Massachusetts Institute of Technology

Biological Engineering Division
Department of Mechanical Engineering
Department of Electrical Engineering and Computer Science

2.797/20.310/3.0536.024, Fall 2006
MOLECULAR, CELLULAR, & TISSUE BIOMECHANICS

Problem Set #8

Issued: 11/17/06
Due: 12/7/06

You only need to turn in <u>Problems 2 and 3</u>. Solutions will be distributed for the others.

True-False

1. A specimen that continues to creep (deform) under a constant stress, as long as the stress is applied, could be modeled as a Kelvin (or Standard Linear Solid) material.

2. If the applied stress is held constant with time, a purely elastic material will exhibit a constant, steady strain.

3. For a poroelastic material subjected to an oscillatory load at a single frequency, the strain is in phase with the applied load.

4. When subjected to confined compression, a tissue specimen is reduced in volume due to the expulsion of liquid through the boundaries.

5. For unconfined compression of a poroelastic material, the following constitutive law applies:

$$\sigma_{11}^{tot} = 2G\varepsilon_{11} + \lambda\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) - p$$

True-False

1. A specimen that continues to creep (deform) under a constant stress, as long as the stress is applied, could be modeled as a Kelvin (or Standard Linear Solid) material.
False. A Kelvin material will creep to a certain extent, but will eventually reach a new equilibrium.

2. If the applied stress is held constant with time, a purely elastic material will exhibit a constant, steady strain.
True. A purely elastic material will deform immediately upon the application of stress and will change its deformation only when the applied stress changes.

3. For a poroelastic material subjected to an oscillatory load at a single frequency, the strain is in phase with the applied load.
False. A poroelastic material, like a viscoelastic material, is dissipative so that the load will lead the deformation.

4. When subjected to confined compression, a tissue specimen is reduced in volume due to the expulsion of liquid through the boundaries.
True. In confined compression, the upper boundary is usually porous and interstitial fluid will leak out.

5. For unconfined compression of a poroelastic material, the following constitutive law applies:

$$\sigma_{11} = 2G\varepsilon_{11} + \lambda\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) - p$$

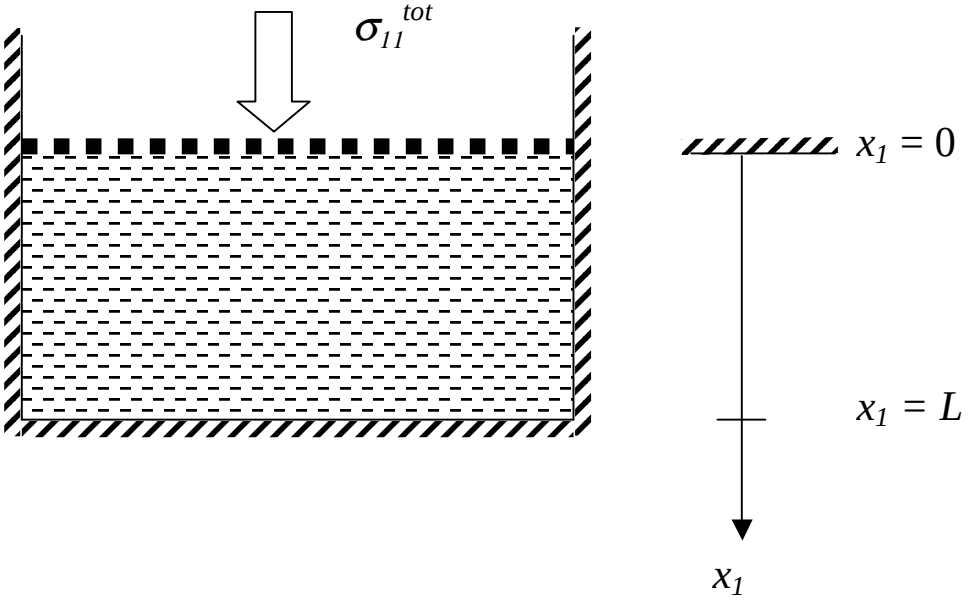
True. The only difference between this expression and that for an elastic material is the addition of the fluid pressure, p.

Problem 1: Linear, isotropic, homogeneous, poroelastic material

Consider a Poroelastic tissue specimen subjected to confined compression.

