11 Lorenz equations

In this lecture we derive the Lorenz equations, and study their behavior.

The equations were first derived by writing a severe, low-order truncation of the equations of R-B convection.

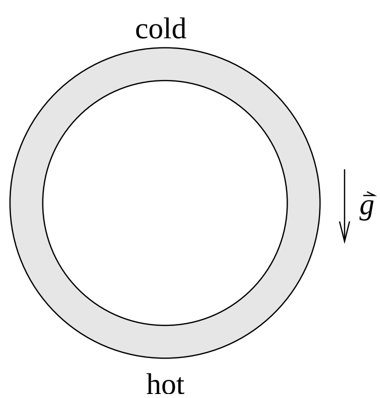
One motivation was to demonstrate SIC for weather systems, and thus point out the impossibility of accurate long-range predictions.

Our derivation emphasizes a simple physical setting to which the Lorenz equations apply, rather than the mathematics of the low-order truncation.

See Strogatz, Ch. 9, for a slightly different view. This lecture derives from Tritton, Physical Fluid Dynamics, 2nd ed. The derivation is originally due to Malkus and Howard.

11.1 Physical problem and parameterization

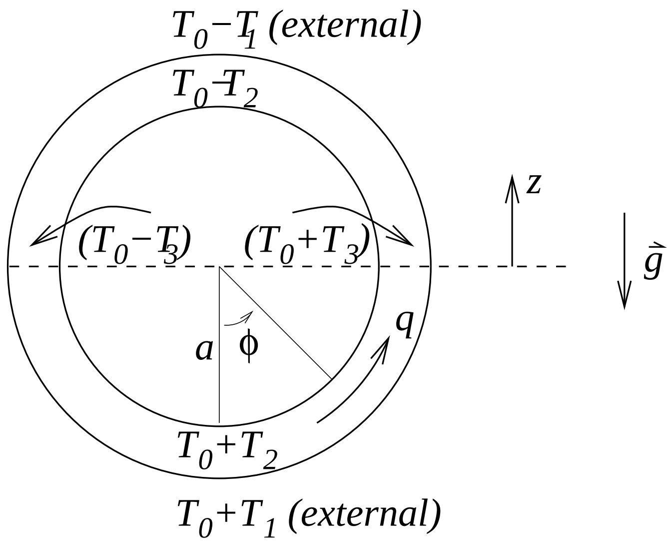
We consider convection in a vertical loop or torus, i.e., an empty circular tube:



We expect the following possible flows:

- Stable pure conduction (no fluid motion)
- Steady circulation
- Instabilities (unsteady circulation)

The precise setup of the loop:



ϕ = position round the loop.

External temperature T_E varies linearly with height:

$$T_E = T_0 - T_1 z/a = T_0 + T_1 \cos\phi \tag{24}$$

Let a be the radius of the loop. Assume that the tube's inner radius is much smaller than a.

Quantities inside the tube are averaged cross-sectionally:

$$\begin{aligned} \text{velocity} &= q = q(\phi, t) \\ \text{temperature} &= T = T(\phi, t) \qquad \text{(inside the loop)} \end{aligned}$$

As in the Rayleigh-Bénard problem, we employ the Boussinesq approximation (here, roughly like incompressiblity) and therefore assume

$$\frac{\partial \rho}{\partial t} = 0.$$

Thus mass conservation, which would give $\nabla \cdot \vec{u}$ in the full problem, here gives

$$\frac{\partial q}{\partial \phi} = 0. \tag{25}$$

Thus motions inside the loop are equivalent to a kind of solid-body rotation, such that

$$q = q(t).$$

The temperature $T(\phi)$ could in reality vary with much complexity. Here we assume it depends on only two parameters, T_2 and T_3, such that

$$T - T_0 = T_2 \cos\phi + T_3 \sin\phi. \tag{26}$$

Thus the temperature difference is

- $2T_2$ between the top and bottom, and
- $2T_3$ between sides at mid-height.

T_2 and T_3 vary with time:

$$T_2 = T_2(t), \qquad T_3 = T_3(t)$$

11.2 Equations of motion

11.2.1 Momentum equation

Recall the Navier-Stokes equation for convection:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} = -\frac{1}{\rho} \vec{\nabla} p - \vec{g} \alpha \Delta T + \nu \nabla^2 \vec{u}$$

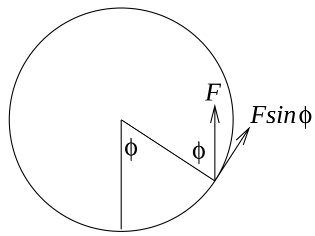
We write the equivalent equation for the loop as

$$\frac{\partial q}{\partial t} = -\frac{1}{\rho a} \frac{\partial p}{\partial \phi} + g\alpha (T - T_0) \sin\phi - \Gamma q. \tag{27}$$

The terms have the following interpretation:

- $\vec{u} \rightarrow q$
- $\vec{u} \cdot \nabla \vec{u} \rightarrow 0$ since $\partial q / \partial \phi = 0$.

