15 SP 2003

13.024 - Numerical Methods in Incompressible Fluid Mechanics

Lecture Notes – Version 3

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INCOMPRESSIBLE FLUID MECHANICS BACKGROUND

$$\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w$$

Conservation of Mass, Continuity Equation

$$\operatorname{div}\vec{V} = \nabla \cdot \vec{V} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Newtonian Dynamics, Navier-Stokes Equations

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \vec{V}$$

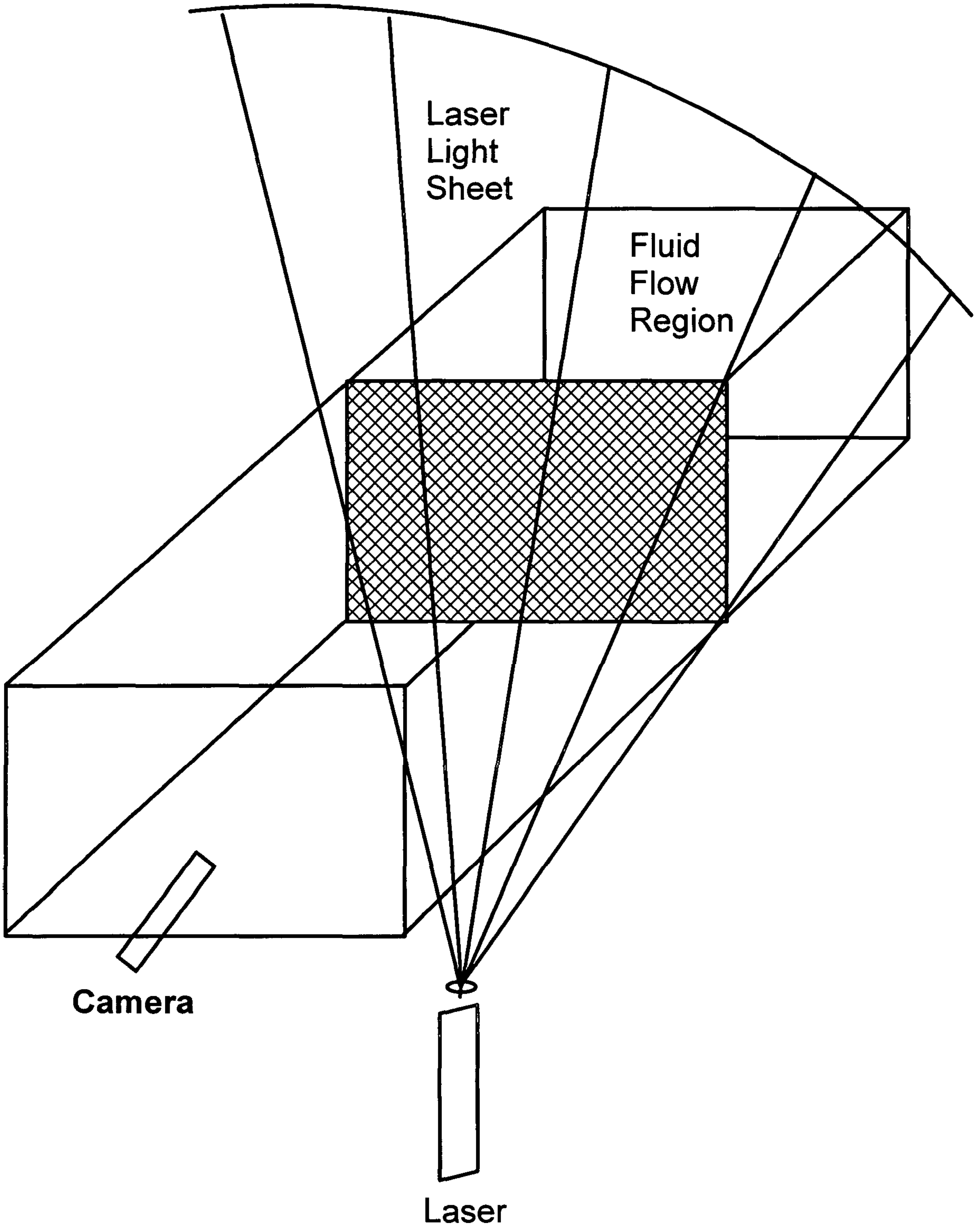
P is the dynamic pressure. The total pressure, P_T, is the sum of the dynamic pressure and the hydrostatic pressure, $-\rho\, g\, z$, where z is positive upwards. $P_T = P - \rho\, g\, z$.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

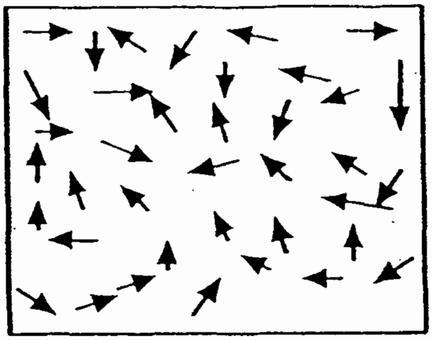
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

PARTICLE IMAGE VELOCIMETRY



PIV Example



u and v can be measured, so $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are known.

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

If PIV is done on multiple planes inside a fluid domain, then $\frac{\partial w}{\partial z}$ is known over the whole domain. At a rigid boundary, $w = 0$ and, in principle, w can be found anywhere by:

$$w = \int_{boundary}^{z} \frac{\partial w}{\partial z} dz$$

Example

In a domain bounded by $0 \le z \le 4$, $u = (3e^z - ze^z - 3)\sin x$ and $v = 0$ over a range of x and in $0 \le z \le 2$. In this sub-domain,

$$\frac{\partial u}{\partial x} = (3e^z - ze^z - 3)\cos(x)$$

$$\frac{\partial w}{\partial z} = -(3e^z - ze^z - 3)\cos(x)$$

$$w = \int_0^z \frac{\partial w}{\partial z} dz = -\left[3e^z - 3 - ze^z + e^z - 1 - 3z\right]\cos(x)$$

$$w = \int_0^z \frac{\partial w}{\partial z} dz = -\left[4e^z - 4 - ze^z - 3z\right]\cos(x)$$

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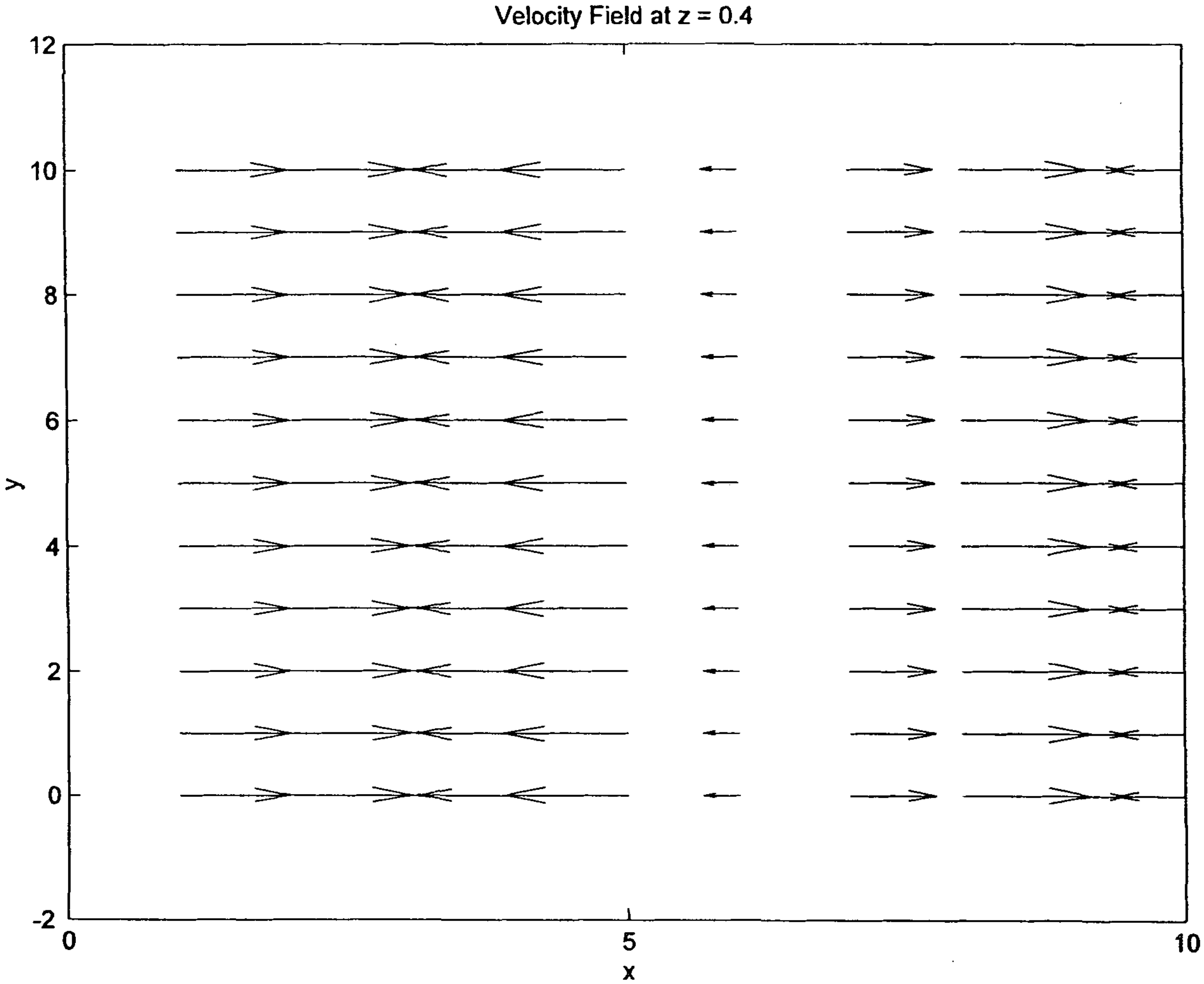

