Introduction to Numerical Analysis for Engineers

- Fundamentals of Digital Computing
  - Digital Computer Models
  - Convergence, accuracy and stability
  - Number representation
  - Arithmetic operations
  - Recursion algorithms
- Error Analysis
  - Error propagation – numerical stability
  - Error estimation
  - Error cancellation
  - Condition numbers

Error Propagation

Spherical Bessel Functions

Differential Equation

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}(x^2-n(n+1))y=0$$

Solutions

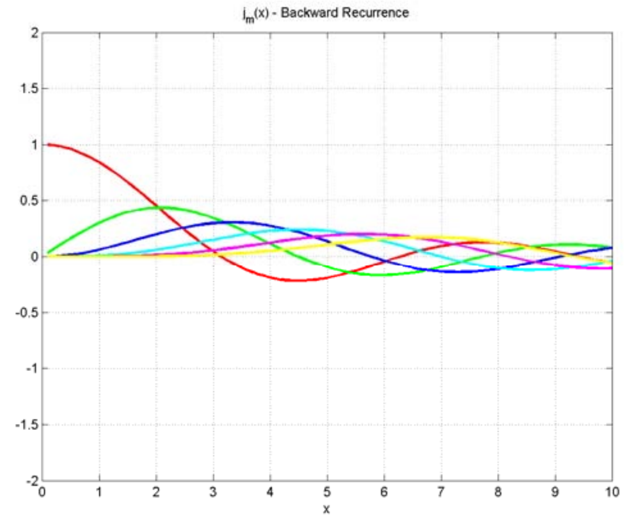
$$j_n(x)y_n(x)$$

n	$j_n(x)$	$y_n(x)$
0	$\frac{\sin x}{x}$	$-\frac{\cos x}{x}$
1	$\frac{\sin x}{x^2}-\frac{\cos x}{x}$	$-\frac{\cos x}{x^2}-\frac{\sin x}{x}$

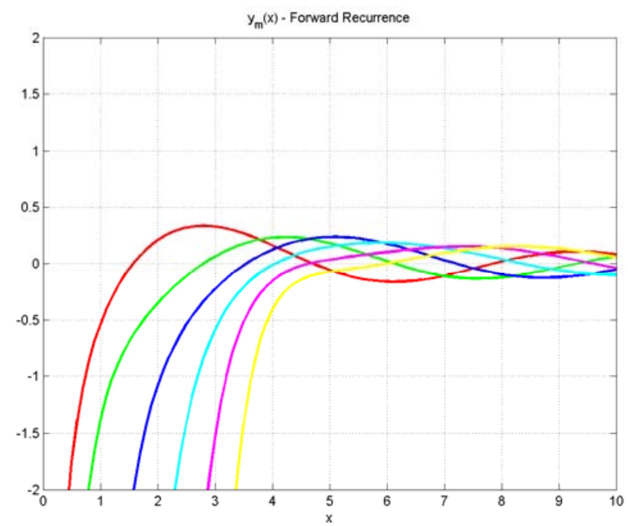
$$j_n(x)\rightarrow 0\begin{cases} n\rightarrow\infty \\ x\rightarrow 0\end{cases}$$

$$y_n(x)\rightarrow -\infty\begin{cases} n\rightarrow\infty \\ x\rightarrow 0\end{cases}$$