- $\nabla p \rightarrow \frac{1}{a}\frac{\partial p}{\partial \phi}$ by transformation to polar coordinates.
- A factor of $\sin\phi$ modifies the buoyancy force $F = g\alpha(T - T_0)$ to obtain the tangential component:



  The sign is chosen so that hot fluid rises.

- Γ is a generalized friction coefficient, corresponding to viscous resistance proportional to velocity.

Now substitute the expression for $T - T_0$ (equation (26)) into the momentum equation (27):

$$\frac{\partial q}{\partial t} = -\frac{1}{\rho a}\frac{\partial p}{\partial \phi} + g\alpha(T_2\cos\phi + T_3\sin\phi)\sin\phi - \Gamma q$$

Integrate once round the loop, with respect to ϕ, to eliminate the pressure term:

$$2\pi\frac{\partial q}{\partial t} = g\alpha\int_0^{2\pi}(T_2\cos\phi\sin\phi + T_3\sin^2\phi)\mathrm{d}\phi - 2\pi\Gamma q.$$

The pressure term vanished because

$$\int_0^{2\pi}\frac{\partial p}{\partial \phi}\mathrm{d}\phi = 0,$$

i.e., there is no net pressure gradient around the loop.

The integrals are easily evaluated:

$$\int_0^{2\pi}\cos\phi\sin\phi\,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi\Big|_0^{2\pi} = 0$$

and

$$\int_0^{2\pi}\sin^2\phi\,\mathrm{d}\phi = \pi.$$

Then, after dividing by 2π, the momentum equation is

$$\frac{\mathrm{d}q}{\mathrm{d}t} = -\Gamma q + \frac{g\alpha T_3}{2} \tag{28}$$

where we have written $\mathrm{d}q/\mathrm{d}t$ instead of $\partial q/\partial t$ since $\partial q/\partial\phi = 0$.

We see that the motion is driven by the horizontal temperature difference, $2T_3$.

11.2.2 Temperature equation

We now seek an equation for changes in the temperature T. The full temperature equation for convection is

$$\frac{\partial T}{\partial t} + \vec{u}\cdot\vec{\nabla}T = \kappa\nabla^2 T$$

where κ is the heat diffusivity.

We approximate the temperature equation by considering only cross-sectional averages within the loop:

$$\frac{\partial T}{\partial t} + \frac{q}{a}\frac{\partial T}{\partial\phi} = K(T_E - T) \tag{29}$$

Here we have made the following assumptions:

- RHS assumes that heat is transferred through the walls at rate $K(T_{\text{external}} - T_{\text{internal}})$.
- Conduction round the loop is negligible (i.e., no $\nabla^2 T$).
- $\frac{q}{a}\frac{\partial T}{\partial\phi}$ is the product of averages, not (as it should be) the average of a product; i.e., q is taken to be uncorrelated to $\partial T/\partial\phi$.

Recall that we parameterized the internal temperature with two time-dependent variables, $T_2(t)$ and $T_3(t)$. We also have the external temperature T_E varying linearly with height. Specifically:

$$\begin{aligned} T_E &= T_0 + T_1\cos\phi \\ T - T_0 &= T_2\cos\phi + T_3\sin\phi \end{aligned}$$

Subtracting the second from the first,

$$T_E - T = (T_1 - T_2)\cos\phi - T_3 \sin\phi.$$

Substitute this into the temperature equation (29):

$$\frac{\mathrm{d}T_2}{\mathrm{d}t}\cos\phi + \frac{\mathrm{d}T_3}{\mathrm{d}t}\sin\phi - \frac{q}{a}T_2\sin\phi + \frac{q}{a}T_3\cos\phi = K(T_1 - T_2)\cos\phi - KT_3\sin\phi.$$

Here the partial derivatives of T have become total derivatives since T_2 and T_3 vary only with time.

Since the temperature equation must hold for all ϕ, we may separate $\sin\phi$ terms and $\cos\phi$ terms to obtain

$$\begin{aligned}
\sin\phi: \qquad \frac{\mathrm{d}T_3}{\mathrm{d}t} - \frac{qT_2}{a} &= -KT_3 \\
\cos\phi: \qquad \frac{\mathrm{d}T_2}{\mathrm{d}t} + \frac{qT_3}{a} &= K(T_1 - T_2)
\end{aligned}$$

These two equations, together with the momentum equation (28), are the three o.d.e.'s that govern the dynamics.

