15 Lyapunov exponents

Whereas fractals quantify the geometry of strange attractors, Lyaponov exponents quantify the sensitivity to initial conditions that is, in effect, their most salient feature.

In this lecture we point broadly sketch some of the mathematical issues concerning Lyaponov exponents. We also briefly describe how they are computed. We then conclude with a description of a simple model that shows how both fractals and Lyaponov exponents manifest themselves in a simple model.

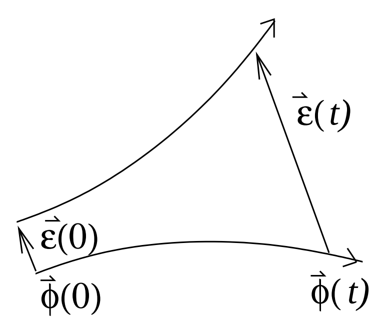
15.1 Diverging trajectories

Lyapunov exponents measure the rate of divergence of trajectories on an attractor.

Consider a flow $\vec{\phi}(t)$ in phase space, given by

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \vec{F}(\vec{\phi})$$

If instead of initiating the flow at $\vec{\phi}(0)$, it is initiated at $\vec{\phi}(0)+\varepsilon(0)$, sensitivity to initial conditions would produce a divergent trajectory:



Here $|\vec{\varepsilon}|$ grows with time. To the first order,

$$\frac{\mathrm{d}(\vec{\phi}+\vec{\varepsilon})}{\mathrm{d}t} \simeq \vec{F}(\vec{\phi}) + M(t)\,\vec{\varepsilon}$$

where

$$M_{ij}(t) = \left. \frac{\partial F_i}{\partial \phi_j} \right|_{\vec{\phi}(t)} .$$

We thus find that

$$\frac{\mathrm{d}\vec{\varepsilon}}{\mathrm{d}t} = M(t)\,\vec{\varepsilon}. \tag{31}$$

Consider the example of the Lorenz model. The Jacobian M is given by

$$M(t) = \begin{bmatrix} -P & P & 0 \\ -Z(t)+r & -1 & -X(t) \\ Y(t) & X(t) & -b \end{bmatrix}.$$

We cannot solve for $\vec{\varepsilon}$ because of the unknown time dependence of $M(t)$. However one may numerically solve for $\vec{\phi}(t)$, and thus $\vec{\varepsilon}(t)$, to obtain (formally)

$$\vec{\varepsilon}(t) = L(t)\,\vec{\varepsilon}(0).$$

15.2 Example 1: M independent of time

Consider a simple 3-D example in which M is time-independent.

Assume additionally that the phase space coordinates correspond to M's eigenvectors.

Then M is diagonal and

$$L(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}$$

where the λ_i are the eigenvalues of M. (Recall that if $\dot{\vec{\varepsilon}} = M\vec{\varepsilon}$, then $\vec{\varepsilon}(t) = e^{Mt}\vec{\varepsilon}(0)$, where, in the coordinate system of the eigenvectors, $e^{Mt} = L(t)$.)

As t increases, the eigenvalue with the largest real part dominates the flow $\vec{\varepsilon}(t)$.

To express this formally, let L^* be the conjugate (Hermitian) transpose of L, i.e.

$$L^*_{ij} = L_{ji}.$$

Also let

$$\mathrm{Tr}(L) = \text{diagonal sum } = \sum_{i=j} L_{ij}.$$

Then

$$\mathrm{Tr}[L^*(t)L(t)] = e^{(\lambda_1+\lambda_1^*)t} + e^{(\lambda_2+\lambda_2^*)t} + e^{(\lambda_3+\lambda_3^*)t}$$

Define

$$\overline{\lambda} = \lim_{t\to\infty} \frac{1}{2t} \ln \Big(\mathrm{Tr}[L^*(t)L(t)] \Big)$$

$\overline{\lambda}$ is the *largest Lyapunov exponent*. Its sign is crucial:

$$\begin{aligned} \overline{\lambda} < 0 &\implies \varepsilon(t) \text{ decays exponentially} \\ \overline{\lambda} > 0 &\implies \varepsilon(t) \text{ grows exponentially.} \end{aligned}$$

15.3 Example 2: Time-dependent eigenvalues

Now suppose that $M(t)$ varies with time in such a way that only its eigenvalues, but not its eigenvectors, vary.

Let

$$\vec{\phi} = \begin{bmatrix} X(t) \\ Y(t) \\ Z(t) \end{bmatrix}$$

and consider small displacements $\delta X(t), \delta Y(t), \delta Z(t)$ in the reference frame of the eigenvectors.