$j_n(x)$



$y_n(x)$



Error Propagation

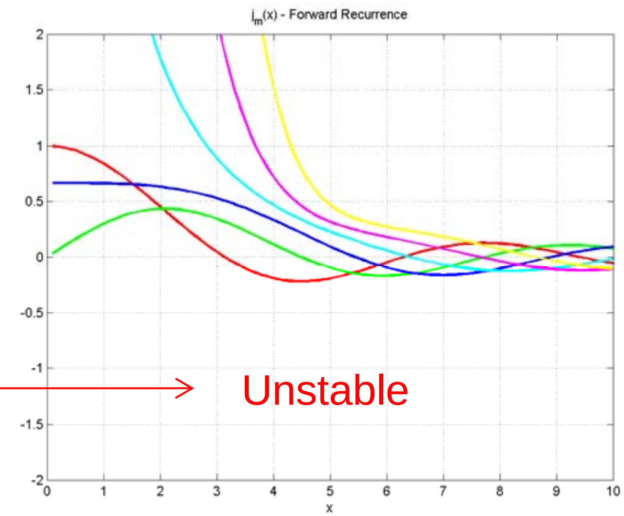
Spherical Bessel Functions

Forward Recurrence

$$j_{n+1}(x) = \frac{2n+1}{x} j_n(x) - j_{n-1}(x)$$

Forward Recurrence

$$\frac{2n+1}{x} j_n(x) \simeq j_{n-1}(x)$$



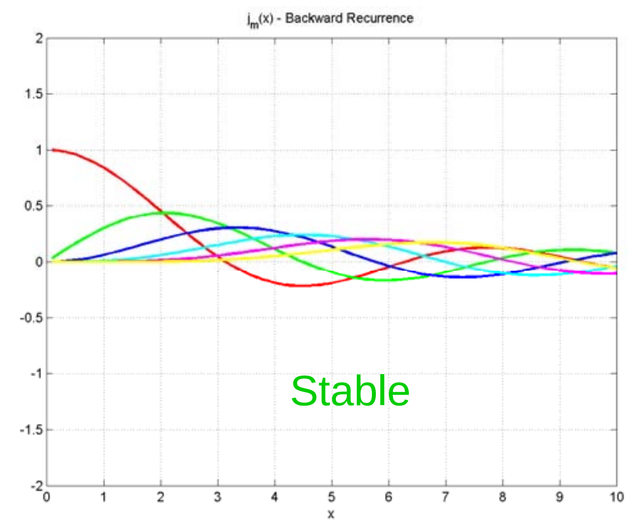
Backward Recurrence

$$j_{n-1}(x) = \frac{2n+1}{x} j_n(x) - j_{n+1}(x)$$

Miller's algorithm

$$j_N(x) = 1\ ,\ j_{N+1}(x) = 0\ ,\ j_0(x) = \frac{\sin x}{x}$$

$N \sim x+20$



Error Propagation

Euler's Method

Differential Equation

$$\frac{dy}{dx} = f(x, y) \; , \; y_0 = p$$

Example

$$f(x, y) = x \;(y = x^2/2 + p)$$

Discretization

$$x_n = nh$$

Finite Difference (forward)

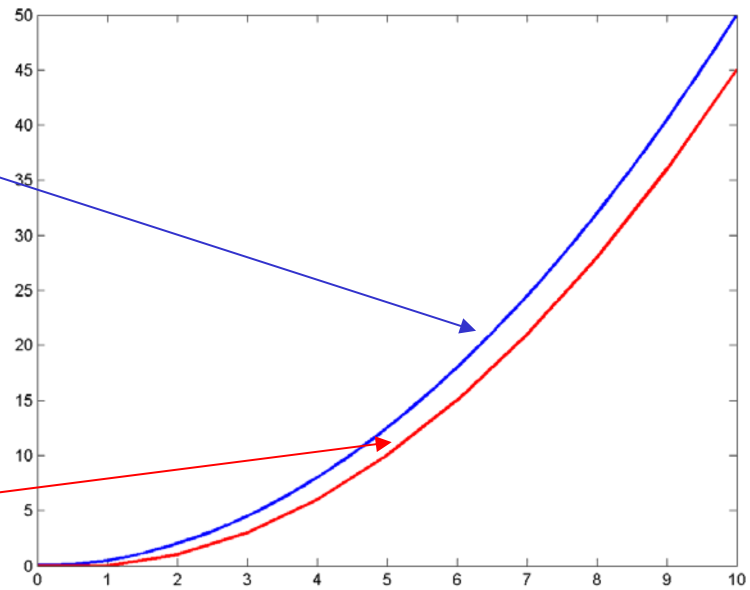
$$\frac{dy}{dx}|_{x=x_n} \simeq \frac{y_{n+1} - y_n}{h}$$

Recurrence

$$y_{n+1} = y_n + hf(nh, y)$$

Central Finite Difference

$$\frac{dy}{dx}|_{x=x_n} \simeq \frac{y_{n+1} - y_{n-1}}{2h}$$



euler.m

Error Analysis
Numerical Instability Example

Evaluate Integral

$$y_n = \int_0^1 \frac{x^n}{x+5} dx \;, n = 0, 2 \ldots \infty$$

Recurrence Relation

$$y_n = \frac{1}{n} - 5 y_{n-1}$$

Proof

$$y_n + 5 y_{n-1} = \int_0^1 \frac{x^n + 5x^{n-1}}{x+5} dx = \int_0^1 \frac{x^{n-1}(x+5)}{x+5} dx = \int_0^1 x^{n-1} dx = \frac{1}{n}$$

3-digit Recurrence

$$\begin{aligned}
y_0 &= \int_0^1 \frac{dx}{x+5} = [\log_e(x+5)]_0^1 = \log_e 6 - \log_e 5 = 0.182 \\
y1 &= 1 - 5y_0 = 1 - 0.910 \simeq 0.0090 \\
y2 &= 0.5 - 5y_1 \simeq 0.050 \\
y3 &= 0.333 - 5y_2 \simeq 0.083 \\
y4 &= 0.25 - 5y_3 \simeq -0.165
\end{aligned}$$

$y3$: > y2 !!

$y4$: < 0 !!