We proceed to simplify by defining

$$T_4(t) = T_1 - T_2(t),$$

which is the difference between internal and external temperatures at the top and bottom—loosely speaking, the extent to which the system departs from a "conductive equilibrium." Substitution yields

$$\begin{aligned}
\frac{\mathrm{d}T_3}{\mathrm{d}t} &= -KT_3 + \frac{qT_1}{a} - \frac{qT_4}{a} \\
\frac{\mathrm{d}T_4}{\mathrm{d}t} &= -KT_4 + \frac{qT_3}{a}
\end{aligned}$$

11.3 Dimensionless equations

Define the nondimensional variables

$$X = \frac{q}{aK}, \qquad Y = \frac{g\alpha T_3}{2a\Gamma K}, \qquad Z = \frac{g\alpha T_4}{2a\Gamma K}$$

Here

$$
\begin{aligned}
X &= \text{dimensionless velocity} \\
Y &= \text{dimensionless temperature difference between up and down currents} \\
Z &= \text{dimensionless departure from conductive equilibrium}
\end{aligned}
$$

Finally, define the dimensionless time

$$t' = tK.$$

Drop the prime on t to obtain

$$
\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}t} &= -PX + PY \\
\frac{\mathrm{d}Y}{\mathrm{d}t} &= -Y + rX - XZ \\
\frac{\mathrm{d}Z}{\mathrm{d}t} &= -Z + XY
\end{aligned}
$$

where the dimensionless parameters r and P are

$$
\begin{aligned}
r &= \frac{g\alpha T_1}{2a\Gamma K} = \text{“Rayleigh number”} \\
P &= \frac{\Gamma}{K} = \text{“Prandtl number”}
\end{aligned}
$$

These three equations are essentially the same as Lorenz's celebrated system, but with one difference. Lorenz's system contained a factor b in the last equation:

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = -\underline{b}Z + XY$$

The parameter b is related to the horizontal wavenumber of the convective motions.

11.4 Stability

We proceed to find the fixed points and evaluate their stability. For now, we remain with the loop equations ($b = 1$).

The fixed points, or steady solutions, occur where

$$\dot{X} = \dot{Y} = \dot{Z} = 0.$$

An obvious fixed point is

$$X^* = Y^* = Z^* = 0,$$

which corresponds, respectively, to a fluid at rest, pure conduction, and a temperature distribution consistent with conductive equilibrium.

Another steady solution is

$$\begin{aligned} X^* &= Y^* = \pm\sqrt{r-1} \\ Z^* &= r-1 \end{aligned}$$

This solution corresponds to flow around the loop at constant speed; the $\pm$ signs arise because the circulation can be in either sense. That $\text{sgn}(X) = \text{sgn}(Y)$ implies that hot fluid rises and cold fluid falls.

Note that the second (convective) solution exists only for $r > 1$. Thus we see that, effectively, $r = \text{Ra}/\text{Ra}_c$, i.e., the convective instability occurs when $\text{Ra} > \text{Ra}_c$.

As usual, we determine the stability of the steady-state solutions by determining the sign of the eigenvalues of the Jacobian.

Let

$$\vec{\phi} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \qquad \phi^* = \begin{pmatrix} X^* \\ Y^* \\ Z^* \end{pmatrix}$$

Then the Jacobian matrix is

$$\left.\frac{\partial \dot{\phi}_i}{\partial \phi_j}\right|_{\phi^*} = \begin{bmatrix} -P & +P & 0 \\ r - Z^* & -1 & -X^* \\ Y^* & X^* & -1 \end{bmatrix}$$

The eigenvalues σ are found by equating the following determinant to zero:

$$\begin{vmatrix} -(\sigma + P) & P & 0 \\ r - Z^* & -(\sigma+1) & -X^* \\ Y^* & X^* & -(\sigma+1) \end{vmatrix} = 0$$

For the steady state without circulation ($X^* = Y^* = Z^* = 0$), we have

$$\begin{vmatrix} -(\sigma + P) & P & 0 \\ r & -(\sigma+1) & 0 \\ 0 & 0 & -(\sigma+1) \end{vmatrix} = 0.$$

