2.20 - Marine Hydrodynamics, Spring 2005

Lecture 8

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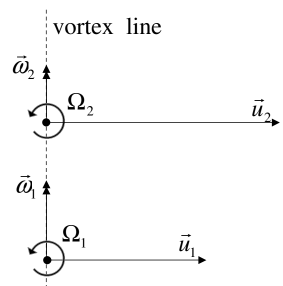
In Lecture 8, paragraph 3.3 we discuss some properties of vortex structures. In paragraph 3.4 we deduce the Bernoulli equation for ideal, steady flow.

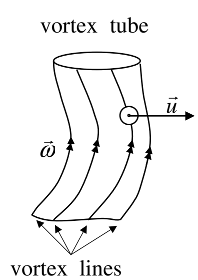
3.3 Properties of Vortex Structures

3.3.1 Vortex Structures

- A **vortex line** is a line everywhere tangent to $\vec{\omega}$.



- A **vortex tube** (filament) is a bundle of vortex lines.



- A **vortex ring** is a closed vortex tube.

  A sketch and two pictures of the production of vortex rings from orifices are shown in Figures 1, 2, and 3 below.

  (Figures 2,3: Van Dyke, *An Album of Fluid Motion* 1982 p.66, 71)

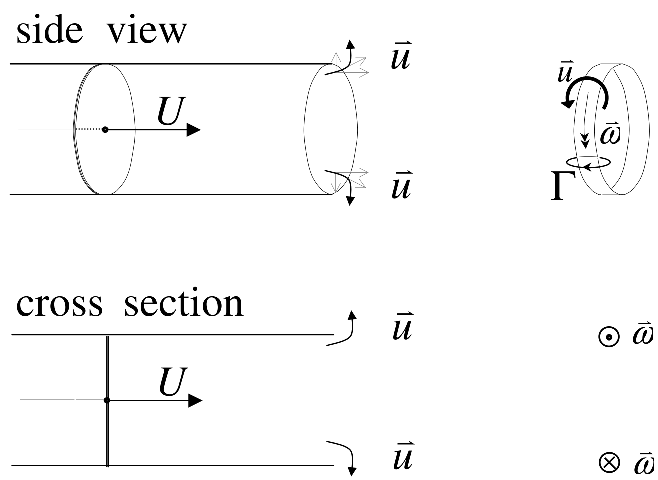


Figure 1: Sketch of vortex ring production

3.3.2 No Net Flux of Vorticity Through a Closed Surface

Calculus identity, **for any vector** $\vec{v}$:

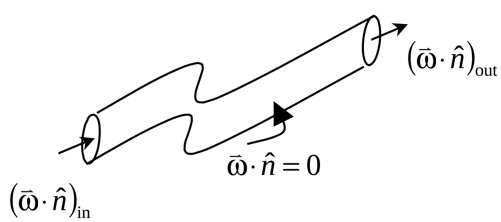
$$\nabla \cdot (\underbrace{\nabla \times \vec{v}}_{\vec{\omega}}) = 0 \Rightarrow$$
$$\nabla \cdot \vec{\omega} = 0 \Rightarrow$$

$$\boxed{\iiint_V \nabla \cdot \vec{\omega} \underset{\substack{\Uparrow \\ \text{Divergence} \\ \text{Theorem}}}{=} \oiint_S \underbrace{\vec{\omega} \cdot \hat{n}}_{\text{vorticity flux}} dS = 0}$$

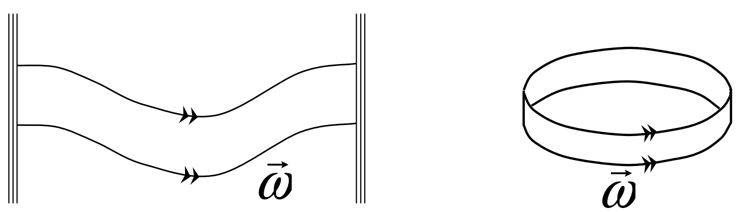
i.e. **The net vorticity flux through a closed surface is zero**.

