5 Forced oscillators and limit cycles

5.1 General remarks

How may we describe a forced oscillator?
The linear equation

$$\ddot{\theta} + \gamma\dot{\theta} + \omega^2\theta = 0 \tag{3}$$

is in general inadequate. Why?

Linearity $\Rightarrow$ if $\theta(t)$ is a solution, then so is $\alpha\theta(t)$, α real. This is incompatible with bounded oscillations (i.e., $\theta_{\max} < \pi$).

We therefore introduce an equation with

- a nonlinearity; and
- an energy source that compensates viscous damping.

5.2 Van der Pol equation

Consider a damping coefficient $\gamma(\theta)$ such that

$$\gamma(\theta) > 0 \qquad \text{for } |\theta| \text{ large}$$

$$\gamma(\theta) < 0 \qquad \text{for } |\theta| \text{ small}$$

Express this in terms of θ^2:

$$\gamma(\theta) = \gamma_0 \left(\frac{\theta^2}{\theta_0^2} - 1 \right)$$

where $\gamma_0 > 0$ and θ_0 is some reference amplitude.

Now, obviously,

$$\gamma > 0 \qquad \text{for } \theta^2 > \theta_0^2$$

$$\gamma < 0 \qquad \text{for } \theta^2 < \theta_0^2$$

Substituting γ into (3), we get

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \gamma_0 \left(\frac{\theta^2}{\theta_0^2} - 1 \right) \frac{\mathrm{d}\theta}{\mathrm{d}t} + \omega^2 \theta = 0$$

This equation is known as the *van der Pol equation.* It was introduced in the 1920's as a model of nonlinear electric circuits used in the first radios.

In van der Pol's (vaccum tube) circuits,

- high current $\implies$ positive (ordinary) resistance; and
- low current $\implies$ negative resistance.

The basic behavior: large oscillations decay and small oscillations grow.

We shall examine this system in some detail. First, we write it in *non-dimensional* form.

We define new units of time and amplitude:

- unit of time $= 1/\omega$
- unit of amplitude $= \theta_0$.

We transform

$$\begin{aligned} t &\rightarrow t'/\omega \\ \theta &\rightarrow \theta'\theta_0 \end{aligned}$$

where θ' and t' are non-dimensional.

Substituting above, we obtain

$$\omega^2 \frac{\mathrm{d}^2\theta'}{\mathrm{d}t'^2}\theta_0 + \gamma_0 \left[\left(\frac{\theta'\theta_0}{\theta_0} \right)^2 - 1 \right] \frac{\mathrm{d}\theta'}{\mathrm{d}t'}\omega\theta_0 + \omega^2\theta'\theta_0 = 0$$

Divide by $\omega^2\theta_0$:

$$\frac{\mathrm{d}^2\theta'}{\mathrm{d}t'^2} + \frac{\gamma_0}{\omega} \left(\theta'^2 - 1 \right) \frac{\mathrm{d}\theta'}{\mathrm{d}t'} + \theta' = 0$$

Now define the dimensionless control parameter

$$\varepsilon = \frac{\gamma_0}{\omega} > 0.$$

Finally, drop primes to obtain

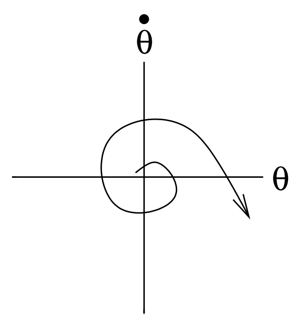
$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \varepsilon(\theta^2 - 1)\frac{\mathrm{d}\theta}{\mathrm{d}t} + \theta = 0. \tag{4}$$

What can we say about the phase portraits?