Backward Recurrence

$$y_{n-1} = \frac{1}{5n} - \frac{y_n}{5}$$

$$\begin{aligned}
y_{10} &\simeq y_9 \Rightarrow y_9 + 5y_9 = 0.1 \Rightarrow y_9 = 0.017 \\
y_8 &= 1/45 - y_9/5 = 0.019 \\
y_7 &= 1/40 - y_8/5 = 0.021 \\
y_6 &= 0.025 \\
&\cdot \\
&\cdot \\
&\cdot \\
y_1 &= 0.088 \\
y_0 &= 0.182
\end{aligned}$$

y_0: Correct

Exercise: Make MATLAB script

Error Propagation

$$y = f(x_1, x_2, \ldots x_n)$$

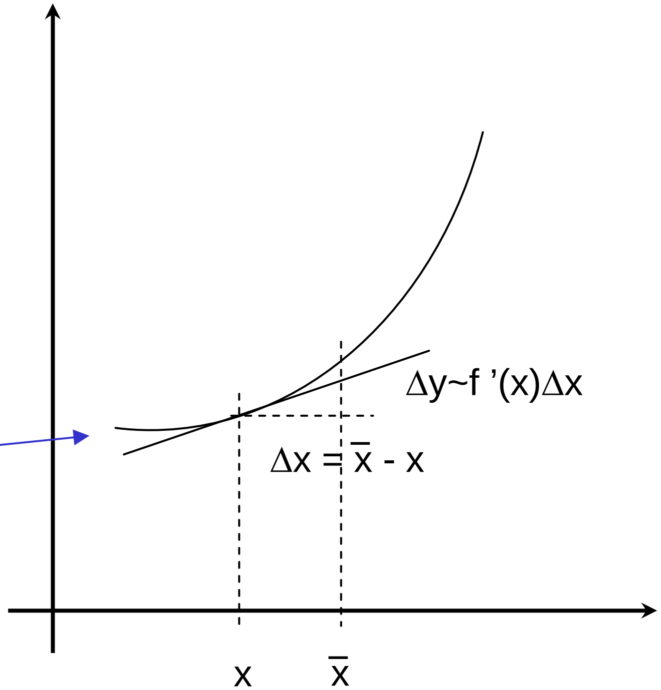
Absolute Errors

$$\epsilon_1, \epsilon_2, \ldots \epsilon_n$$

$$\epsilon_y$$

Function of one variable

$$y = f(x) \quad \bar{y} = f(\bar{x})$$



General Error Propagation Formula

$$\Delta y \simeq \sum_{i=1}^{n} \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} \Delta x_i$$

$$\epsilon_y \leq \sum_{i=1}^{n} \left| \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} \right| |\epsilon_i|$$

Error Propagation Example

Multiplication

$$
\begin{aligned}
y &= x_1 x_2 \\
=> \quad \log y &= \log x_1 + \log x_2 \\
=> \quad \frac{1}{y}\frac{\partial y}{\partial x_i} &= \frac{1}{x_i} \\
=> \quad \frac{\partial y}{\partial x_i} &= \frac{y}{x_i}
\end{aligned}
$$

=>

Error Propagation Formula

$$
\begin{aligned}
\left|\frac{\Delta y}{y}\right| &\leq \sum_{i=1}^{2}\left|\frac{\Delta x_i}{x_i}\right| \\
\alpha_y &\leq \sum_{i=1}^{2}\alpha_i
\end{aligned}
$$

$$y = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$$

$$\boxed{\alpha_y \leq \sum_{i=1}^{n}|m_i|\alpha_i}$$

Relative Errors Add for Multiplication

Error Propagation
Expectation of Errors

Addition

$$y = x_1 + x_2 + \cdots + x_n$$

Truncation

$$\Delta x_i = \bar{x}_i - x_i \leq 0$$

Error Expectation

$$\begin{aligned} E(\Delta x_i) &= -b^{-t}/2 \\ E(\Delta y) &= -nb^{-t}/2 \end{aligned}$$

Rounding

$$\begin{aligned} E(\Delta x_i) &= 0 \\ E(\Delta y) &= -nE(\Delta x_i) = 0 \end{aligned}$$

Standard Error

$$E(\Delta_s y) \simeq \sqrt{\sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}\right)^2 \epsilon_i^2}$$

$$y = x_1 + x_2 + \cdots + x_n$$

$$E(\Delta_s y) \simeq \sqrt{\sum_{i=1}^{n} \epsilon_i^2} = \sqrt{n}\epsilon$$

Standard Error better measure of expected errors

Error Propagation
Error Cancellation

Function of one variable

$$y = f(x) = \sqrt{x^2+1} - x + 200; \quad x = 100 \pm 4$$

$$\begin{aligned} z_1 &= \sqrt{x^2+1} \quad \epsilon_1 = 4 \\ z_2 &= 200 - x \quad \epsilon_2 = 4 \\ y &= z_1 + z_2 \end{aligned}$$