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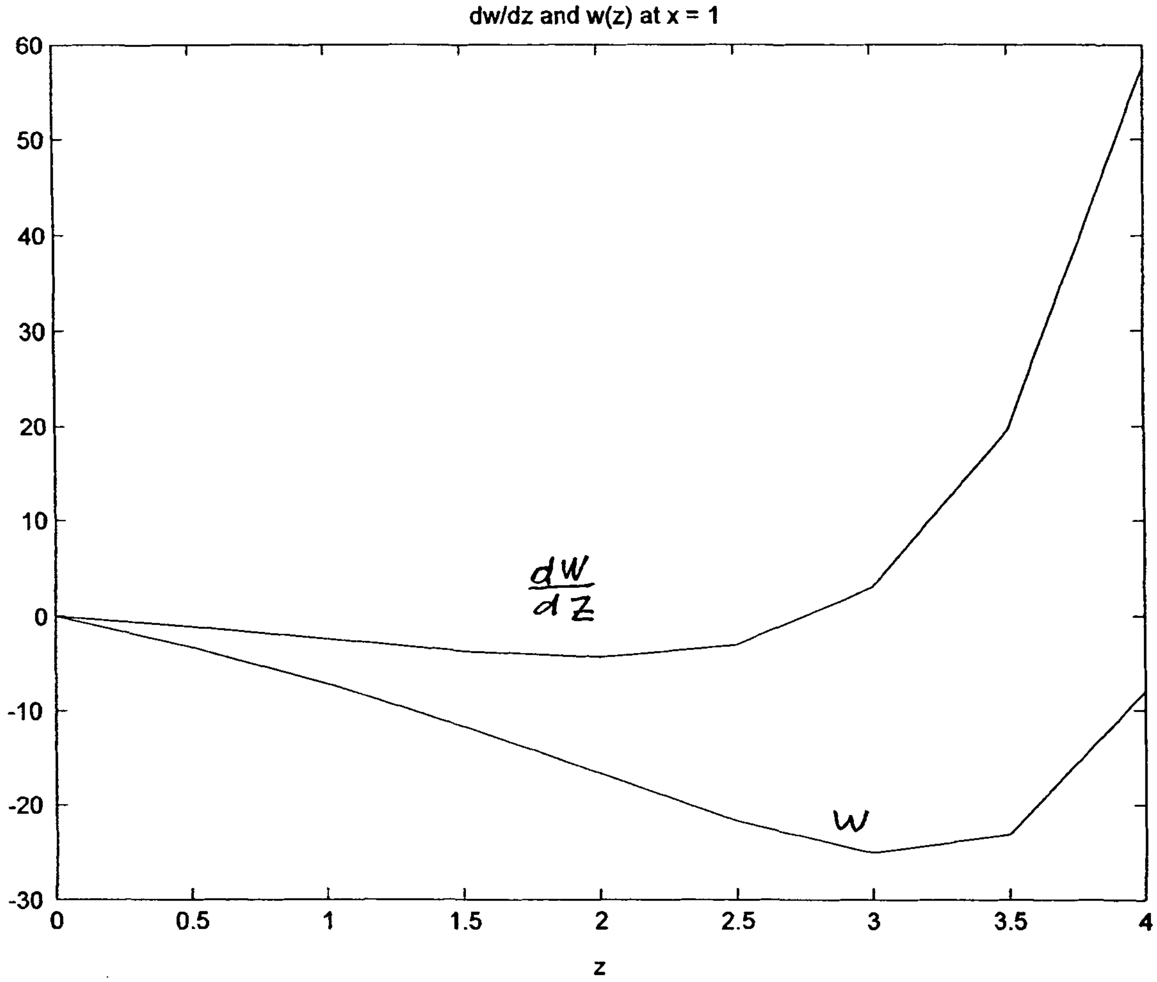

dw/dz and w(z) at x = 1
60
50
40
30
20
10
0
-10
-20
-30
0
0.5
1
1.5
2
2.5
3
3.5
4
z
dW/dZ
W

A More Interesting PIV Example

Consider the following flow for $z > 0$ in a range of x and y. Of course, in an experiment you would not know the mathematical formulation. Rather you would just measure u and v over a set of (x,y,z) points.

$$u(x,y,z) = \left(3e^{0.1z} - 3\cos\, z\right) \sin y \cos x$$

$$v(x,y,z) = \left(3e^{0.1z} - 3\cos\, z\right) \cos y$$

The x and y derivatives of the velocities can be computed numerically from the measurements. If the experiment were done well, they would have values according to the following formulae:

$$\frac{\partial u}{\partial x} = -\left(3e^{0.1z} - 3\cos\, z\right) \sin y \sin x$$

$$\frac{\partial v}{\partial y} = -\left(3e^{0.1z} - 3\cos\, z\right) \sin y$$

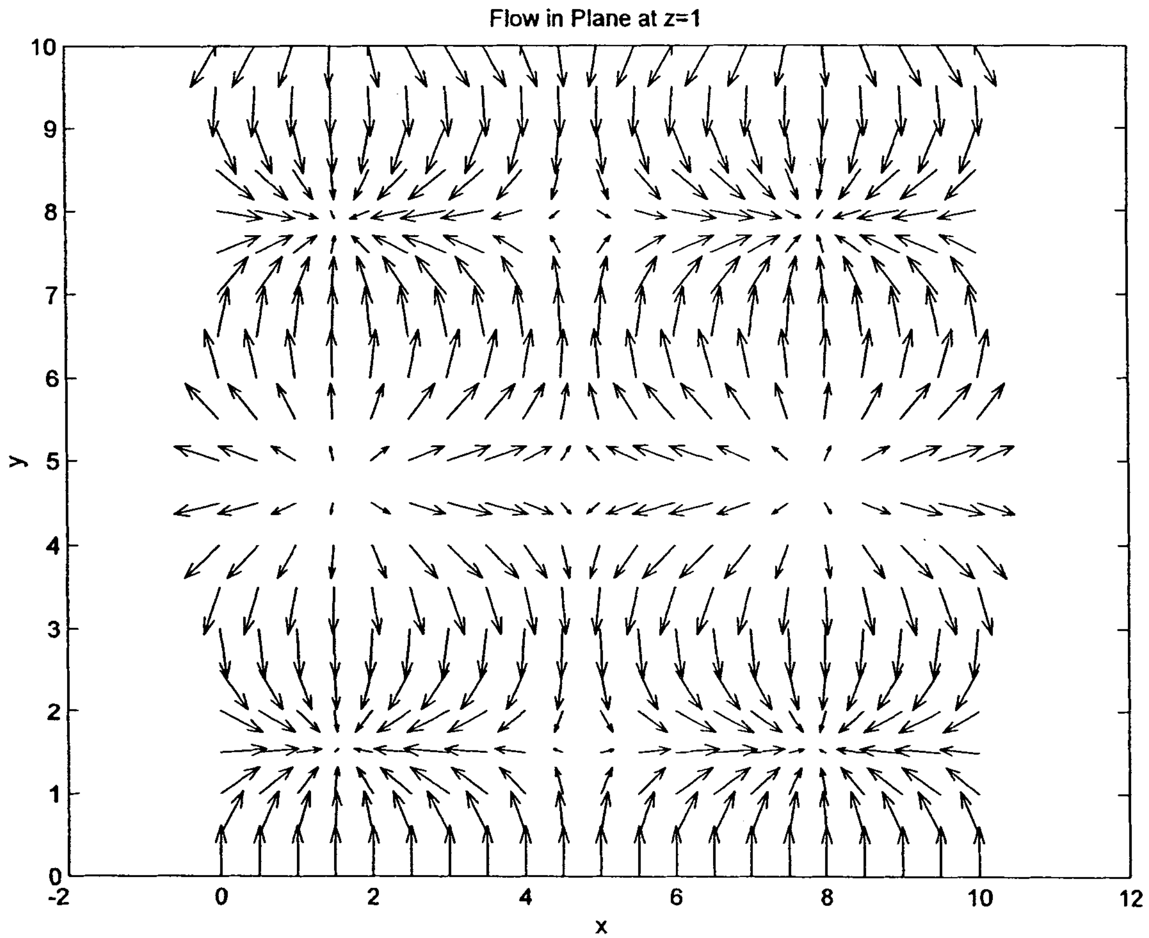
Then, the continuity equation is: $\dfrac{\partial w}{\partial z} = -\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y}$

The experimentally determined values would have the values given by:

$$\frac{\partial w}{\partial z} = \left(3e^{0.1z} - 3\cos\, z\right)\, \sin\, y(\sin\, x + 1)$$

Integrating $\frac{\partial w}{\partial z}$ from 0 to z at a prescribed value of (x,y) would give w(z) there. The values obtained would obey:

$$w = \left(30\, e^{0.1z} - 30 - 3\, \sin\, z\right)\, \sin\, y\, (\sin\, x + 1)$$