- When the amplitude of oscillations is small ($\theta_{\max} < 1$), we have

  $$\varepsilon(\theta_{\max}^2 - 1) < 0 \Rightarrow \text{negative damping}$$

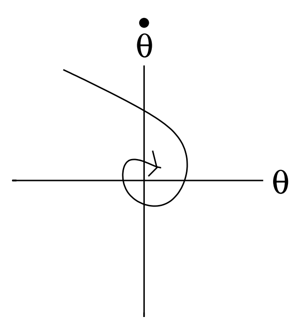
  Thus trajectories spiral outward:



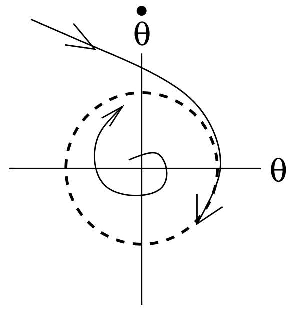
- But when the amplitude of oscillations is large ($\theta_{\max} > 1$),

  $$\varepsilon(\theta_{\max}^2 - 1) > 0 \Rightarrow \text{positive damping}$$

  The trajectories spiral inward:



Intuitively, we expect a closed trajectory between these two extreme cases:



This closed trajectory is called a **limit cycle**.

For $\varepsilon > 0$, the limit cycle is an *attractor* (and is stable).

This is a new kind of attractor. Instead of representing a single fixed point, it represents stable oscillations.

Examples of such stable oscilations abound in nature: heartbeats (**see Figure from Glass**); circadian (daily) cycles in body temperature, etc. Small perturbations always return to the standard cycle.

What can we say about the limit cycle of the van der Pol equation?

With the help of various theorems (see Strogatz, Ch. 7) one can prove the existence and stability of the limit cycle.

We may, however, make substantial progress with a simple energy balance argument.

5.3 Energy balance for small ε

Let $\varepsilon \to 0$, and take θ small. Using our previous expression for energy in the pendulum, the non-dimensional energy is

$$E(\theta, \dot{\theta}) = \frac{1}{2}(\dot{\theta}^2 + \theta^2)$$

The time variation of energy is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{2}(2\dot{\theta}\ddot{\theta} + 2\dot{\theta}\theta)$$

From the van der Pol equation (4), we have

$$\ddot{\theta} = -\varepsilon(\theta^2 - 1)\dot{\theta} - \theta.$$

Substituting this into the expression for $\mathrm{d}E/\mathrm{d}t$, we obtain

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \varepsilon\dot{\theta}^2(1-\theta^2) - \theta\dot{\theta} + \theta\dot{\theta} \qquad (5) \\ &= \varepsilon\dot{\theta}^2(1-\theta^2) \qquad (6) \end{aligned}$$

Now define the average of a function $f(t)$ over one period of the oscillation:

$$\overline{f} \equiv \frac{1}{2\pi}\int_{t_0}^{t_O+2\pi} f(t)\mathrm{d}t.$$

Then the average energy variation over one period is

$$\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} = \frac{1}{2\pi}\int_{t_0}^{t_0+2\pi} \frac{\mathrm{d}E}{\mathrm{d}t}\mathrm{d}t.$$

Substituting equation (6) for $\mathrm{d}E/\mathrm{d}t$, we obtain

$$\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} = \varepsilon\overline{\dot{\theta}^2} - \varepsilon\overline{\dot{\theta}^2\theta^2}.$$

In steady state, the production of energy, $\varepsilon\overline{\dot{\theta}^2}$, is exactly compensated by the dissipation of energy, $\varepsilon\overline{\dot{\theta}^2\theta^2}$. Thus

$$\varepsilon\overline{\dot{\theta}^2} = \varepsilon\overline{\dot{\theta}^2\theta^2}$$

or

$$\overline{\dot{\theta}^2} = \overline{\dot{\theta}^2\theta^2}.$$

Now consider the limit $\varepsilon \to 0$ (from above).
We know the approximate solution:

$$\theta(t) = \rho \sin t,$$

i.e., simple sinusoidal oscillation of unknown amplitude ρ.

We proceed to calculate ρ from the energy balance.