This yields

$$-(\sigma + P)(\sigma + 1)^2 + rP(\sigma + 1) = 0$$

or

$$(\sigma + 1)\left[\sigma^2 + \sigma(P+1) - P(r-1)\right] = 0.$$

There are three roots:

$$\begin{aligned} \sigma_1 &= -1 \\ \sigma_{2,3} &= \frac{-(P+1)}{2} \pm \frac{\sqrt{(P+1)^2 + 4P(r-1)}}{2} \end{aligned}$$

As usual,

$$\begin{aligned} \mathrm{Re}\{\sigma_1, \sigma_2, \textbf{and } \sigma_3\} &< 0 \Longrightarrow \text{stable} \\ \mathrm{Re}\{\sigma_1, \sigma_2, \textbf{or } \sigma_3\} &> 0 \Longrightarrow \text{unstable} \end{aligned}$$

Therefore $X^* = Y^* = Z^* = 0$ is

$$\begin{aligned} \text{stable} \quad & \text{for} \quad 0 < r < 1 \\ \text{unstable} \quad & \text{for} \quad r > 1 \end{aligned}$$

We now calculate the stability of the second fixed point, $X^* = \pm\sqrt{r-1}$, $Y^* = \pm\sqrt{r-1}$, $Z^* = r-1$.

The eigenvalues σ are now the solution of

$$\begin{vmatrix} -(\sigma + P) & P & 0 \\ 1 & -(\sigma+1) & -S \\ S & S & -(\sigma+1) \end{vmatrix} = 0, \qquad S = \pm\sqrt{r-1}.$$

(Explicitly,

$$\begin{aligned} -(\sigma + p)(\sigma+1)^2 - Ps^2 - S^2(\sigma + P) + P(\sigma + 1) &= 0 \\ (\sigma+1)[\sigma^2 + \sigma(P+1)] + \sigma S^2 + 2PS^2 &= 0.) \end{aligned}$$

The characteristic equation is cubic:

$$\sigma^3 + \sigma^2(P+2) + \sigma(P+r) + 2P(r-1) = 0$$

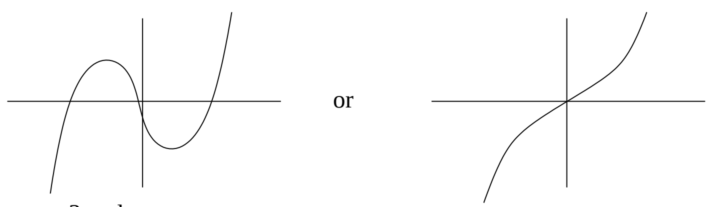
This equation is of the form

$$\sigma^3 + A\sigma^2 + B\sigma + C = 0 \tag{30}$$

where A, B, and C are all real and positive.

Such an equation has either

- 3 real roots; or
- 1 real root and 2 complex conjugate roots, e.g.,



Rearranging equation (30),

$$\sigma \underbrace{(\sigma^2 + B)}_{\text{positive real}} = \underbrace{-A\sigma^2 - C}_{\text{negative real}} < 0.$$

Consequently any real $\sigma < 0$, and we need only consider the complex roots (since only they may yield $\text{Re}\{\sigma\} > 0$).

Let σ_1 be the (negative) real root, and let

$$\sigma_{2,3} = \alpha \pm i\beta.$$

Then

$$(\sigma - \sigma_1)(\sigma - \alpha - i\beta)(\sigma - \alpha + i\beta) = 0$$

and

$$\begin{aligned} A &= -(\sigma_1 + 2\alpha) \\ B &= 2\alpha\sigma_1 + \alpha^2 + \beta^2 \\ C &= -\sigma_1(\alpha^2 + \beta^2) \end{aligned}$$

A little trick:

$$C - AB = 2\alpha \underbrace{\left[(\sigma_1 + \alpha)^2 + \beta^2\right]}_{\text{positive real}}.$$

Since α is the real part of both complex roots, we have

$$\text{sgn}(\text{Re}\{\sigma_{2,3}\}) = \text{sgn}(\alpha) = \text{sgn}(C - AB).$$

Thus instability occurs for $C - AB > 0$, or

$$2P(r-1) - (P+2)(P+r) > 0, .$$

Rearranging,

$$r(2P - P - 2) > 2P + P(P+2)$$

and we find that instability occurs for

$$r > r_c = \frac{P(P+4)}{P-2}.$$

This condition, which exists only for $P > 2$, gives the critical value of r for which steady *circulation* becomes unstable.

Loosely speaking, this is analogous to a transition to turbulence.

