Problem set 2

Suggested reading: Strogatz, pp. 123–155, 168–174; Baker and Gollub, pp. 7–21.

1. The motion of a damped pendulum can be described by

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2 \sin\theta = 0 \tag{1}$$

where θ is the angle the pendulum makes with the vertical ($\theta = 0$ is down), γ is a damping factor and $\omega = \sqrt{g/l}$ is the natural frequency of the pendulum.

Find the fixed points of this system for all $\gamma > 0$ and determine their stability. Sketch the phase portraits in the vicinity of the fixed points for $\gamma \gg \omega$ and $\gamma \ll \omega$. Justify the directions of the trajectories.

2. The equation describing an undamped pendulum is

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0. \tag{2}$$

The phase space for this equation is shown in Fig. 6.7.3 of Strogatz. The phase trajectories appear to cross at ($\theta = n\pi, \dot{\theta} = 0$) for every odd integer n, but the trajectories cannot really cross.

Calculate the time required for the pendulum to travel from one of these apparent crossing-points to another—i.e., calculate the half-period $T_{1/2}$ of a pendulum with initial amplitude π and a vanishing initial velocity. You may find it helpful to find an expression for $\frac{d\theta}{dt}$ from the equation for the energy of a pendulum. A table of integrals may be useful.

3. Equation (2) may be approximated for moderate (as opposed to small) amplitudes by the *nonlinear* equation

$$\frac{d^2\theta}{dt^2} + \omega^2\left(\theta - \frac{1}{6}\theta^3\right) = 0. \tag{3}$$

Find the fixed points for this system and determine their stability. Sketch the phase portrait of this equation, and explain the qualitative differences between this phase portrait and the one for equation (2).

4. (Based on question 2.3 on page 35, Baker and Gollub.) A particle falls a distance $x(t)$ in a gravitational field, with velocity $v(t)$. The system of equations is

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= g \end{aligned}$$

a) Choose a rectangle in phase space with one corner at (x, v) and sides of length Δx and Δv. The corners correspond to four different initial states for four different particles. Where in phase space will these states be after a short time dt? What will be the area enclosed by the quadrilateral defined by the four points, and how does this area compare to the area of the initial rectangle?

b) Add a friction force to the acceleration equation and explain your choice of functional form. Show that areas in phase space now contract.

c) Draw the trajectories in the phase plane for the system in (b).