In class we demonstrated that the displacement $u_1(x_1,t)$ is described by a partial differential equation having the form of a diffusion equation with equivalent "diffusivity" equal to Hk, the product of the confined compression modulus $H = (2G + \lambda)$ and the hydraulic permeability k.

(a) Derive an analogous diffusion equation that describes the spatial and temporal dependence of the fluid pressure p. What is the equivalent "diffusivity"?



(b) A step in displacement is applied at $x_1 = 0$ having amplitude u_0. State the boundary conditions on $u_1(x_1=0,t)$ and $u_1(x_1=L,t)$ and the initial condition $u_1(x_1,t=0)$ that would be used to solve for the displacement $u_1(x_1,t)$ occurring during this "stress relaxation". **(*Do not solve*.)**

(c) A step in stress is applied at $x_1 = 0$ of amplitude σ_0. State the boundary conditions on the displacement (or its slope) and the initial condition on $u_1(x_1,t=0)$ that would be used to solve for the creep displacement $u_1(x_1,t)$. **(*Do not solve*.)**

(d) For the stress relaxation example of part (b), the solution below was provided in one of the slides from class. Use that solution to show (1) that the higher frequency components of the solution decay more rapidly with time, and (2) that the displacement is a linear function of x_1 as $t \to \infty$. What is the expression for the slowest ("n=1") decay time; i.e., the *stress relaxation* time, in terms of material and geometric constants?

$$u_1(x_1,t) = u_0\left(1 - \frac{x_1}{L}\right) - \sum_n A_n \sin\left(\frac{n\pi x_1}{L}\right)\exp\left(-\frac{t}{\tau_n}\right)$$

$$\tau_n = \frac{L^2}{n^2\pi^2 Hk}$$

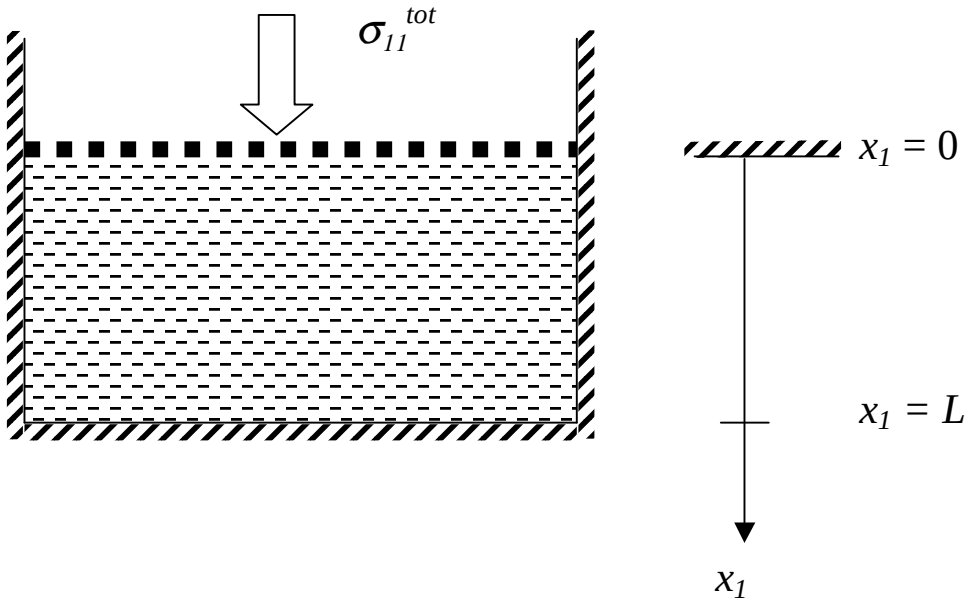
Problem 2: Measuring H and k

You wish to perform a simple set of experiments on a sample of cartilage to obtain values for the hydraulic permeability k and confined compression modulus H using an apparatus of the type shown in the sketch. The sample is placed into a compression chamber with rigid, non-permeable sides and bottom. On top of the sample is placed a permeable but rigid platen to which a vertical force can be applied.