Summary: The rest state, $X^* = Y^* = Z^* = 0$, is

$$\begin{array}{rll} \text{stable} & \text{for} & 0 < r < 1 \\ \text{unstable} & \text{for} & r > 1. \end{array}$$

The convective state (steady circulation), $X^* = Y^* = \pm\sqrt{r-1}$, $Z^* = r - 1$, is

$$\begin{array}{rll} \text{stable} & \text{for} & 1 < r < r_c \\ \text{unstable} & \text{for} & r > r_c. \end{array}$$

What happens for $r > r_c$?

Before addressing that interesting question, we first look at contraction of volumes in phase space.

11.5 Dissipation

We now study the "full" equations, with the parameter b, such that

$$\dot{Z} = -bZ + XY, \qquad b > 0.$$

The rate of volume contraction is given by the Lie derivative

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \sum_i \frac{\partial \dot{\phi}_i}{\partial \phi_i}, \qquad i = 1, 2, 3, \qquad \phi_1 = X, \phi_2 = Y, \phi_3 = Z.$$

For the Lorenz equations,

$$\frac{\partial \dot{X}}{\partial X} + \frac{\partial \dot{Y}}{\partial Y} + \frac{\partial \dot{Z}}{\partial Z} = -P - 1 - b.$$

Thus

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -(P + 1 + b)V$$

which may be solved to yield

$$V(t) = V(0)e^{-(P+1+b)t}.$$

The system is clearly dissipative, since $P > 0$ and $b > 0$.

The most common choice of parameters is that chosen by Lorenz

$$\begin{aligned} P &= 10 \\ b &= 8/3 \qquad \text{(corresponding to the first wavenumber to go unstable).} \end{aligned}$$

For these parameters,

$$V(t) = V(0)e^{-\frac{41}{3}t}.$$

Thus after 1 time unit, volumes are reduced by a factor of $e^{-\frac{41}{3}} \sim 10^{-6}$. The system is therefore *highly* dissipative.

11.6 Numerical solutions

For the full Lorenz system, instability of the convective state occurs for

$$r > r_c = \frac{P(P + 3 + b)}{P - 1 - b}$$

For P=10, b=8/3, one has

$$r_c = 24.74.$$

In the following examples, $r = 28$.

Time series of the phase-space variables are shown in

Tritton, Fig 24.2, p. 397

- $X(t)$ represents variation of velocity round the loop.
  - Oscillations around each fixed point X_+^* and X_-^* represent variation in speed but the same direction.
  - Change in sign represents change in direction.
- $Y(t)$ represents the temperature difference between up and downggoing currents. Intuitively, we expect some correlation between $X(t)$ and $Y(t)$.
- $Z(t)$ represents the departure from conductive equilibrium. Intuitively, we may expect that pronounced maxima of Z (i.e., overheating) would foreshadow a change in sign of X and Y, i.e., a destabilization of the sense of rotation.

Projection in the Z-Y plane, showing oscillations about the unstable convective fixed points, and flips after maxima of Z:

BPV, Fig. VI.12

A 3-D perspective, the famous "butterfly:"

BPV, Fig. VI.14

Note the system is symmetric, being invariant under the transformation $X \to -X$, $Y \to -Y$, $Z \to Z$.

A slice (i.e., a Poincaré section) through the plane $Z = r - 1$, which contains the convective fixed points:

BPV, Fig. VI.15

- The trajectories lie on roughly straight lines, indicating the attractor dimension $d \simeq 2$.
- These are really closely packed sheets, with (as we shall see) a fractal dimension of 2.06.
- $d \simeq 2$ results from the strong dissipation.

Since $d \simeq 2$, we can construct, as did Lorenz, the first return map

$$z_{k+1} = f(z_k),$$

where z_k is the kth maximum of $Z(t)$. The result is

BPV, Fig. VI.16

(These points intersect the plane $XY - bZ = 0$, which corresponds to $\dot{Z} = 0$.)

The first-return map shows that the dynamics can be approximated by a 1-D map. It also reveals the stability properties of the fixed point $Z = r - 1$:

BPV, Fig. VI.17

Finally, sensitivity to initial conditions is documented by

BPV, Fig. VI.18

11.7 Conclusion

The Lorenz model shows us that the apparent unpredictability of turbulent fluid dynamics is deterministic. Why?

Lorenz's system is much simpler than the Navier-Stokes equations, but it is essentially contained within them.

Because the simpler system exhibits deterministic chaos, surely the Navier-Stokes equations contain sufficient complexity to do so also.

Thus any doubt concerning the deterministic foundation of turbulence, such as assuming that turbulence represents a failure of deterministic equations, is now removed.

A striking conclusion is that only a few (here, three) degrees of freedom are required to exhibit this complexity. Previous explanations of transitions to turbulence (e.g., Landau) had invoked a successive introduction of a large number of degrees of freedom.