(a) No net vorticity flux through a vortex tube:

$$(\text{Vorticity Flux})_{in} = (\text{Vorticity Flux})_{out} \Rightarrow (\vec{\omega} \cdot \hat{n})_{in}\ \delta \text{A}_{in} = (\vec{\omega} \cdot \hat{n})_{out}\ \delta \text{A}_{out}$$

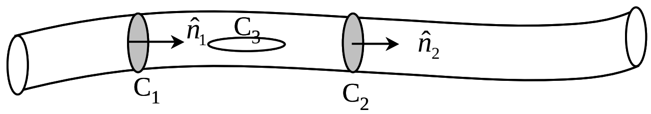


(b) Vorticity cannot stop anywhere in the fluid. It either traverses the fluid beginning or ending on a boundary or closes on itself (vortex ring).



3.3.3 **Conservation of Vorticity Flux**

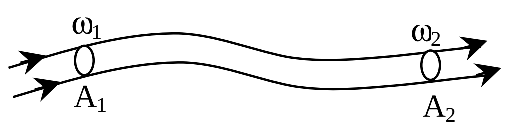
$$0 = \Gamma_3 = \oint_{C_3} \vec{v} \cdot d\vec{x} = \iint_{S_3} \vec{\omega} \cdot \hat{n} dS = 0$$



$$\Gamma_1 = \oint_{C_1} \vec{v} \cdot d\vec{x} = \iint_{S_1} \vec{\omega} \cdot \hat{n}_1 dS = \iint_{S_2} \vec{\omega} \cdot \hat{n}_2 dS = \Gamma_2$$

Therefore, circulation is the same in all circuits embracing the same vortex tube. For the special case of a vortex tube with 'small' area:

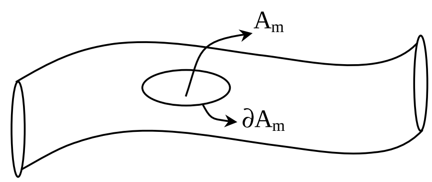
$$\Gamma = \omega_1 A_1 = \omega_2 A_2$$



An application of the equation above is displayed in the figure below:

3.3.4 Vortex Structures are Material Structures

Consider a material patch A_m on a vortex tube at time t.



By definition,

$$\vec{\omega} \cdot \hat{n} = 0 \text{ on } \ A_n$$

Then,

$$\Gamma_{\partial A_m} = \oint_{\partial A_m} \vec{v} \cdot d\vec{x} = \iint_{A_m} \vec{\omega} \cdot \hat{n} ds = 0$$

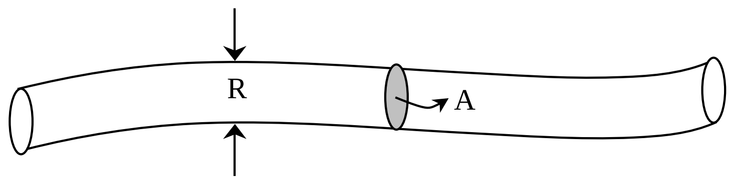
At time $t + \Delta t$, A_m moves, and for an ideal fluid under the influence of conservative body forces, Kelvin's theorem states that

$$\Gamma_{\partial A_m} = 0$$

So, $\vec{\omega} \cdot \hat{n} = 0$ on A_m still, i.e., A_m still on the vortex tube. Therefore, the vortex tube is a material tube for an ideal fluid under the influence of conservative forces. In the same manner it can be shown that a vortex line is a material line, i.e., it moves with the fluid.

3.3.5 Vortex stretching

Consider a small vortex filament of length L and radius R, where by definition $\vec{\omega}$ is tangent to the tube.



$$\Gamma \underset{\substack{\uparrow \\ \text{Stokes} \\ \text{Theorem}}}{=} \omega A \underset{\substack{\uparrow \\ \text{Kelvins'} \\ \text{Theorem}}}{=} \text{constant (in time)}$$