Flow in Plane at z=1
10
9
8
7
6
5
4
3
2
1
0
y
-2
0
2
4
6
8
10
12
x

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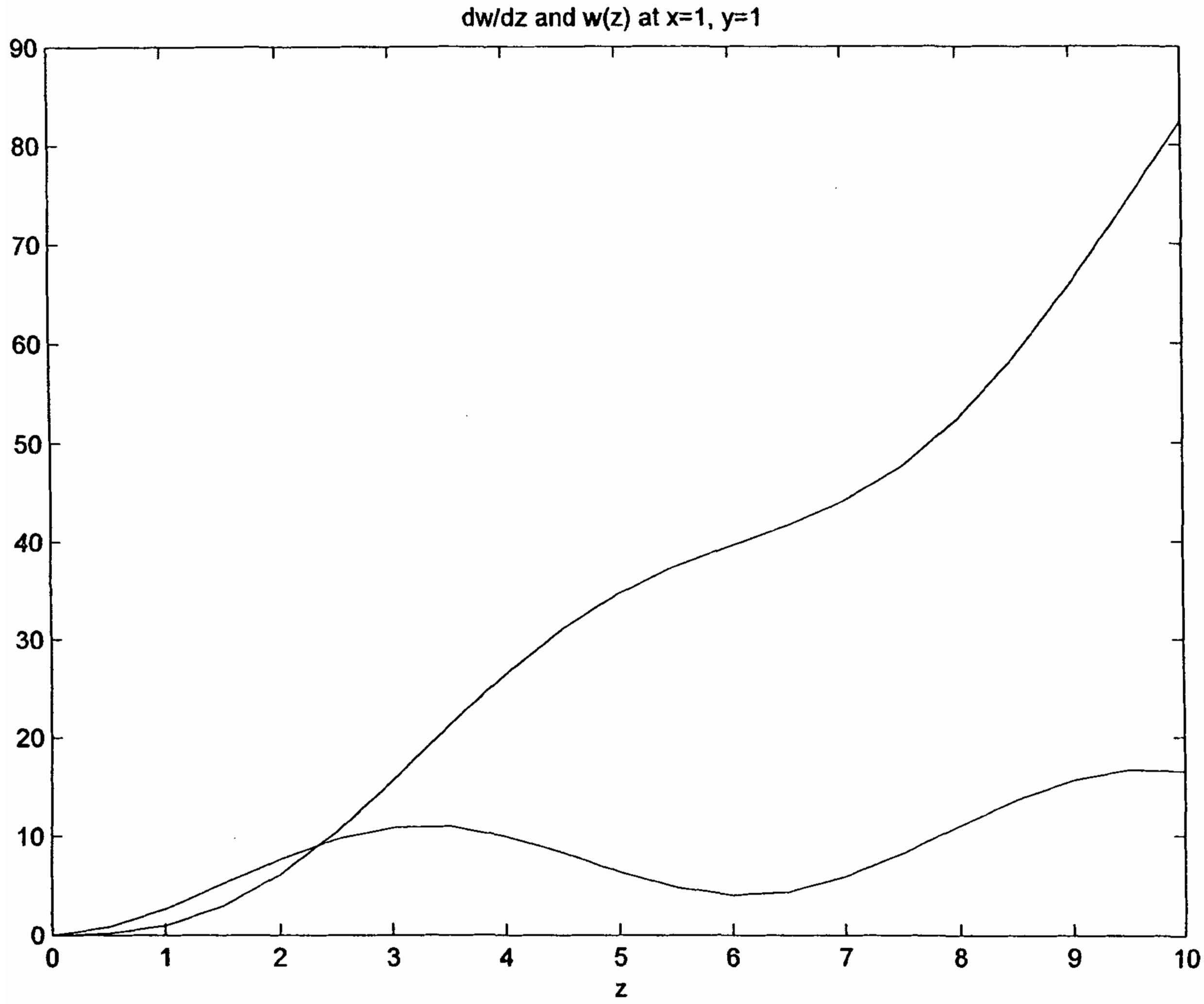

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AVERAGED NAVIER-STOKES EQUATIONS

$$\vec{V} = \overline{\vec{V}} + \vec{v'}$$

$$\frac{\partial\left(\overline{\vec{V}} + \vec{v'}\right)}{\partial t} + \left[\left(\overline{\vec{V}} + \vec{v'}\right)\cdot\nabla\right]\left(\overline{\vec{V}} + \vec{v'}\right) = -\frac{1}{\rho}\nabla\left(\overline{P} + p'\right) + \nu\nabla^2\left(\overline{\vec{V}} + \vec{v'}\right)$$

Take Average of above equation:

$$\frac{\partial\overline{\vec{V}}}{\partial t} + \overline{\left[\left(\overline{\vec{V}} + \vec{v'}\right)\cdot\nabla\right]\left(\overline{\vec{V}} + \vec{v'}\right)} = -\frac{1}{\rho}\nabla\overline{P} + \nu\nabla^2\overline{\vec{V}}$$

$$\left[\left(\overline{\vec{V}} + \vec{v'}\right)\cdot\nabla\right]\left(\overline{\vec{V}} + \vec{v'}\right) = (\overline{\vec{V}}\cdot\nabla)\overline{\vec{V}} + (\overline{\vec{V}}\cdot\nabla)\vec{v'} + (\vec{v'}\cdot\nabla)\overline{\vec{V}} + (\vec{v'}\cdot\nabla)\vec{v'}$$

$$\overline{\left[\left(\overline{\vec{V}} + \vec{v'}\right)\cdot\nabla\right]\left(\overline{\vec{V}} + \vec{v'}\right)} = (\overline{\vec{V}}\cdot\nabla)\overline{\vec{V}} + \overline{(\vec{v'}\cdot\nabla)\vec{v'}}$$

Thus, the Reynolds-Averaged Equation is:

$$\frac{\partial\overline{\vec{V}}}{\partial t} + (\overline{\vec{V}}\cdot\nabla)\overline{\vec{V}} + \overline{(\vec{v'}\cdot\nabla)\vec{v'}} = -\frac{1}{\rho}\nabla\overline{P} + \nu\nabla^2\overline{\vec{V}}$$

The "Reynolds Stress Term" is:

$$\begin{aligned}\overline{(\vec{v'}\cdot\nabla)\vec{v'}} \;=\; & \vec{i}\overline{\left(u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z}\right)} + \\ & \vec{j}\overline{\left(u'\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y} + w'\frac{\partial v'}{\partial z}\right)} + \\ & \vec{k}\overline{\left(u'\frac{\partial w'}{\partial x} + v'\frac{\partial w'}{\partial y} + w'\frac{\partial w'}{\partial z}\right)}\end{aligned}$$

EmB 3

THE PRESSURE EQUATION FOR AN INCOMPRESSIBLE FLUID

Start with the Navier-Stokes Equation

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \vec{V}$$

Take its divergence.

Because $\nabla \cdot \vec{V} = 0$, the only non-zero terms are:

$$\mathrm{div}\,(\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla^2 P$$

Working out the details of the LHS and interchanging the LHS and the RHS results in:

$$-\frac{1}{\rho}\nabla^2 P = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + 2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + 2\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}$$

$$\nabla^2 P = -\rho\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + 2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + 2\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}\right\}$$

The pressure, P, satisfies Poisson's Equation driven by products of the spatial derivatives of the velocity. This is different than the common Bernoulli Equations because here the flow can be unsteady and rotational.

EMB 4

The Vorticity equation

$$\text{vorticity} = \varpi \equiv \text{curl}\,\vec{V} \equiv \nabla \times \vec{V}$$

Start with the Navier Stokes equation:

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \vec{V}$$

Take the curl of this equation, term by term:

$$\frac{\partial \varpi}{\partial t} + (\vec{V} \cdot \nabla)\varpi + (\varpi \cdot \nabla)\vec{V} = \nu \nabla^2 \varpi$$

$$\frac{D\varpi}{\partial t} = -(\varpi \cdot \nabla)\vec{V} + \nu \nabla^2 \varpi$$

The first term on the right hand side is the rotation and stretching of the vorticity by the non-uniform velocity field.

FMb 5

Inviscid Fluid Mechanics, Euler's Equation

Set the viscosity, μ and the kinematic viscosity, ν to zero. Apply these "settings" to the Navier Stokes Equation.

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla P$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z}$$

Bernoulli Theorems for Inviscid Flow

Theorem 1 - Irrotational Flow

Vector Identity

$$(\vec{V}\cdot\nabla)\vec{V} = \frac{1}{2}\nabla(|\vec{V}|^2) - \vec{V}\times(\nabla\times\vec{V}) = \frac{1}{2}\nabla(|\vec{V}|^2) - \vec{V}\times\varpi$$

$$\text{Let} \quad H = \frac{1}{2}(|\vec{V}|)^2 + \frac{P}{\rho}$$

$$\text{For Irrotational Flow,} \quad \vec{V} = \nabla\phi \text{ and } \nabla\times\vec{V} = 0$$

$$\frac{\partial}{\partial t}\nabla\phi + \nabla H \equiv \nabla(\frac{\partial\phi}{\partial t} + H) = 0$$

$$\frac{\partial\phi}{\partial t} + H = f(t)$$

The function $f(t)$ can be absorbed into ϕ by letting $\phi = \phi' + \int_{t_o}^{t} f(t)dt$ and $\vec{V} = \nabla\phi'$.

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi'}{\partial t} + f(t) \quad \text{so,} \quad \frac{\partial\phi'}{\partial t} + H = 0$$

Finally rename ϕ' as ϕ.

$$\frac{\partial\phi}{\partial t} + H = 0$$

Remember that the total pressure, $P_T = P - \rho g z$.

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\vec{V})^2 + \frac{P_T}{\rho} + gz = 0$$

Theorem 2 - Steady Flow

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla P$$

$$\frac{\partial}{\partial t}\nabla\phi + \frac{1}{2}\nabla\left(|\vec{V}|^2\right) - \vec{V} \times \varpi + \frac{1}{\rho}\nabla P = 0$$

$$\frac{\partial}{\partial t}\nabla\phi + \nabla H - \vec{V} \times \varpi = 0$$

Thus, for steady flow:

$$\vec{V} \times \varpi = \nabla H$$

Streamlines and vortex lines are perpendicular to ∇H.
Along either a streamline or a vortex line, H is a constant. So on any one of these lines,

$$H = \frac{1}{2}(\vec{V})^2 + \frac{P}{\rho} = \frac{1}{2}(\vec{V})^2 + \frac{P_T}{\rho} + gz = \text{constant}$$

If the flow is both steady and irrotational, H is the same everywhere because $\nabla H = 0$.

