

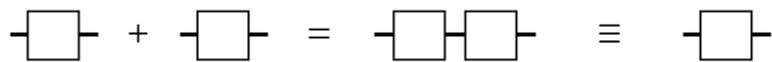
Junction elements in network models.

Classify by number of ports and examine the possible structures that result.

Using only one-port elements, no more than two elements can be assembled.



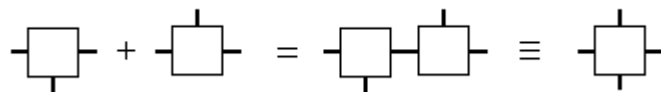
Combining two two-ports yields another two-port.



At most two one-port elements could be connected.



Thus three-port junction elements are *necessary*. They are also *sufficient* insofar as junctions with more than three ports may be constructed from assemblages of three-ports.



An assembly of two three-port elements yields a four-port element.

Using two- and three-port junctions, an arbitrary number of one-port (or n-port) elements may be assembled into a system.

Three-port junction elements

A junction structure or junction element is treated as a system with certain special properties which are derived from energetic considerations.

An ideal junction structure is power-continuous:

power is transmitted instantaneously without storing or dissipating energy.

instantaneous power out equals instantaneous power in.

Power flow in and out of a three-port may be characterized by six real-valued wave-scattering variables

u_1, u_2, u_3 , define the input power flow

v_1, v_2, v_3 , define the output power flow

Using vector notation:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad (\text{A.1})$$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad (\text{A.2})$$

The input and output power flows are the square of the length of these vectors, their inner products.

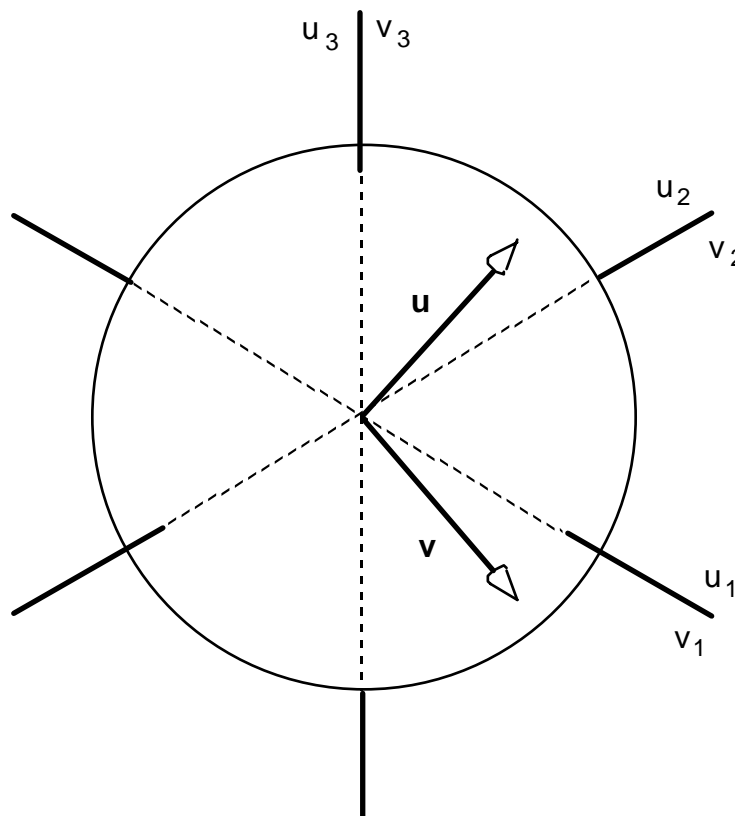
$$P_{\text{in}} = \sum_{i=1}^3 u_i^2 = \mathbf{u}^t \mathbf{u} \quad (\text{A.3})$$

$$P_{\text{out}} = \sum_{i=1}^3 v_i^2 = \mathbf{v}^t \mathbf{v} \quad (\text{A.4})$$

Constitutive equations of the junction structure may be written in vector form as an algebraic relation between input and output scattering variables.

$$\mathbf{v} = \mathbf{f}(\mathbf{u}) \quad (\text{A.5})$$

This algebraic relation is constrained by the requirement that power in equal power out. Geometrically, the length of the vector \mathbf{v} must equal the length of the vector \mathbf{u} . The input and output vectors are both confined to the surface of a sphere.



Consequently, for any two particular values of \mathbf{u} and \mathbf{v} , the algebraic relation $\mathbf{f}(\cdot)$ is equivalent to a *rotation operator* and equation A.5 may be re-written as follows.

$$\mathbf{v} = \mathbf{S}(\mathbf{u}) \mathbf{u} \quad (\text{A.6})$$

where the square matrix \mathbf{S} is known as a *scattering matrix*.

\mathbf{S} need not be a constant matrix, but may in general depend on the power flux through the junction, hence the notation $\mathbf{S}(\mathbf{u})$. However, \mathbf{S} is subject to important restrictions. In particular,

$$\mathbf{v}^t \mathbf{v} = \mathbf{u}^t \mathbf{S}^t \mathbf{S} \mathbf{u} = \mathbf{u}^t \mathbf{u} \quad (\text{A.7})$$

\mathbf{S} is an orthogonal matrix: the vectors formed by each of its rows (or columns) are (i) orthogonal and (ii) have unit magnitude; its transpose is its inverse.

$$\mathbf{S}^t \mathbf{S} = \mathbf{1} \quad (\text{A.8})$$

(This is simply a mathematical statement of the conclusion that \mathbf{S} is a rotation operator.)