Then, analogous to equation (31), and again assuming that phase space coordinates correspond to M's eigenvectors,

$$\begin{bmatrix} \delta\dot{X}(t) \\ \delta\dot{Y}(t) \\ \delta\dot{Z}(t) \end{bmatrix} = \begin{bmatrix} A[\phi(t)] & 0 & 0 \\ 0 & B[\phi(t)] & 0 \\ 0 & 0 & C[\phi(t)] \end{bmatrix} \begin{bmatrix} \delta X(t) \\ \delta Y(t) \\ \delta Z(t) \end{bmatrix}.$$

Here A, B, C are the time-dependent eigenvalues (assumed to be real).

The solution for $\delta X(t)$ is

$$\delta X(t) = \delta X(0)\ \exp\left[\int_0^t \mathrm{d}t' A[\phi(t')]\right]$$

Rearranging and dividing by t,

$$\frac{1}{t}\ln\left|\frac{\delta X(t)}{\delta X(0)}\right| = \frac{1}{t}\int_0^t \mathrm{d}t' A[\phi(t')]$$

The RHS represents the time-average of the eigenvalue A. We assume that for sufficiently long times this average is equivalent to an average of A for all possible flows ϕ evaluated at the same time.

In other words, we assume that the flow is *ergodic*.

We denote this average by angle brackets:

$$\begin{aligned}
\langle A\rangle &= \phi\text{-average of } A[\phi(t)] \\
&= \text{time-average of } A[\phi(t)] \\
&= \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathrm{d}t' A[\phi(t')] \\
&= \lim_{t\to\infty}\frac{1}{t}\ln\left|\frac{\delta X(t)}{\delta X(0)}\right|
\end{aligned}$$

$\langle A\rangle$ is one of the three Lyapunov exponents for $\phi(t)$.

More sophisticated analyses show that the theory sketched above applies to the general case in which both eigenvectors and eigenvalues vary with time.

15.4 Numerical evaluation

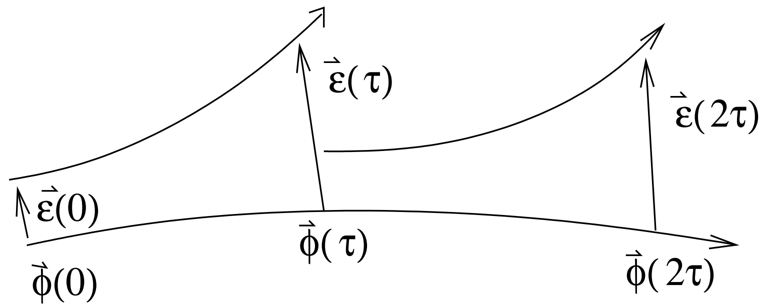
Lyaponov exponents are almost always evaluated numerically.

The most obvious method is the one used in the problem sets: For some $\vec{\varepsilon}(0)$, numerically evaluate $\vec{\varepsilon}(t)$, and then find $\overline{\lambda}$ such that

$$|\vec{\varepsilon}(t)| \simeq |\vec{\varepsilon}(0)|e^{\overline{\lambda}t}.$$

This corresponds to the definition of $\langle A\rangle$ above.

A better method avoids saturation at the size of the attractor by successively averaging small changes over the same trajectory:



Here $\vec{\varepsilon}$ is renormalized at each step such that

$$\vec{\varepsilon}(\tau) = \vec{\varepsilon}(0)e^{\gamma_1\tau}$$

$$\vec{\varepsilon}(2\tau) = \frac{\vec{\varepsilon}(\tau)}{|\vec{\varepsilon}(\tau)|}e^{\gamma_2\tau}$$

The largest Lyaponov exponent is given by the long-time average:

$$\overline{\lambda} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\gamma_i = \lim_{n\to\infty}\frac{1}{n\tau}\sum_{i}^{n}\ln|\vec{\varepsilon}(i\tau)|$$

Experimental data poses greater challenges, because generally we have only a single time series $X(t)$.

One way is to compare two intervals on $X(t)$, say

$$[t_1, t_2] \qquad \text{and} \qquad [t_1', t_2'],$$

where $X(t)$ is nearly the same on both intervals.

Then the comparison of $X(t)$ beyond t_2 and t_2' may yield the largest Lyaponov exponent.

Another way is to reconstruct phase space by, say, the method of delays. Then all trajectories that pass near a certain point may be compared to see the rate at which they diverge.

15.5 Lyaponov exponents and attractors in 3-D

Consider an attractor in a 3-D phase space. There are 3 Lyaponov exponents.

Their signs depend on the type of attractor:

Type	Signs of Lyapunov exponents
Fixed point	$(-,-,-)$
Limit cycle	$(-,-,0)$
Torus T^2	$(-,0,0)$
Strange attractor	$(-,0,+)$

If the attractor is a fixed point, all three exponents are negative.