Vorticity Dynamics and Kelvin's Circulation Theorem

$$\text{Circulation} = \Gamma = \oint \vec{V} \cdot d\mathbf{r} = \int_{\mathbf{S}} \varpi \cdot \mathbf{ds}$$

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\vec{V}$$

Identity for an incompressible fluid: $\nabla^2\vec{V} = -\nabla \times \varpi$

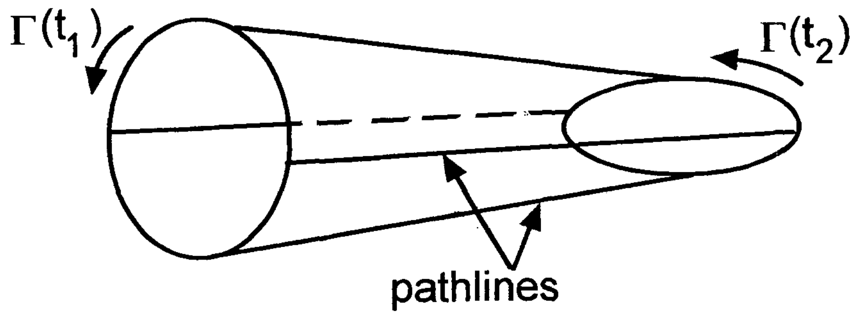
$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla P - \nu\nabla \times \varpi$$

Following a closed material curve,

$$\begin{aligned}
\frac{d\Gamma}{dt} &= \oint \frac{D\vec{V}}{Dt} \cdot d\mathbf{r} = -\oint \frac{1}{\rho}\nabla \mathbf{P} \cdot \mathbf{dr} - \nu \oint (\nabla \times \varpi) \cdot \mathbf{dr} \\
&= -\nu \oint (\nabla \times \varpi) \cdot d\mathbf{r}
\end{aligned}$$

<u>Kelvin's Circulation Theorem</u>

In an inviscid fluid, $\dfrac{d\Gamma}{dt} = 0$



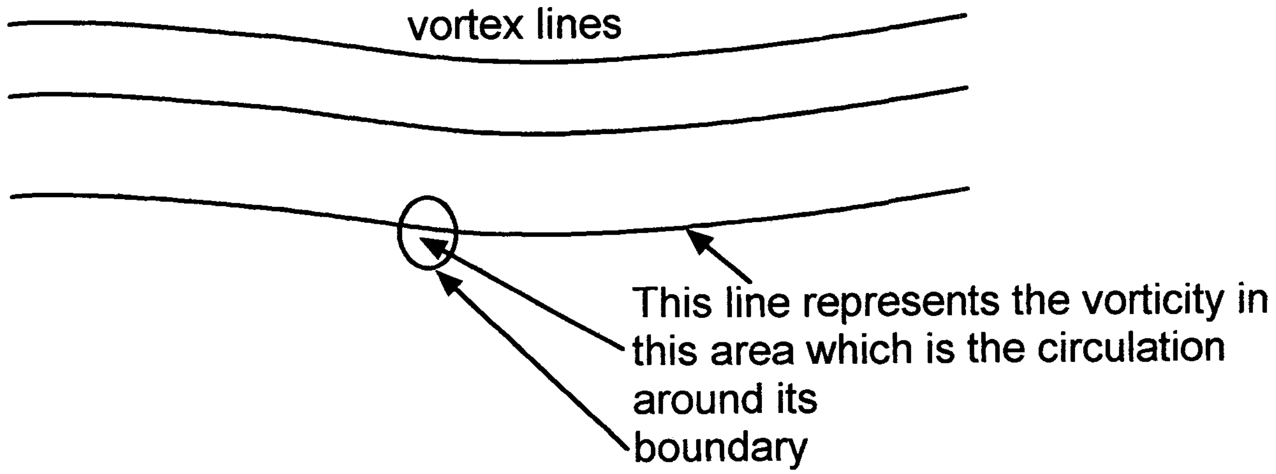
Corollary: In an inviscid fluid with no circulation (such as starting from rest) the circulation remains zero.

Practical Implication

In a high Reynolds number streaming flow, fluid which has not passed close to a boundary or a free surface has negligible vorticity.

Therefore, in high Reynolds number streaming flows, vorticity is limited to boundary layers, separated zones, and wakes.

Concept of Vortex Lines

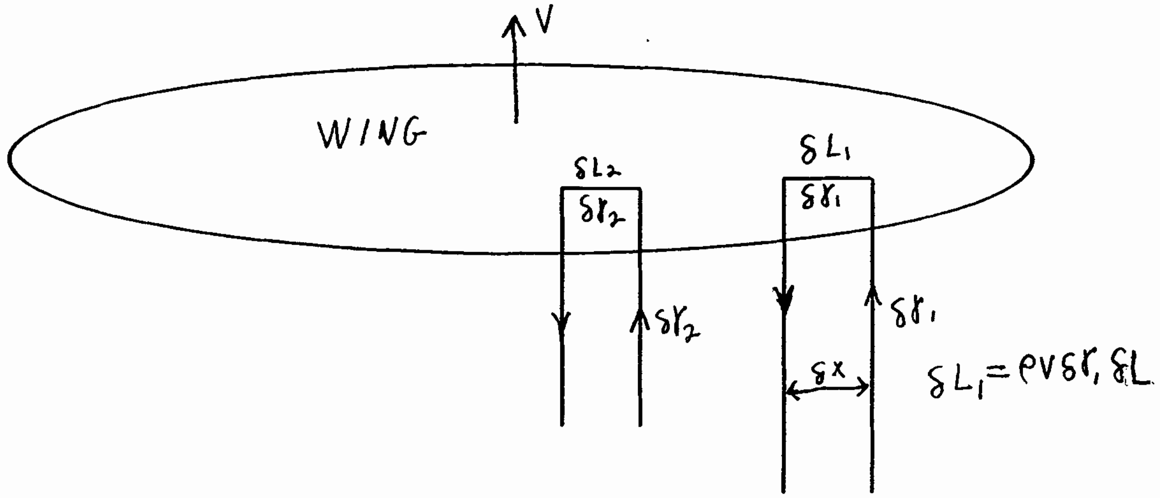


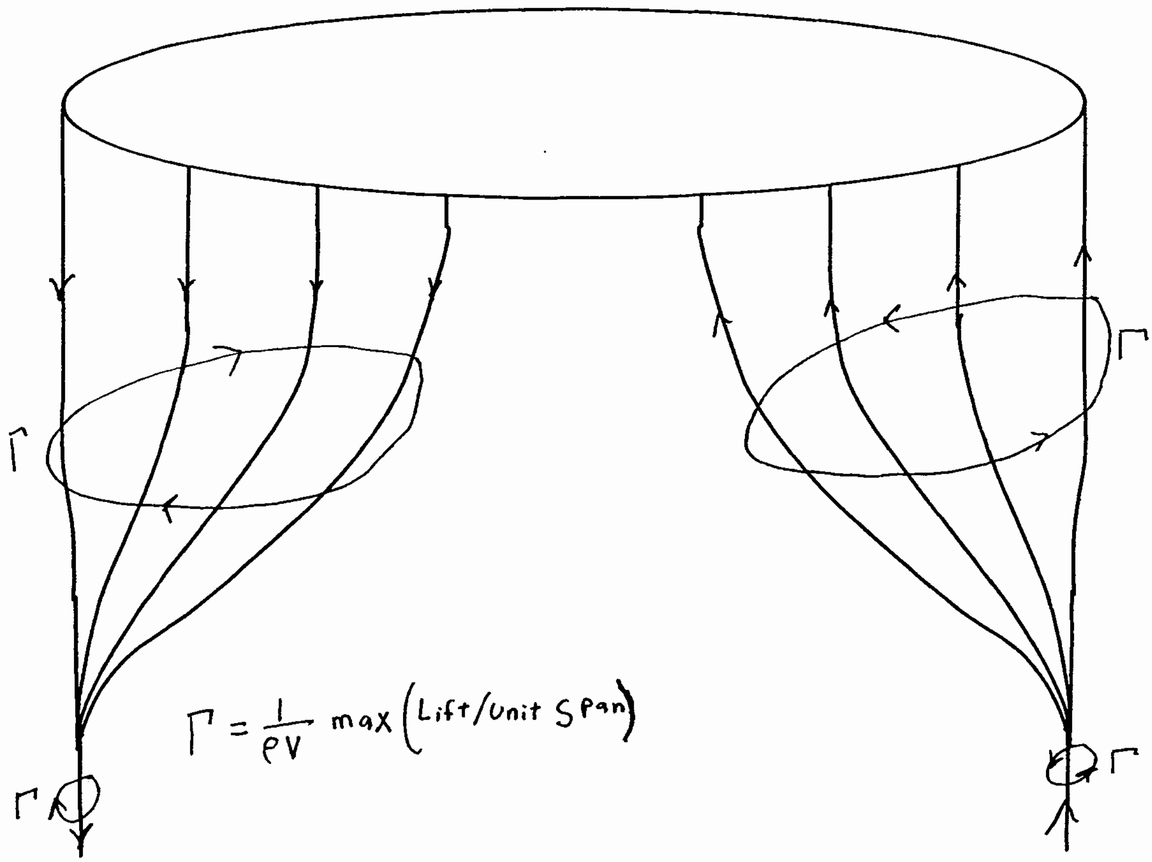
Each line represents circulation around an area which is the same as the vorticity inside the area.

$$\varpi = \mathrm{curl}\,\vec{V} = \nabla \times \vec{V}$$

$$\text{Therefore:}\quad \mathrm{div}\,\varpi = \nabla \cdot \varpi = 0$$

The vortex field is solenoidal. Vortex lines are continuous. They can have curves and turns,but they cannot have ends in the fluid.