The average rate of energy production is

$$\overline{\dot{\theta}^2} \simeq \frac{1}{2\pi}\int_{t_0}^{t_0+2\pi} \rho^2 \cos^2 t dt = \frac{1}{2}\rho^2.$$

The average rate of energy dissipation is

$$\overline{\dot{\theta}^2\theta^2} \simeq \frac{1}{2\pi}\int_{t_0}^{t_0+2\pi} \rho^4 \sin^2 t \cos^2 t dt = \frac{1}{8}\rho^4.$$

The energy balance argument gives

$$\frac{1}{2}\rho^2 = \frac{1}{8}\rho^4.$$

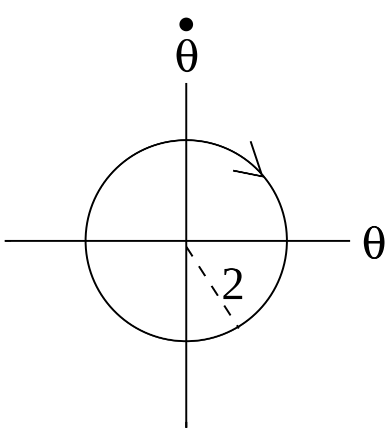
Therefore

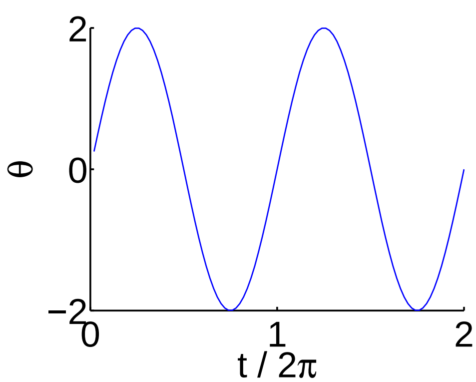
$$\rho = 2.$$

We thus find that, independent of $\varepsilon = \gamma_0/\omega$, we have the following approximate solution for $\varepsilon \ll 1$:

$$\theta(t) \simeq 2\sin t.$$

That is, we have a limit cycle with an amplitude of 2 dimensionless units. Graphically,





Further work (see, e.g., Strogatz) shows that this limit cycle is stable.

5.4 Limit cycle for ε large

The case of ε large requires a different analysis. We follow the argument given in Strogatz (p. 212).

First, we introduce an unconventional set of phase plane variables (not $\dot{x} = y, \dot{y} = \ldots$). That is, the phase plane coordinates will not be θ and $\dot{\theta}$.

Recall the van der Pol equation (4), but write in terms of $x = \theta$:

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0. \tag{7}$$

Notice that

$$\ddot{x} + \varepsilon\dot{x}(x^2 - 1) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{x} + \varepsilon\left(\frac{1}{3}x^3 - x\right)\right].$$

Let

$$F(x) = \frac{1}{3}x^3 - x \tag{8}$$

and

$$w = \dot{x} + \varepsilon F(x). \tag{9}$$

Then, using (8) and (9), we have

$$\dot{w} = \ddot{x} + \varepsilon\dot{x}(x^2 - 1).$$

Substituting the van der Pol equation (7), this gives

$$\dot{w} = -x \tag{10}$$

Now rearrange equation (9) to obtain

$$\dot{x} = w - \varepsilon F(x) \tag{11}$$

We have thus parameterized the system by x and w. However we make one more change of variable. Write

$$y = w/\varepsilon.$$

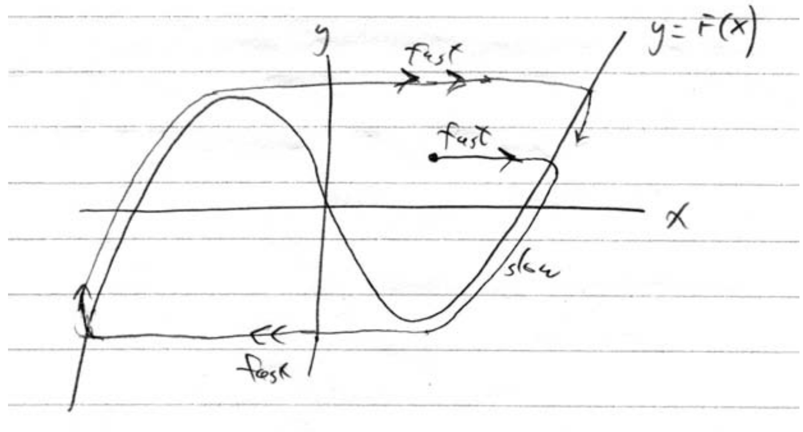
Then (10) and (11) become

$$\dot{x} = \varepsilon[y - F(x)] \tag{12}$$

$$\dot{y} = -\frac{1}{\varepsilon}x \tag{13}$$

Now consider a trajectory in the x-y plane.