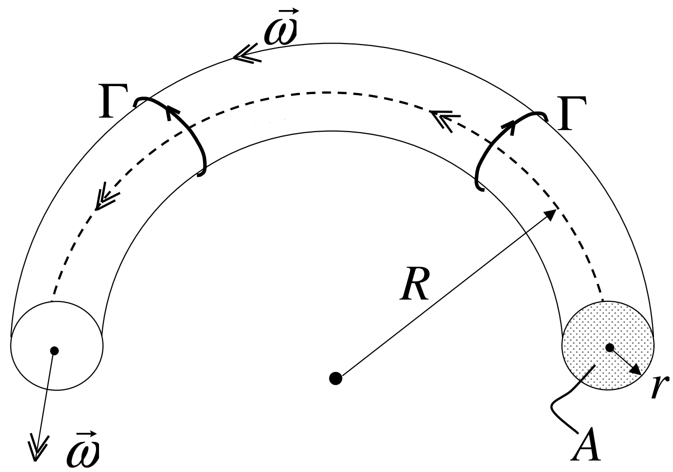
But tube is material with volume = AL = πR^{2}L = constant in time (continuity)

$$\therefore \frac{\Gamma}{\text{Volume}} = \frac{\omega A}{LA} = \frac{\omega}{L} = \text{ constant}$$

As a vortex stretches, L increases, and since the volume is constant (from continuity), A and R decrease, and due to the conservation of the angular momentum, ω increases. In other words,

$$\begin{aligned} \text{Vortex stretching } &\Leftrightarrow L\uparrow \Rightarrow \omega\uparrow \text{ (conservation of angular momentum)} \\ &\Rightarrow A \text{ and } R\uparrow \text{ (continuity)} \end{aligned}$$

3.3.6 **Summary on Vortex Structures**



Vortex ring length	$L = 2\pi R$	[L]
Cross sectional area	$A = \pi r^2$	[L^2]
Vortex ring volume	$\boxed{\forall = AL \underset{\substack{\Uparrow \\ \text{continuity}}}{=} const}$	[L^3]
Vorticity	$\vec{\omega} = \nabla \times \vec{v}$	[T^{-1}]
Circulation	$\boxed{\Gamma \underset{\substack{\Uparrow \\ \text{Kelvin's theorem}}}{=} const}$	[L^2T^{-1}]
	$\boxed{\Gamma \underset{\substack{\Uparrow \\ \text{vorticity flux through A}}}{=} \omega A = const}$	[L^2T^{-1}]
	$\boxed{\Gamma \propto Ur = const}$	[L^2T^{-1}]

Continuity relates length ratios

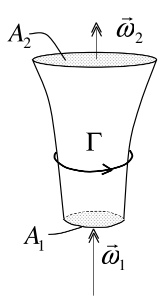
$$\forall = LA = const \begin{cases} A \quad \propto \quad \dfrac{\forall}{L} \qquad \therefore \text{ as } L \uparrow \quad A \downarrow \\ \\ r \quad \propto \quad \sqrt{\dfrac{\forall}{L}} \qquad \therefore \text{ as } L \uparrow \quad r \downarrow \end{cases}$$

Kelvin's theorem + Continuity relate length ratios to Γ, ω, U

$$Ur \propto \Gamma = const \quad \rightarrow \quad U \propto \frac{\Gamma}{r} \quad \overset{r\propto\sqrt{\forall/L}}{\longrightarrow} \quad U \propto \Gamma\sqrt{\frac{L}{\forall}} \qquad \therefore \text{ as } L \uparrow \quad U \uparrow$$

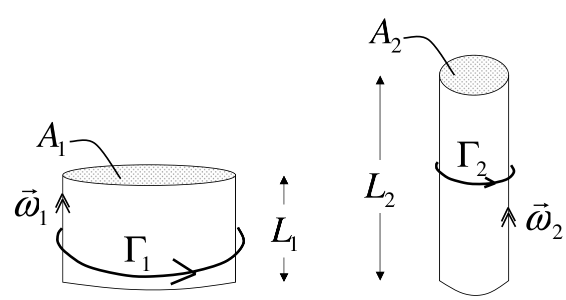
$$\omega A \propto \Gamma = const \quad \rightarrow \quad \omega \propto \frac{\Gamma}{A} \quad \overset{A\propto\forall/L}{\longrightarrow} \quad \omega \propto \Gamma\frac{L}{\forall} \qquad \therefore \text{ as } L \uparrow \quad \omega \uparrow$$