V
WING
δL₂
δγ₂
δγ₂
δL₁
δγ₁
δγ₁
δx
δL₁ = ρVδγ₁ δL


Γ
Γ
Γ
Γ
Γ = 1/ρV max (Lift/Unit Span)

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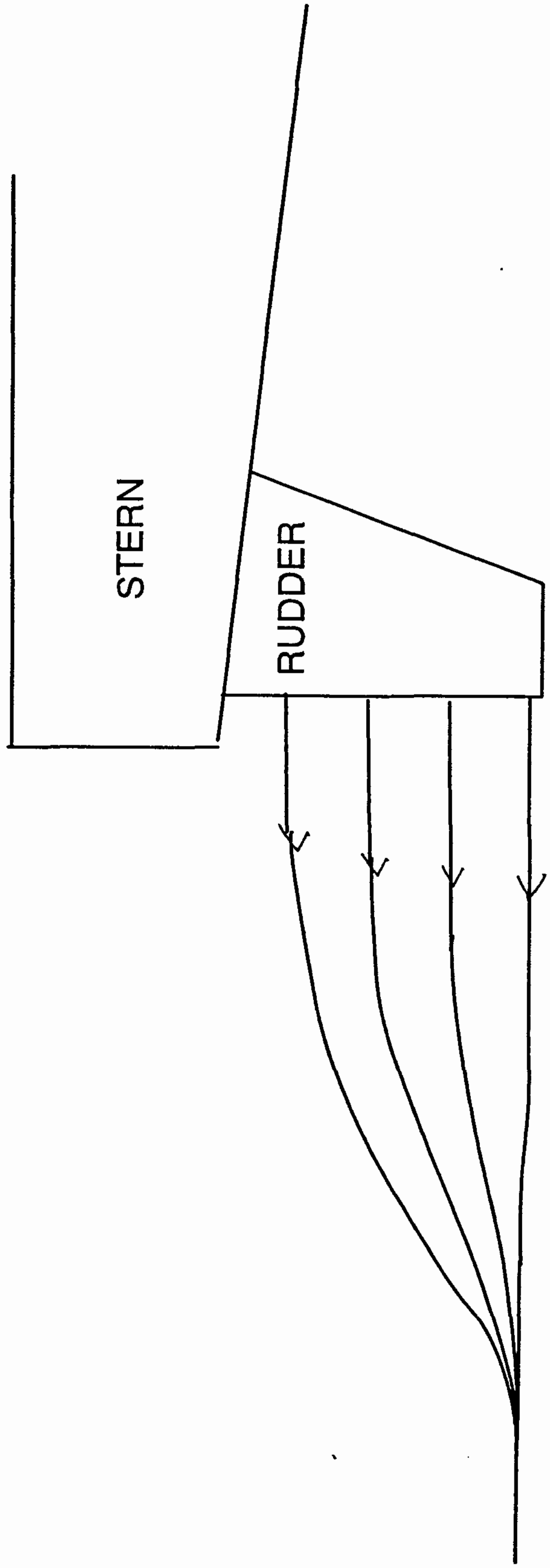


Potential Flows and Mostly Potential Flows

For an irrotational fluid $\nabla \times \vec{V} = 0$

This means that there exists a *velocity potential*, ϕ, such that,

$$\vec{V} = \nabla\phi$$

For an incompressible fluid $\nabla \cdot \vec{V} = 0$

Thus, $\nabla \cdot (\nabla\phi) = 0$

$$\nabla^2\phi = 0$$

For a completely potential flow, the velocity potential satisfies Laplace's equation.

For an incompressible flow that is "nearly" irrotational except in boundary layers and wakes, the flow outside these boundary layers and wakes is approximately described by a velocity potential that satisfies Laplace's equation.

Green Functions, Green's Theorem and Boundary Integral Equations

The following development is for three-dimensional flows. The development is similar for two-dimensional flows except that two dimensional source functions are involved and the dimensionality of some integrals and associated constants are different.

Green's Theorem

If ϕ and ψ both satisfy Laplace's equation ($\nabla^2\phi = 0,\ \ \nabla^2\psi = 0$), then:

$$\int\int_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] dS = \int\int\int_V \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right] dV = 0$$

Green Functions A three-dimensional (ξ, η, ζ) space is considered with a "sink" at location (x, y, z). The "sink" has the velocity potential ψ_s,

$$\psi_s = \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} = \frac{1}{r}$$

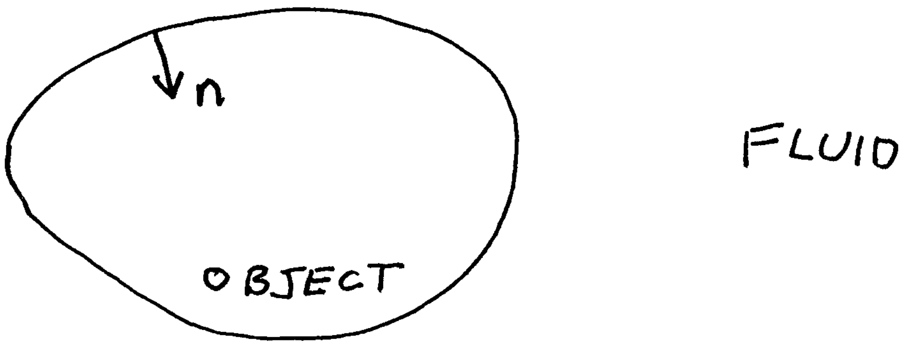
$$\text{where: } r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

Importantly, $\nabla^2\psi_s = 0$ both for the differentiations done in (ξ, η, ζ) space as well as for the differentiations done in (x, y, z) space. The following development can be formulated either way and we will choose to differentiate over (ξ, η, ζ).

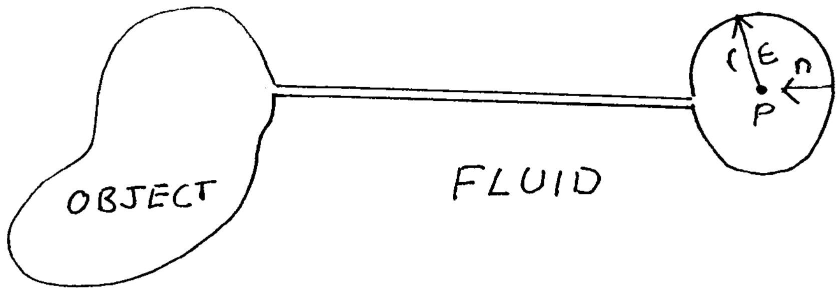
A Green Function, G is:

$$G = \psi_s(x, y, z, \xi, \eta, \zeta) + \psi_r(x, y, z, \xi, \eta, \zeta)$$

where, $\nabla^2\phi_r = 0$ in the fluid domain, and (x, y, z) is called the point P.



If P is outside the fluid domain, the bracketed terms on the right hand side in Green's Theorem are zero. However, if P is inside the fluid domain, $\nabla^2\psi \neq 0$ at P. Then, if P is enclosed by a small sphere of radius ϵ, which is excluded from the fluid domain, Green's Theorem applies in the modified fluid domain. However, now the integrals include an integral about the small sphere.

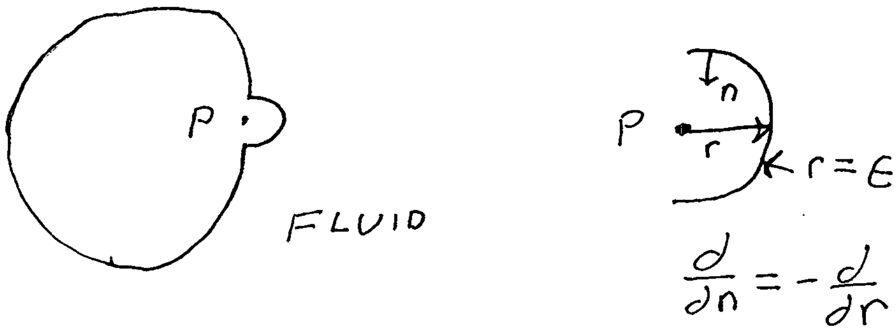


$$\int\int_{S+sphere}\left[\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds_{\xi,\eta,\zeta} = 0$$

$$\int\int_{sphere}\phi\frac{\partial G}{\partial n}ds = -\phi[P(x,y,z)]\frac{-1}{\epsilon^2}4\pi\epsilon = 4\pi\phi[P(x,y,x)]$$

$$\int\int_{S}\left[\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds_{\xi,\eta,\zeta} = -4\pi\phi[P(x,y,z)]$$

If $P(x,y,z)$ is on the boundary, the integral is not defined. However, if we replace the real boundary by one which has an infinitesmal hemisphere syrrounding P, the Green Function integral is zero because the functions have no singularities in the revised fluid domain.



$$\int\int_{S+hemisphere}\left[\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds_{\xi,\eta,\zeta} = 0$$

$$\int\int_{hemisphere}\phi\frac{\partial G}{\partial n}ds = -\phi[P(x,y,z)]\frac{-1}{\epsilon^2}2\pi\epsilon = 2\pi\phi[P(x,y,x)]$$

$$\int\int_{S}\left[\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds_{\xi,\eta,\zeta} = -4\pi\phi[P(x,y,z)]$$

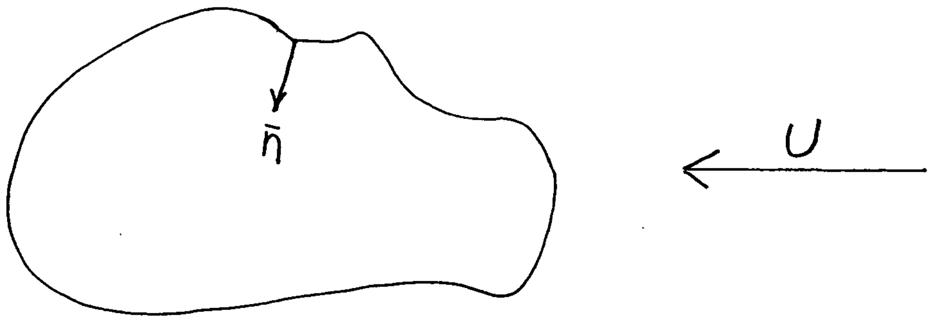
Putting the preceding parts together, if a closed fluid domain of surface S is considered with $\vec{n}$ being the outward normal vector (out of the fluid) and ψ is taken as G with proper exclusion of the singular point of G when (x, y, z) is inside the domain or on its boundary,

$$\int\int_S \left[\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] dS = \begin{cases} 0 & (x, y, z) \text{ outside } S \\ -2\pi\phi(x, y, x) & (x, y, z) \text{ on } S \\ -4\pi\phi(x, y, x) & (x, y, z) \text{ inside } S \end{cases}$$

The integral is over the closed area in (ξ, η, ζ). When the singular point is on the surface, an infinitesimally small circle surrounding the singular point is excluded from the integral.