Symmetric three-port junction elements

To evaluate \mathbf{S} , we assume an additional *symmetry property* in the mathematical sense of invariance under the action of a group operator.

Specifically, assume the constitutive equation is invariant under the operation of permuting or exchanging the ports of the junction structure.

In terms of the coefficients of the scattering matrix \mathbf{S} :

$$\mathbf{S} = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix} \quad (\text{A.9})$$

The power output along any port depends upon the power input along all three ports, itself and its two neighbors.

Each of the neighboring ports contribute the same fraction, b , of their input power

That fraction may be different from the portion, a , of the power input which is "reflected" back out the same port.

Each of the three ports is the same in this regard.

Using equation A.9 in equation A.8 yields three equations for the coefficients a and b , but only two are distinct.

$$a^2 + 2b^2 = 1 \quad (\text{A.10})$$

$$b^2 + 2ab = 0 \quad (\text{A.11})$$

These equations admit only two solutions,

$$a = 1/3; b = -2/3 \quad (\text{A.12})$$

or

$$a = -1/3; b = 2/3 \quad (\text{A.13})$$

so the symmetry condition constrains the scattering matrix to be a constant.

There are only two possible symmetric, power-continuous, three-port junctions. They are both linear.

Transform back to effort and flow variables.

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \quad (\text{A.14})$$

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \quad (\text{A.15})$$

The relation between efforts and wave-scattering variables is as follows.

$$\mathbf{e} = (\mathbf{u} - \mathbf{v}) c = c (\mathbf{1} - \mathbf{S}) \mathbf{u} \quad (\text{A.16})$$

where c is a scaling constant. Using equation A.12 for the value of \mathbf{S}

$$\mathbf{S} = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix} \quad (\text{A.17})$$

it can be seen that

$$\mathbf{e} = c \begin{bmatrix} 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \end{bmatrix} \mathbf{u} \quad (\text{A.18})$$

The efforts on each of the ports are equal. This is a common effort or type zero junction.

$$e_1 = e_2 = e_3 \quad (\text{A.19})$$

The relation between flows and wave-scattering variables is as follows.

$$\mathbf{f} = (\mathbf{u} + \mathbf{v})/c = 1/c (\mathbf{1} + \mathbf{S}) \mathbf{u} \quad (\text{A.20})$$

Using the same value for \mathbf{S}

$$\mathbf{f} = \frac{1}{c} \begin{bmatrix} 4/3 & -2/3 & -2/3 \\ -2/3 & 4/3 & -2/3 \\ -2/3 & -2/3 & 4/3 \end{bmatrix} \mathbf{u} \quad (\text{A.21})$$

Summing down the columns, it can be seen that the relation between flows on the ports is

$$f_1 + f_2 + f_3 = 0 \quad (\text{A.22})$$

This is a continuity equation¹. It is a *derived* property of the zero junction.

¹ Note that power on all ports of the junction has been defined to be positive inwards.

Using the other of the two values of \mathbf{S} (equation A.13).

$$\mathbf{S} = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \quad (\text{A.23})$$

From equation A.20 we find

$$\mathbf{f} = \frac{1}{c} \begin{bmatrix} 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \end{bmatrix} \mathbf{u} \quad (\text{A.24})$$

The flows on each of the ports of this junction are equal. This is a common flow or type one junction.

$$f_1 = f_2 = f_3 \quad (\text{A.25})$$

Using equation A.16

$$\mathbf{e} = c \begin{bmatrix} 4/3 & -2/3 & -2/3 \\ -2/3 & 4/3 & -2/3 \\ -2/3 & -2/3 & 4/3 \end{bmatrix} \mathbf{u} \quad (\text{A.26})$$

Summing down the columns, it can be seen that the relation between efforts is

$$e_1 + e_2 + e_3 = 0 \quad (\text{A.27})$$

This is a generalized compatibility equation. It is a *derived* property of the zero junction.

Equating effort and voltage, flow with current, we have derived Kirchhoff's current and voltage laws from energetic "first principles".

We have also shown that an exactly analogous pair of equations can be derived for any of the energetic media and that *they must always be linear*.

Conservation principles and symmetric junctions

In terms of efforts and flows, power continuity constrains the constitutive equations for a junction as follows.

$$\sum e_i f_i = 0 \quad (\text{A.28})$$

The bond graph zero-junction is associated with a generalization of Kirchhoff's node (current) law, a statement of flow (current) continuity that constrains the constitutive equations for a zero junction as follows.

$$\sum f_i = 0 \quad (\text{A.29})$$

But *these two constraints are not sufficient to define the zero junction* .

A junction element known as a *circulator* (used in microwave circuit theory) satisfies the requirements of power continuity (equation A.28) and current continuity (equation A.29), but it is quite distinct from a zero junction.

The *defining* property of a zero junction is its symmetry — the effort is invariant under the operation of permuting or exchanging the ports of the junction structure.

$$e_i = e_j \quad \forall i, j \quad (\text{A.30})$$

Combined with this condition, either of the previous two constraints implies the other. Thus, suppressing the subscript on the common effort, the power continuity condition may be written as follows.

$$e \sum f_i = 0 \quad (\text{A.31})$$

If this relation is to be identically true for all values of the (common) effort, the flow continuity condition must be satisfied.

A similar argument holds for the one-junction.

Thus symmetry is the fundamental property, continuity or compatibility are not.