Max. error $E(\Delta y) \; = \; 8$

Stand. error $E(\Delta_s y) \; = \; 4\sqrt{2} = 5.6$

$$\frac{\partial z_1}{\partial x} = \frac{x}{\sqrt{x^2+1}}, \quad \frac{\partial z_2}{\partial x} = -1$$

$$\Delta y \simeq \frac{dz_1}{dx}\Delta x + \frac{dz_2}{dx}\Delta x$$

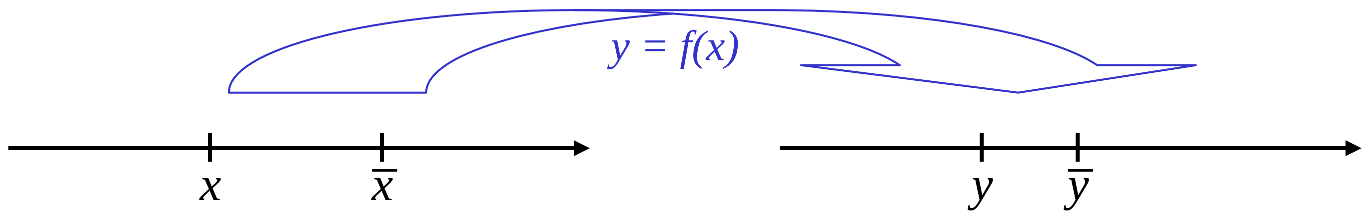
$$= \left(\frac{x}{\sqrt{x^2+1}} - 1 \right) \Delta x \simeq_{x \gg 1} \frac{-1}{x^2}\Delta x$$

Error cancellation

$$x = 100 \pm 4 \Rightarrow |\Delta y| \leq 4\, 10^{-4} \leq 0.5\, 10^{-3}$$

$$y = 200.005 \pm 0.5\, 10^{-3}$$

Error Propagation
Condition Number



$\bar{x} = x(1 + \alpha)$ $\quad\quad$ $\bar{y} = y(1 + \beta)$

Problem Condition Number

$$
\begin{aligned}
K_P &= \frac{|\beta|}{|\alpha|} \\
&= \left| \frac{f(\bar{x}) - f(x)}{f(x)} \right| / \left| \frac{\bar{x} - x}{x} \right| \\
&= \left| \frac{f(\bar{x}) - f(x)}{\bar{x} - x} \right| \times \left| \frac{x}{f(x)} \right| \\
&\simeq \left| x \frac{f'(x)}{f(x)} \right|
\end{aligned}
$$

$$K_P \gg 1$$

Problem ill-conditioned

Error cancellation example

$$y = f(x) \;=\; \sqrt{x^2+1} - x + 200; \quad x = 100 \pm 4$$

$$K_P \;=\; \left| 100 \, \frac{-10^{-4}}{200.005} \right| = 0.5\ 10^{-4}$$

Well-conditioned problem

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Error Propagation
Condition Number

Problem Condition Number

$$y = \sqrt{x^2+1} - x$$

$$x = 100 \Rightarrow y = 0.5 \cdot 10^{-2}$$

$$y' = \frac{x}{\sqrt{x^2+1}} - 1 \simeq -\frac{1}{x^2} = -10^{-4}$$

$$K_P = \left|100 \frac{-1 \cdot 10^{-4}}{0.5 \cdot 10^{-4}}\right| = 2.0$$

4 Significant Digits

$$\bar{y} = \sqrt{0.1 \cdot 10^5 + 1} - 0.1 \cdot 10^3$$

$$= \sqrt{(0.1000 + 0.00001) \cdot 10^5} - 0.1 \cdot 10^3 = 0$$

$$|\beta| = \left|\frac{\bar{y} - y}{y}\right| = \frac{0.5 \cdot 10^{-2}}{0.5 \cdot 10^{-2}} = 1$$

$$|\alpha| = \left|\frac{\bar{x} - x}{x}\right| \leq \frac{1}{2} 10^{1-t} \simeq \frac{1}{2} 10^{-3}$$

Algorithm Condition Number

$$K_A = \frac{|\beta|}{|\alpha|} \simeq 2000$$

K_A is algorith condition number, which may be much larger than the K_P due to limited number representation.

Solution

- Higher precision
- Rewrite algorithm

Well-conditioned Algorithm

$$y = \frac{1}{\sqrt{x^2+1} + x}$$

$$\bar{y} = \frac{1}{0.1 \cdot 10^3 + 0.1 \cdot 10^3} = 0.5 \cdot 10^{-}2$$

$$|\beta| \simeq 0 \Rightarrow K_A \simeq 0 \ll 1$$