Example of method of solution

Generate integral equation on surface of an object in a uniform flow.



Suppose uniform flow onto an object is known

$\frac{\partial \phi}{\partial n}$ is known.

$\Phi = -Ux + \phi$

Boundary condition: $\frac{\partial \Phi}{\partial n} = 0$,

$-U\,\hat{\imath} \cdot \vec{n} + \frac{\partial \phi}{\partial n} = 0, \quad \frac{\partial \phi}{\partial n} = U\,\hat{\imath} \cdot \vec{n}$

$$\int\int_S \left[\phi \frac{\partial G}{\partial n} - G\,U\,\hat{\imath} \cdot \vec{n}\right] dS = -2\pi\phi$$

$$\int\int_S \phi \frac{\partial G}{\partial n} dS + 2\pi\phi = \int\int_S G\,U\,\hat{\imath} \cdot \vec{n} dS$$

Right hand side is known in integral equation for ϕ on boundary.

Solve for values of ϕ on boundary (panel methods).

Then ϕ and $\frac{\partial \phi}{\partial n}$ are known on boundary.

Green's Theorem then gives ϕ in all space.

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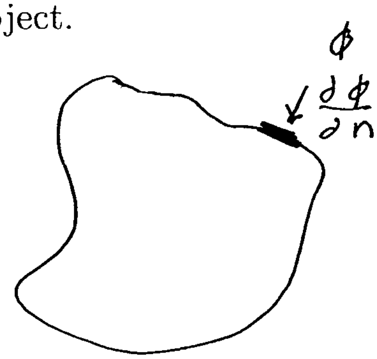
Interpretation of Boundary Integral Equation in terms of source and Dipole Layers

$$\int\int_S \left[\frac{1}{4\pi}\frac{\partial\phi}{\partial n}G - \frac{1}{4\pi}\phi\,\frac{\partial G}{\partial n}\right] dS = \begin{cases} 0 & (x,y,z) \text{ outside } S \\ \phi(x,y,x) & (x,y,z) \text{ inside } S \end{cases}$$

"Inside S" means inside the fluid and outside "S'' means outside the fluid. G is the potential of a "unit sink" and -$\partial g/\partial n$ is the potential of a unit dipole.

The Green's Theorem integrals are integrals of sink distributions per unit area of $(1/4\pi)\partial\phi/\partial n$ over the object and of dipole distributions of strength per unit area of $\phi/4\pi$ over the object.

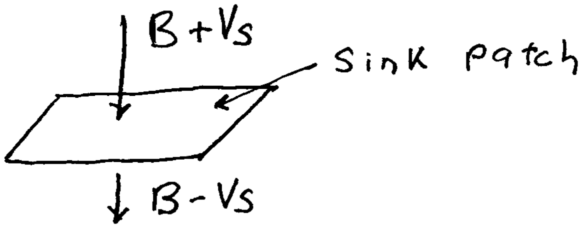
The sinks



Consider the effect of a unit sink.

$$\phi = \frac{1}{r}, \quad V_n = \frac{1}{r^2}, \quad \text{influx } = \frac{1}{r^2}4\pi r^2 = 4\pi$$

Now, look at the small patch of area A on the surface:



B is the effect of the integrals on the remainder of the object. Call the sink strength per unit area σ. Total sink Strength on the patch is σA.

Net influx, based on the velocities is $2AV_s$.

$$2AV_s = 4\pi\sigma A \qquad\qquad 2V_s = 4\pi\sigma$$

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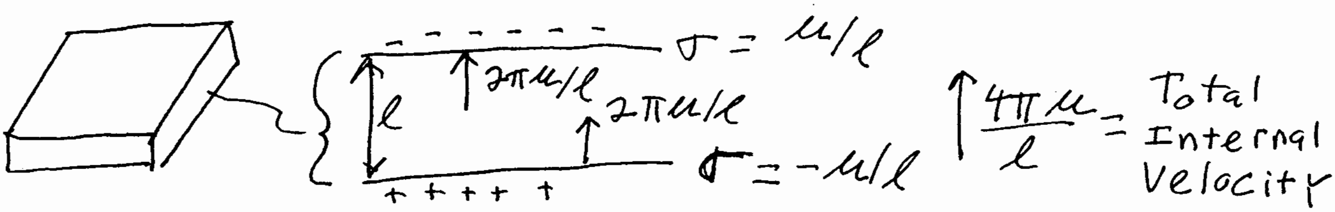
$2V_s$ is the jump in normal velocity. This must equal the normal velocity, $\partial\phi/\partial n$ in the fluid at the boundary since $V = 0$ inside the object.

$$\frac{\partial \phi}{\partial n} = 4\pi\sigma \qquad\qquad \sigma = \frac{1}{4\pi}\frac{\partial \phi}{\partial n}$$

Now consider an infinitesmal dipole patch of strength μ



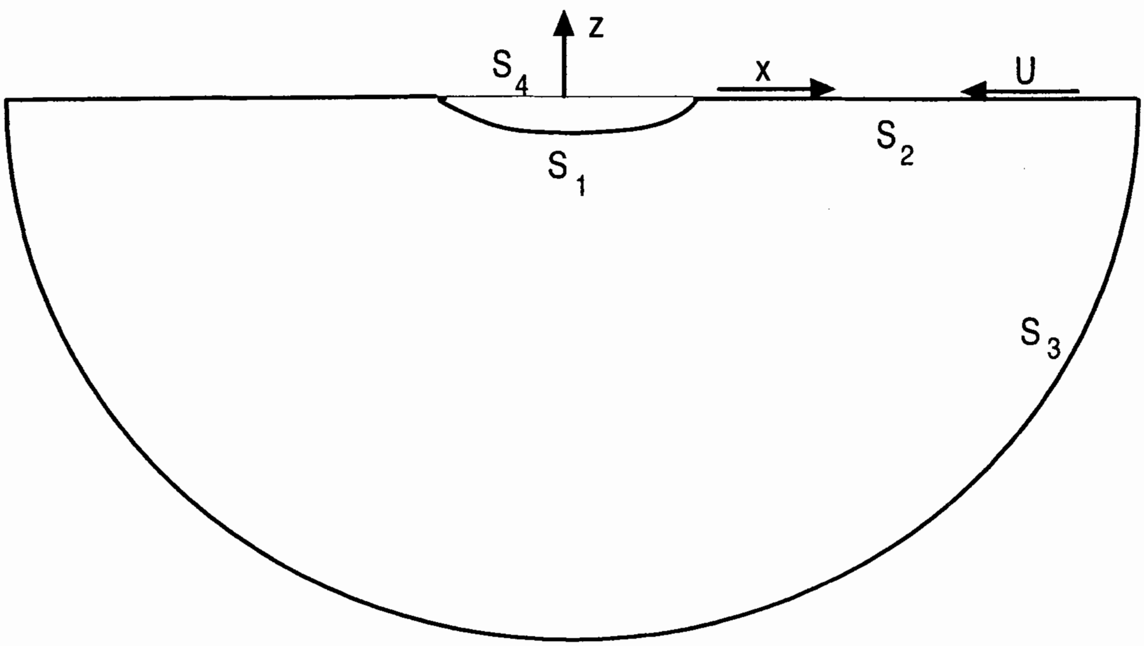
Inside the infinistemally thin dipole layer of thickness ℓ,

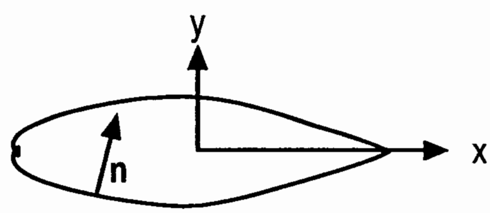
$$v = \frac{4\pi\mu}{\ell} \qquad\qquad \phi \text{ in fluid} - \phi_{\text{inside object}} = 4\pi\mu$$

since: $\quad \phi_{\text{inside object}} = 0$,

$$\phi_{\text{in fluid}} = 4\pi\mu \qquad\qquad \mu = \frac{1}{4\pi}\phi_{\text{in fluid}}$$

Kelvin-Neumann Problem





The Kelvin–Neumann Problem

$$\phi = \frac{1}{4\pi} \int\int_{(S_1+S_2+S_3)} \left[G\frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n} \right] dS$$

ϕ is the perturbation potential (does not include $-Ux$).

The integral over S_1, which is the part of the ship hull below the waterline is of the same form as a Green's theorem or "panel method" integral for any finite size body, except here the top is open.

ϕ and G decay with distance from the ship fast enough for the integral over S_3 to vanish.