For this problem, design an appropriate experiment or set of experiments that will allow you to compute individual values for k and H. You may specify either a time-varying (or static) force or displacement for the upper platen. Assume that all displacements are purely uni-directional (in the x_1-direction), that the sample has homogeneous properties, and that it satisfies the following expressions derived in class:

1) $U_1 = -k\dfrac{\partial p}{\partial z}$ 2) $\sigma_{11}^{tot} = H\varepsilon_{11} + p$ 3) $\dfrac{\partial \sigma_{11}^{tot}}{\partial x_1} = 0$

4) $U_1 = -\dfrac{\partial u_1}{\partial t}$ 5) $\varepsilon_{11} = \dfrac{\partial u_1}{\partial x_1}$



where U_1 is the x_1-component of velocity, u_1 is displacement of the cartilage matrix, p is hydrostatic pressure, and σ_1^{tot} is total stress.

There is no single correct answer to this problem; there are a variety of schemes that will work. All you need to do is describe <u>one</u> experimental procedure, and then provide the appropriate analysis indicating how H and k are to be computed from the experimental measurements.

Problem 3: Arterial wall poroelasticity

As the wall of an artery expands and contracts due to arterial pressure variations, there is a tendency for fluid to be periodically drawn into and expelled from the tissue comprising the wall. In this problem, you will model the arterial wall as a poroelastic material and analyze this fluid motion. This is of special interest in the context of arterial disease since this represents one method by which lipids normally found in the blood plasma might enter the arterial wall where they could react with extracellular matrix proteins and form the nucleus for lipid aggregation (see e.g., Yin et al., A model for the initiation and growth of extracellular lipid liposomes in arterial intima. Am J Physiol. 1997 Feb;272(2 Pt 2):H1033-46.).

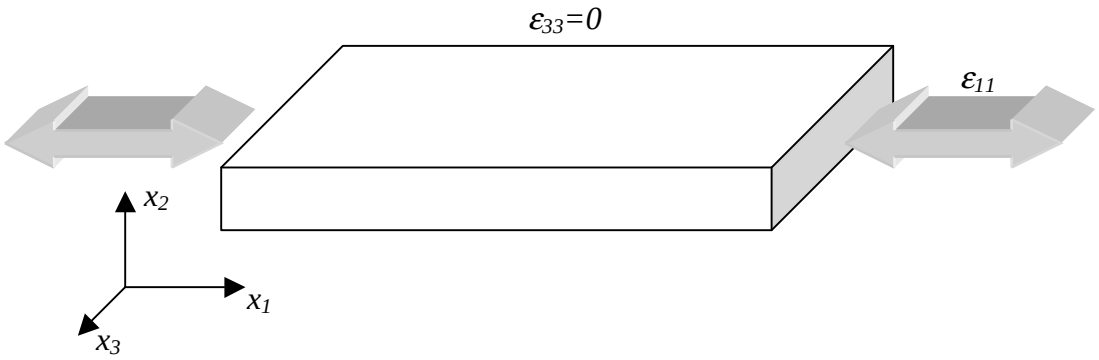
Assume here that:

- the wall is thin compared to the radius of the artery, so that you can treat the wall locally as though it were a flat plate. With this assumption, the change in vessel

circumference with time can be expressed in terms of a time-varying value of $\boldsymbol{\varepsilon_{11} = \varepsilon_0 \sin(\omega t)}$ (see Fig. 1).

- since the total length of the arterial segment does not change during a cardiac cycle, you may assume that $\boldsymbol{\varepsilon_{33} = 0}$. Consequently, there will be a tendency for ε_{22} to vary with time. Note that this does not imply that $\tau_{33} = 0$.

- the bottom surface is impermeable, so that all the fluid inflow and outflow occurs through the top surface ($x_2 = h$). The top surface can be assumed to be exposed to a **constant pressure $p = 0$** for all times, so that the fluid flow into and out of the arterial wall is driven entirely by the imposed time varying strain ε_{11}.



a) First consider the tissue to be a homogeneous, isotropic, incompressible ($\nu = 0.5$) and **linearly elastic** (not poroelastic) material, and obtain expressions for $\varepsilon_{22}(x_2,t)$ and $u_2(x_2,t)$. Note, in particular, whether or not ε_{22} depends upon x_2.

b) Now treat the tissue as a **poroelastic** rather than an elastic material, with known values for the shear modulus G and Lame' constant λ. **Show** that u_2 satisfies the following relationship:

$$\frac{\partial u_2}{\partial t} = Hk\frac{\partial^2 u_2}{\partial x_2^2}$$

(which is the same as that which governs one-dimensional confined compression) where $H = 2G+\lambda$ and k is the hydraulic permeability.