Example 1:



$A_1 < A_2$
$\omega_1 > \omega_2$

Example 2:



$L_1 < L_2$	Given
$A_1 > A_2$	From continuity only
$\Gamma_1 = \Gamma_2$	From Kelvin's theorem
$\omega_1 < \omega_2$	From Kelvin's theorem + continuity
$U_1 < U_2$	From Kelvin's theorem + continuity

3.4 Bernoulli Equation for Steady ($\frac{\partial}{\partial t} = 0$), Ideal($\nu = 0$), Rotational flow

$p = f\left(\vec{v}\right)$	Viscous flow: Navier-Stokes' Equations (Vector Equations)
$p = f(\lvert\vec{v}\rvert)$	**Ideal flow: Bernoulli Equation (Scalar equation)**

Steady, inviscid Euler equation (momentum equation):

$$\vec{v} \cdot \nabla \vec{v} = -\nabla \left(\frac{p}{\rho} + gy \right) \tag{1}$$

From Vector Calculus we have

$$\begin{aligned}
\nabla \left(\vec{u} \cdot \vec{v} \right) &= \left(\vec{u} \cdot \nabla \right) \vec{v} + \left(\vec{v} \cdot \nabla \right) \vec{u} + \vec{u} \times \left(\nabla \times \vec{v} \right) + \vec{v} \times \left(\nabla \times \vec{u} \right) \Rightarrow \\
\nabla \left(\tfrac{1}{2} \left| \vec{v} \right|^2 \right) &= \vec{v} \cdot \nabla \vec{v} + \vec{v} \times \left(\nabla \times \vec{v} \right) \Rightarrow \\
\vec{v} \cdot \nabla \vec{v} &= \nabla \left(\frac{v^2}{2} \right) - \vec{v} \times \left(\nabla \times \vec{v} \right) \text{ where } v^2 \equiv \vec{v} \cdot \vec{v} = \left| \vec{v} \right|^2
\end{aligned}$$

From the previous identity and Equation (1) we obtain

$$\vec{v} \quad \cdot (1) \;\; \rightarrow \vec{v} \cdot \nabla \left(\frac{v^2}{2} \right) - \underbrace{\vec{v} \cdot \underbrace{\vec{v} \times \left(\nabla \times \vec{v} \right)}_{\perp \vec{v}}}_{0} = -\vec{v} \cdot \nabla \left(\frac{p}{\rho} + gy \right)$$

$$\vec{v} \quad \cdot \text{ momentum } (1) \rightarrow \text{ energy}$$

Therefore,

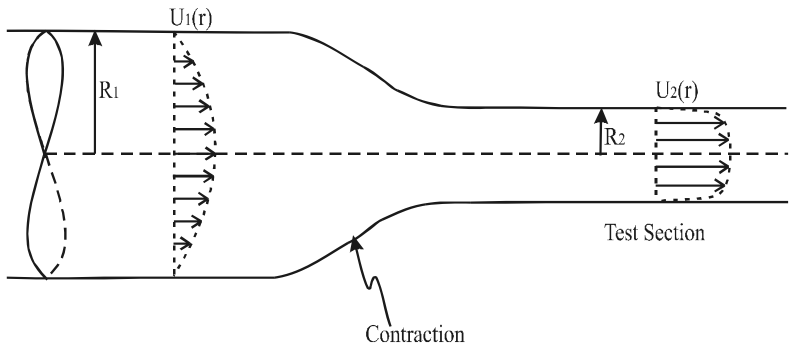
$$\underset{\text{streamline}}{\vec{v} \cdot \nabla \left(\tfrac{v^2}{2} + \tfrac{p}{\rho} + gy \right)} = 0 = \underset{\text{pathline}}{\tfrac{D}{Dt} \left(\tfrac{v^2}{2} + \tfrac{p}{\rho} + gy \right)}$$

$$\text{i.e., } \tfrac{v^2}{2} + \tfrac{p}{\rho} + gy = \text{ constant on a streamline}$$

In general, $\frac{v^2}{2} + \frac{p}{\rho} + gy = F\left(\Psi\right)$ where Ψ is a tag for a particular streamline.