The integral over S_2, which is the free surface external to the ship is special. We consider it here and call it ϕ_2.

$$\phi_2 = \frac{1}{4\pi} \int\int_{S_2} \left[G\frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n} \right] dx\, dy$$

Since $\mathbf{n}$ is a unit vector in the z direction on the mean free surface,

$$\phi_2 = \frac{1}{4\pi} \int\int_{S_2} \left[G\frac{\partial \phi}{\partial z} - \phi \frac{\partial G}{\partial z} \right] dx\, dy$$

On the mean free surface, $\frac{\partial \phi}{\partial z} = -\frac{U^2}{g}\frac{\partial^2 \phi}{\partial x^2}$ and we choose G (the Kelvin-Neumann Green function) such that it satisfies the same boundary condition, $\frac{\partial G}{\partial z} = -\frac{U^2}{g}\frac{\partial^2 G}{\partial x^2}$

Then, applying these boundary conditions, ϕ_2 becomes,

$$\phi_2 = -\frac{1}{4\pi}\frac{U^2}{g} \int\int_{S_2} \left[G\frac{\partial^2 \phi}{\partial x^2} - \phi \frac{\partial^2 G}{\partial x^2} \right] dx\, dy$$

$$\phi_2 = -\frac{1}{4\pi}\frac{U^2}{g} \int\int_{S_2} \frac{\partial}{\partial x}\left[G\frac{\partial \phi}{\partial x} - \phi \frac{\partial G}{\partial x} \right] dx\, dy$$

Now, the integral over x can be done. The contributions at $x = \pm\infty$ vanish so the result is:

$$\phi_2 = \frac{1}{4\pi}\frac{U^2}{g}\int_{fore}\left[G\frac{\partial\phi}{\partial x} - \phi\frac{\partial G}{\partial x}\right]dy - \frac{1}{4\pi}\frac{U^2}{g}\int_{aft}\left[G\frac{\partial\phi}{\partial x} - \phi\frac{\partial G}{\partial x}\right]dy$$

The curve of the waterline is called C, with the part forward of the maximum beam called C_f and the part aft of this is called C_a. Consider the integrals taken along C in the counterclockwise direction. Then, dy is positive on the forebody (C_f) and negative on the afterbody (C_a).

$$\phi_2 = \frac{1}{4\pi}\frac{U^2}{g}\int_C\left[G\frac{\partial\phi}{\partial x} - \phi\frac{\partial G}{\partial x}\right]dy$$

If ν is the vector in the horizontal plane that is perpendicular to the waterline and pointed out of the fluid into the ship, and $d\ell$ is the differential of arc length along C, along the waterline $dy = -\nu_x d\ell$, so,

$$\phi = \frac{1}{4\pi}\int\int_{S_1}\left[G\frac{\partial\phi}{\partial n} - \phi\frac{\partial G}{\partial n}\right]dS - \frac{1}{4\pi}\frac{U^2}{g}\int_C\left[G\frac{\partial\phi}{\partial x} - \phi\frac{\partial G}{\partial x}\right]\nu_x d\ell$$

Now, consider a potential function, ϕ' defined in the region bounded by S_1 and S_4. In other words, ϕ' is some function of space such that $\nabla^2\phi' = 0$ in the region where it is defined outside the actual fluid. The boundary condition we impose on ϕ' on S_4 is the same as the one we impose on ϕ on S_2. On S_1 we impose $\phi' = \phi$. $\frac{\partial\phi'}{\partial z} = -\frac{U^2}{g}\frac{\partial^2\phi'}{\partial x^2}$ on S_4

In the fluid region,

$$0 = \frac{1}{4\pi}\int\int_{(S_1+S_4)}\left[G\frac{\partial\phi'}{\partial n'} - \phi'\frac{\partial G}{\partial n'}\right]dS$$

$\mathbf{n}' = -\mathbf{n}$ pointed into the fluid on S_1 and n' is a unit vector in the z direction on S_4.

Call the contribution to ϕ' from the integral on S_4 by ϕ'_4.

$$\phi'_4 = \frac{1}{4\pi}\int\int_{S_4}\left[G\frac{\partial\phi'}{\partial z} - \phi'\frac{\partial G}{\partial z}\right]dx\,dy = -\frac{1}{4\pi}\frac{U^2}{g}\int\int_{S_4}\left[G\frac{\partial^2\phi'}{\partial x^2} - \phi'\frac{\partial^2 G}{\partial x^2}\right]dx\,dy$$

$$\phi'_4 = -\frac{1}{4\pi}\frac{U^2}{g}\int\int_{S_4}\frac{\partial}{\partial x}\left[G\frac{\partial\phi'}{\partial x} - \phi'\frac{\partial G}{\partial x}\right]dx\,dy$$

Carrying out the integral over x gives,

$$\phi_4' = -\frac{1}{4\pi}\frac{U^2}{g}\int_{fore}\left[G\frac{\partial\phi'}{\partial x} - \phi'\frac{\partial G}{\partial x}\right]dy + \frac{1}{4\pi}\frac{U^2}{g}\int_{aft}\left[G\frac{\partial\phi'}{\partial x} - \phi'\frac{\partial G}{\partial x}\right]dy$$

$$\phi_4' = -\frac{1}{4\pi}\frac{U^2}{g}\int_C\left[G\frac{\partial\phi'}{\partial x} - \phi'\frac{\partial G}{\partial x}\right]dy = \frac{1}{4\pi}\frac{U^2}{g}\int_C\left[G\frac{\partial\phi'}{\partial x} - \phi'\frac{\partial G}{\partial x}\right]\nu_x d\ell$$

$$0 = \frac{1}{4\pi}\int\int_{S_1}\left[G\frac{\partial\phi'}{\partial n'} - \phi'\frac{\partial G}{\partial n'}\right]dS + \frac{1}{4\pi}\frac{U^2}{g}\int_C\left[G\frac{\partial\phi'}{\partial x} - \phi'\frac{\partial G}{\partial x}\right]\nu_x d\ell$$

$$0 = \frac{1}{4\pi}\int\int_{S_1}\left[G\frac{\partial\phi'}{\partial n'} + \phi\frac{\partial G}{\partial n}\right]dS + \frac{1}{4\pi}\frac{U^2}{g}\int_C\left[G\frac{\partial\phi'}{\partial x} - \phi\frac{\partial G}{\partial x}\right]\nu_x d\ell$$

To this, we add the equation for ϕ derived before:

$$\phi = \frac{1}{4\pi}\int\int_{S_1}\left[G\frac{\partial\phi}{\partial n} - \phi\frac{\partial G}{\partial n}\right]dS - \frac{1}{4\pi}\frac{U^2}{g}\int_C\left[G\frac{\partial\phi}{\partial x} - \phi\frac{\partial G}{\partial x}\right]\nu_x d\ell$$

The sum is:

$$\phi = \frac{1}{4\pi}\int\int_{S_1} G\left[\frac{\partial\phi}{\partial n} + \frac{\partial\phi'}{\partial n'}\right]dS - \frac{1}{4\pi}\frac{U^2}{g}\int_C G\left[\frac{\partial\phi}{\partial x} - \frac{\partial\phi'}{\partial x}\right]\nu_x d\ell$$

$\frac{1}{4\pi}\left[\frac{\partial\phi}{\partial n} + \frac{\partial\phi'}{\partial n'}\right]$ is the source (actually it is a sink) strength σ.

The normal derivative of $\frac{1}{4\pi}\phi$ jumps at the interface by the source strength, σ. The tangential derivative of ϕ is continuous across the interface because ϕ is continuous. $\frac{\partial\phi}{\partial x}$ jumps across the interface by the jump in the normal derivative times n_x. Therefore,

$$\phi = \int\int_{S_1} G\,\sigma\,dS - \frac{U^2}{g}\int_C G\,\sigma\,n_x\,\nu_x,d\ell$$

The Kelvin–Neumann Green Function

The Kelvin Neumann Green Function, $G^k(x,y,z)$ is the velocity potential for a source located at (a,b,c) and moving at speed U and which satisfies the linearized free surface boundary condition:

$$G^k_{xx}(x,y,0) + vG^k_z = 0, \qquad\qquad v = \frac{g}{U^2}$$

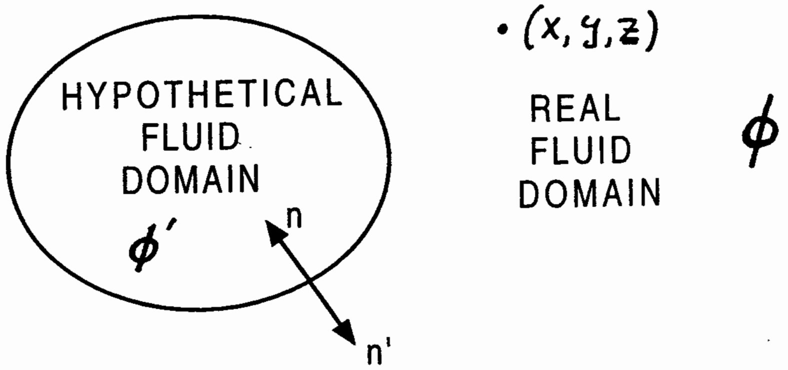
This function is:

$$\begin{aligned}
G^k(x,y,z) &= -\frac{1}{r} + \frac{1}{r_1} + \\
& \frac{4v}{\pi}\int_0^{\pi/2} d\theta \oint_0^{\infty} \frac{e^{k(z+c)}\cos[k(x-a)\cos\theta]\cos[k(y-b)\sin\theta]}{k\cos^2\theta - v} dk + \\
& 4v\int_0^{\pi/2} e^{v(z+c)\sec^2\theta}\sin[v(x-a)\sec\theta]\cos[v(y-b)\sin\theta\sec^2\theta]\sec^2\theta d\theta
\end{aligned}$$

where:

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2, \qquad r_1^2 = (x-a)^2 + (y-b)^2 + (z+c)^2$$

Source Only and Dipole Only Distributions



$$\mathbf{n} = -\mathbf{n}' \qquad \frac{\partial}{\partial \mathbf{n}} = -\frac{\partial}{\partial \mathbf{n}'}$$

ϕ is the velocity potential in the fluid.

ϕ' is a function that satisfies $\nabla^2\phi' = 0$ in the region inside the object.

