X_0 + \frac{X_1}{(1+i)} + \frac{X_2}{(1+i)^2} + ... + \frac{X_T}{(1+i)^T}$$

$$PW_{Dep}[X(T)] = X_0 + X\left[\frac{1}{(1+i)} + \frac{1}{(1+i)^2} + ... + \frac{1}{(1+i)^T}\right]$$

- **In both the independent and dependent cases the mean values are the same ($2,885).**
- **The standard deviation in the dependent case ($2,938) is 73% larger than that of the independent case ($1,700).**
- **$P_{Dep}(PW < 0) = P(Z < -(2885/2938)) = P(Z < -0.98) = 0.16$**
- **Compare with $P_{Ind}(PW < 0) = 0.04$ (slide 32)**
- **These two cases are considered as bounding the problem.**