Assumptions: **Ideal** fluid, **Steady** flow, **Rotational** in general.

3.4.1 **Example: Contraction in Water or Wind Tunnel**

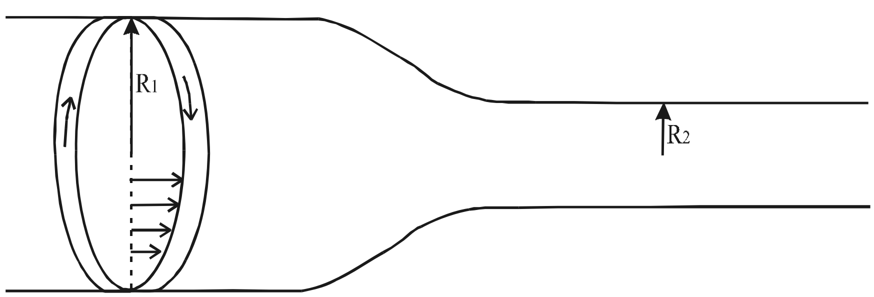


Contraction Ratio: $\gamma = R_1/R_2 >> 1$ ($\gamma = O(10)$ for wind tunnel ; $\gamma = O(5)$ for water tunnel)

Let $\bar{U}_1$ and $\bar{U}_2$ denote the average velocities at sections 1 and 2 respectively.

1. From continuity: $\bar{U}_1\left(\pi R_1^2\right) = \bar{U}_2\left(\pi R_2^2\right) \rightarrow \dfrac{\bar{U}_2}{\bar{U}_1} = \left(\dfrac{R_1}{R_2}\right)^2 = \gamma^2 >> 1$

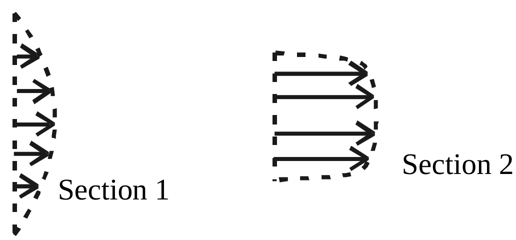
2.



Since $\dfrac{\partial u}{\partial r} \neq 0$, $\vec{\omega} \neq 0 \rightarrow$ vortex ring.

$$\omega/L = \quad \text{constant} \quad \Rightarrow \begin{cases} \dfrac{\omega_1}{2\pi R_1} = \dfrac{\omega_2}{2\pi R_2} \rightarrow \dfrac{\omega_2}{\omega_1} = \dfrac{R_2}{R_1} \sim \dfrac{1}{\gamma} << 1 \\ \\ \text{since } \omega \sim \dfrac{\partial u}{\partial r} \rightarrow \left(\dfrac{\partial u}{\partial r}\right)_2 << \left(\dfrac{\partial u}{\partial r}\right)_1 \end{cases}$$

i.e.,



3. Near the center, let $U_1 = \bar{U}_1 (1 + \varepsilon_1)$ and $U_2 = \bar{U}_2 (1 + \varepsilon_2)$ where ε_1 and ε_2 measure the relative velocity fluctuations. Apply the Bernoulli equation along a reference average streamline

$$P_1 + \tfrac{1}{2}\rho \bar{U}_1^2 = P_2 + \tfrac{1}{2}\rho \bar{U}_2^2 \tag{2}$$

Apply Bernoulli Equation to a particular streamline

$$P_1 + \tfrac{1}{2}\rho \left[\bar{U}_1 (1 + \varepsilon_1)\right]^2 = P_2 + \tfrac{1}{2}\rho \left[\bar{U}_2 (1 + \varepsilon_2)\right]^2$$

From (2) and (3) we obtain

$$\varepsilon_1 \bar{U}_1^2 = \varepsilon_2 \bar{U}_2^2 + O(\varepsilon^2) \rightarrow \frac{\varepsilon_2}{\varepsilon_1} \sim \frac{\bar{U}_1^2}{\bar{U}_2^2} \sim \frac{1}{\gamma^4} << 1$$