First, draw the *nullcline* for x, that is, the curve showing where $\dot{x} = 0$. This is the cubic curve $y = F(x)$.



Now imagine a trajectory starting not too close to $y = F(x)$, i.e.. suppose

$$y - F(x) \sim 1.$$

Then from the equations of motion (12) and (13),

$$\begin{aligned} \dot{x} &\sim \varepsilon \gg 1 \\ \dot{y} &\sim 1/\varepsilon \ll 1 \qquad \text{assuming } x \sim 1. \end{aligned}$$

Thus the horizontal velocity is large and the vertical velocity is small.
$\Rightarrow$ trajectories move horizontally.
The y-nullcline shows that the vertical velocity vanishes for $x = 0$.)

Eventually the trajectory is so close to $y = F(x)$ such that

$$y - F(x) \sim \frac{1}{\varepsilon^2}$$

implying that

$$\dot{x} \sim \dot{y} \sim \frac{1}{\varepsilon}.$$

Thus the trajectory crosses the nullcline (vertically, since $\dot{x} = 0$ on the nullcline).

Then $\dot{x}$ changes sign, we still have $\dot{x} \sim \dot{y} \sim 1/\varepsilon$, and the trajectories crawl slowly along the nullcline.

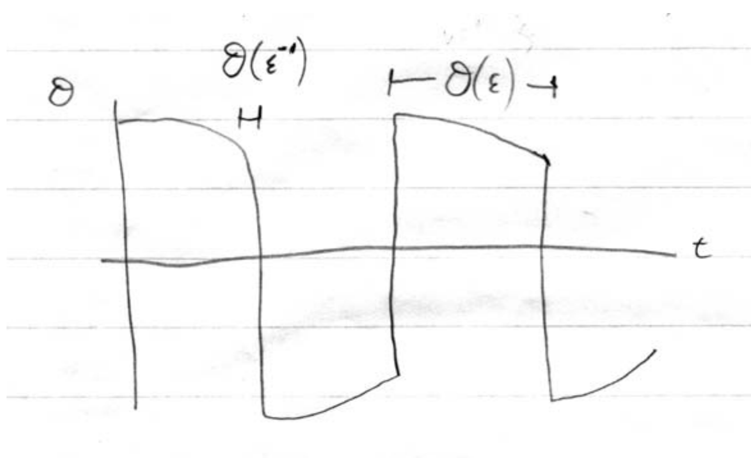
What happens at the knee (the minimum of $F(x)$)?
The trajectories jump sideways again, as may be inferred from the symmetry $x \to -x, \;\; y \to -y$.

The trajectory closes to form the limit cycle.

Summary: The dynamics has two widely separated time scales:

- The crawls: $\Delta t \sim \varepsilon \qquad (\dot{x} \sim 1/\varepsilon)$
- The jumps: $\Delta t \sim 1/\varepsilon \qquad (\dot{x} \sim \varepsilon)$

A time series of $x(t) = \theta(t)$ shows a classic **relaxation oscillation**:



Relaxation oscillations are periodic processes with two time scales: a slow buildup is followed by a fast discharge.

Examples include

- stick-slip friction (earthquakes, avalanches, bowed violin strings, etc.)
- nerve cells, heart beats (large literature in mathematical biology...)

5.5 A final note

Limit cycles exist only in nonlinear systems. Why?

A linear system $\dot{\vec{x}} = A\vec{x}$ can have closed periodic orbits, but not an *isolated* orbit.

That is, linearity requires that if $\vec{x}(t)$ is a solution, so is $\alpha\vec{x}(t)$, $\alpha \neq 0$.

Thus the amplitude of a periodic cycle in a linear system depends on the initial conditions.

The amplitude of a limit cycle, however, is independent of the initial conditions.