If it is a limit cycle with one frequency, only two are negative, and the third is zero. The zero-exponent corresponds to the direction of flow—which can neither be expanding nor contracting.

Of the other cases in the table below, the most interesting is that of a strange attractor:

- The largest exponent is, by definition, positive.
- There must also be a zero-exponent corresponding to the flow direction.
- The smallest exponent must be negative—and of greater magnitude than the largest, since volumes must be contracting.

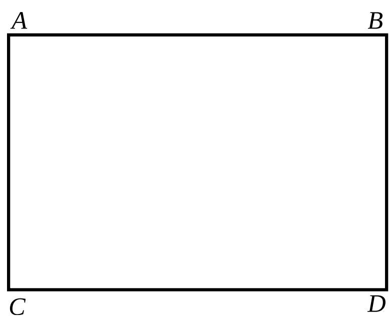
15.6 Smale's horseshoe attractor

We have seen that

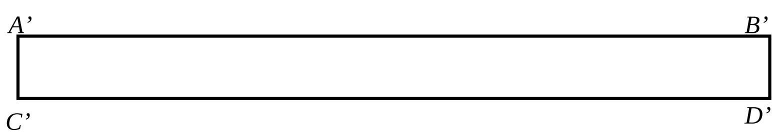
- Lyaponov exponents measure "stretching."
- Fractal dimensions measure "folding."

Smale's *horseshoe attractor* exemplifies both, and allows easy quantification.

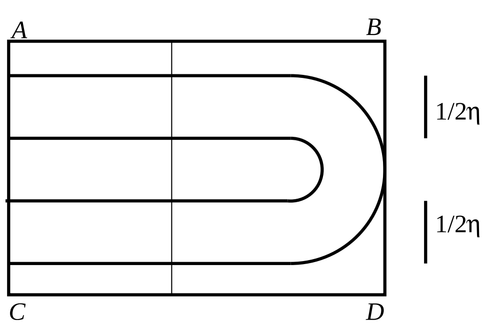
Start with a rectangle:



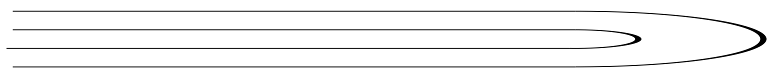
Stretch by a factor of 2; squash by a factor of $1/(2\eta)$, $\eta > 1$:



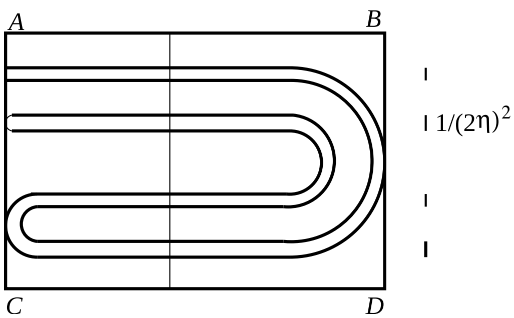
Now fold like a horseshoe and put back in $ABCD$:



Now iterate the process. Stretch and squash:



Fold and place back in $ABCD$:



Each dimension is successively scaled by its own multiplier, called a *Lyaponov number*:

$$\begin{aligned}\Lambda_1 &= 2 \qquad (x - \text{stretch}) \\ \Lambda_2 &= \frac{1}{2\eta} \qquad (y - \text{squash})\end{aligned}$$

Area contraction is given by

$$\Lambda_1 \Lambda_2 = 1/\eta.$$

The *Lyapunov exponents* are

$$\begin{aligned}\lambda_1 &= \ln \Lambda_1 \\ \lambda_2 &= \ln \Lambda_2\end{aligned}$$

Note also that vertical cuts through the attractor appear as the early iterations of a Cantor set.

To obtain the fractal dimension, we use the definition

$$D = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}.$$

Taking the initial box height to be unity, the ε, N pairs for the number N of segments of length ε required to cover the attractor is

ε	N
1	1
$1/(2\eta)$	2
$1/(2\eta)^2$	4
$\dots$	$\dots$
$1/(2\eta)^m$	2^m

Therefore the dimension D of the Cantor set is

$$D = \frac{\ln 2}{\ln 2\eta}.$$

The dimension D' of the attractor in the plane $ABCD$ is

$$D' = 1 + \frac{\ln 2}{\ln 2\eta},$$

where we have neglected the "bend" in the horseshoe (i.e., we've assumed the box's width is much greater than its height.

Note that,

$$\text{as } \eta \to 1, \quad D' \to 2,$$

because iterates nearly fill the plane. Conversely,

$$\text{as } \eta \to \infty, \quad D' \to 1,$$

meaning that the attractor is nearly squashed to a simple line.