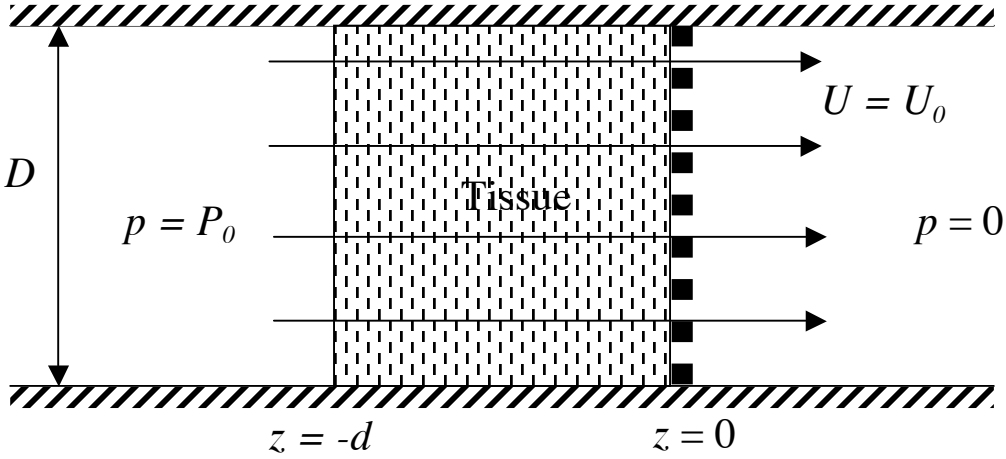
c) Consider the limit of $\omega \to \infty$ and obtain an expression for $u_2(x_2,t)$. (Hint: Think in physical terms what happens in this limiting case, and <u>do not</u> simply set $\partial u_2 / \partial t$ to zero!)

d) Consider the limit of $\omega \to 0$ and obtain an expression for $u_2(x_2,t)$. (Hint: What boundary condition must be satisfied at $x_2 = h$?)

e) What are the two boundary conditions needed to solve the equation you obtained in (b)? Sketch (but **do not solve**) the solution $u_2(x_2,t)$ for the intermediate case, when ω is neither very large nor very small.

Problem 4: Permeability measurements (from recitation)

This problem emphasizes a general issue concerning the measurement of hydraulic permeability of hydrated soft tissues and other poroelastic media: applied pressure gradients and the resulting fluid flow can cause consolidation or deformation of the tissue. This might give rise to a *nonlinear permeability* that may ultimately be important to include in a model.



A cylindrical disk of porous, hydrated tissue has thickness d and diameter D, and is held within a chamber that confines the disk at its radial periphery. The tissue is supported by a rigid, porous filter located at the position $z = 0$. A constant pressure drop P_0 is applied across the tissue from left to right, resulting in a constant fluid flow velocity U_0 with respect to the tissue (the rigid filter prevents the tissue from moving, but does not impeded fluid flow).

The applied pressure drop and resulting fluid flow cause a compression of the tissue against the rigid filter. You are to find the resulting steady state, z-dependent strain and displacement profiles using the 1-dimensional poroelastic model for tissue derived in class.

(a) As done in class, write expressions for (1) conservation of momentum, (2) Darcy's law, and (3) total stress versus strain (including hydrostatic pressure) in terms of the total stress σ_{zz}, strain ε_{zz}, displacement u_z, and pressure p.

The fourth equation, mass conservation, is given here as:

$$U = \frac{-\partial u_z}{\partial t} + U_0$$

where the term U_0 corresponds to the possibility of a constant flow of fluid even when $\partial u_z / \partial t = 0$, as is the case here.

(b) Combine your equations (1)-(3) with the above equation to find the differential equation for u_z in terms of the constant velocity U_0, the hydraulic permeability k, and the confined compression modulus $H = (2G + \lambda)$.

(c) For the case of steady flow ($\partial / \partial t = 0$), integrate your differential equation to find an expression for the displacement u_z in terms of two integration constants. Find the two constants from the boundary conditions:

(i) zero displacement at $z = 0$

(ii) zero strain ($\partial u_z / \partial z$) at $z = -d$

(d) Write your final expressions for $u_z(z)$ and $\varepsilon_{zz}(z)$ within the tissue. Sketch u_z and ε_{zz} as functions of z within the tissue ($-d < z < 0$).