For a field point in the fluid domain, the following equations apply:

$$\phi = \frac{1}{4\pi}\int\int G\frac{\partial \phi}{\partial n}dS - \frac{1}{4\pi}\int\int \phi\frac{\partial G}{\partial n}dS$$

$$0 = \frac{1}{4\pi}\int\int G\frac{\partial \phi'}{\partial n'}dS - \frac{1}{4\pi}\int\int \phi'\frac{\partial G}{\partial n'}dS$$

$$0 = \frac{1}{4\pi}\int\int G\frac{\partial \phi'}{\partial n'}dS + \frac{1}{4\pi}\int\int \phi'\frac{\partial G}{\partial n}dS$$

$$\phi = \frac{1}{4\pi}\int\int G\left(\frac{\partial \phi}{\partial n} + \frac{\partial \phi'}{\partial n'}\right)dS - \frac{1}{4\pi}\int\int (\phi - \phi')\frac{\partial G}{\partial n}ds$$

Suppose ϕ' is chosen as the harmonic function whose values on S are the same as ϕ. The hypothetical interior flow would have the same tangential velocity on the object as the real outer flow. Then:

$$\phi = \frac{1}{4\pi} \int\int G \left(\frac{\partial \phi}{\partial n} + \frac{\partial \phi'}{\partial n'} \right) dS$$

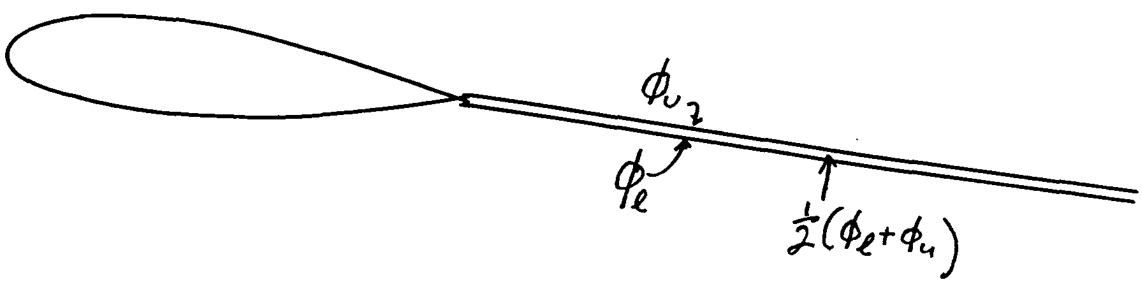
This is a representation for ϕ in terms of surface sources only.

This representation does not apply to lifting flows since they have wakes across which the potential jumps.

Now consider the case for which ϕ' is chosen such that on S, $\frac{\partial \phi}{\partial n} = -\frac{\partial \phi'}{\partial n'}$. The normal velocity is continuous across the surface for this case. Then:

$$\phi = -\frac{1}{4\pi} \int\int (\phi - \phi') \frac{\partial G}{\partial n} ds$$

This is a distribution of dipoles on the object surface S.



FMB8C (1)

Green's Theorem in Two Dimensions

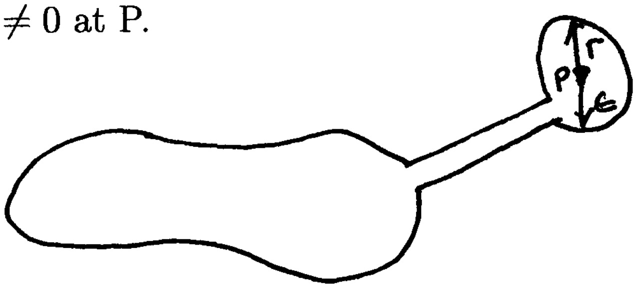
For two dimensional flow, the source potential is $\ln r$ and the Green function becomes:

$$G(x, y, \xi, \eta) = \ln r + \psi_r(x, y, \xi, \eta) \qquad \text{where: } \nabla^2\psi_r = 0$$

The analysis proceeds exactly the same as in the 3D case. When a point $P = (x, y)$, which is the local origin for $\ln r$, is outside the fluid domain,

$$\int_s \left[\phi \frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds = 0$$

When the point P is inside the fluid domain, Green's Theorem is valid in a domain in which the point P is excluded by a small circle, *circle* ϵ surrounding it since $\nabla^2 \neq 0$ at P.



Then: $$-\int_{circle\,\epsilon} \left[\phi \frac{\partial G}{\partial r} - G\frac{\partial \phi}{\partial r}\right] ds + \int_S \left[\phi \frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds = 0$$

Here: $$-\int_{circle\,\epsilon} \left[\phi \frac{\partial G}{\partial r} - G\frac{\partial \phi}{\partial r}\right] ds = -2\pi\phi(P)$$

Therefore: $$\int_S \left[\phi \frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds = \begin{cases} 0 & (x,y) \text{ outside } S \\ \pi\phi(x,y) & (x,y) \text{ on } S \\ 2\pi\phi(x,y) & (x,y) \text{ inside } S \end{cases}$$

FMB8c(2)

Sometimes, the two-dimensional Green function is taken as:

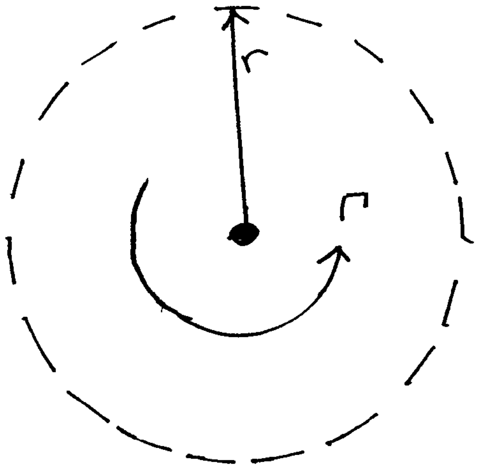
$$G(x, y, \xi, \eta) = -\ln r + \psi_r(x, y, \xi, \eta) \qquad \text{where: } \nabla^2 \psi_r = 0$$

Then,

$$\int_S \left[\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n}\right] ds = \begin{cases} 0 & (x, y) \text{ outside } S \\ -\pi \phi(x, y) & (x, y) \text{ on } S \\ -2\pi \phi(x, y) & (x, y) \text{ inside } S \end{cases}$$

F MBV1

Force on a Vortex



$$\mathbf{n} = -\hat{\imath}\cos\theta - \hat{k}\sin\theta$$

$$u_v = -\frac{\Gamma}{2\pi r}\sin\theta \qquad\qquad w_v = \frac{\Gamma}{2\pi r}\cos\theta$$

$$P = -\frac{\rho}{2}\left\{\left(U - \frac{\Gamma}{2\pi r}\sin\theta\right)^2 + \left(\frac{\Gamma}{2\pi r}\cos\theta\right)^2 - U^2\right\}$$

$$P = -\frac{\rho}{2}\left\{-\frac{\Gamma U}{\pi r}\sin\theta + \left(\frac{\Gamma}{2\pi r}\right)^2\right\}$$

$$\begin{aligned} F_P &= \int_0^{2\pi} P\mathbf{n}\, ds \\ &= -\int_0^{2\pi} \frac{\rho\Gamma}{2\pi r}\left(-U\sin\theta + \frac{\Gamma}{4\pi r}\right)\left(-\hat{\imath}\cos\theta - \hat{k}\sin\theta\right) r d\theta \\ &= \hat{k}\int_0^{2\pi} -\frac{\rho U\Gamma}{2\pi r}\sin^2\theta\, r\, d\theta = -\hat{k}\frac{\rho U\Gamma}{2} \end{aligned}$$

Momentum influx $\equiv M_{in}$ $\qquad\qquad U_{in} = \hat{\imath}U\cdot\mathbf{n} = -U\,\cos\theta$

$$F_M = M_{in} = \rho\int_0^{2\pi}\hat{k}w_v U_{in} r\, d\theta = -\hat{k}\frac{\rho U\Gamma}{2}$$

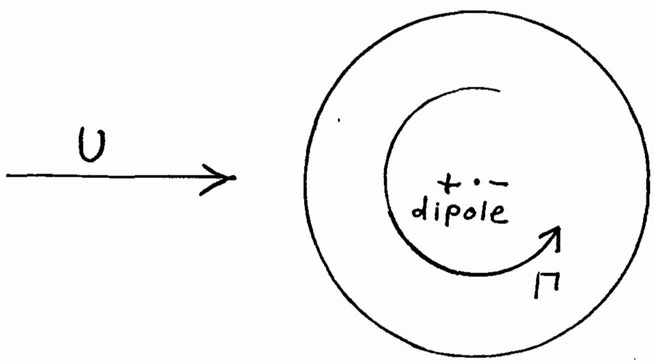
$$F_{total} = F_P + F_M = -\hat{k}\rho U\Gamma$$

Lift on a Vortex in a Cylinder

When a vortex is in a uniform stream, to determine the lift force both the pressure force and the momentum influx into a circular cylinder must be considered.

If the vortex is in a flow whose streamlines form a cylinder around it, there is no momentum influx so the pressure force is the complete force.

A closed circle in a stream can be represented by a dipole.



The velocity potential of a 2D dipole is $\phi_d = A\frac{x}{x^2+z^2}$.

For the flow to make a circle of radius equal to 1 in a stream of speed U, $A = U$. the $x-$ and $z-$directed speeds on the circle of radius 1 due to the dipole are:

$$u_d = U(z^2 - x^2) = U(\sin^2\theta - \cos^2\theta) = -U\cos 2\theta$$

$$w_d = -2Uzx = -2U\sin\theta\cos\theta = -U\sin 2\theta$$

The speeds obn the circle due to the vortex are:

$$u_v = -\frac{\Gamma}{2\pi}\sin\theta \qquad\qquad w_v = \frac{\Gamma}{2\pi}\cos\theta$$

The pressure on the circle is:

$$\begin{aligned} P &= -\frac{\rho}{2}\left[\left(U - U\cos 2\theta - \frac{\Gamma}{2\pi}\sin\theta\right)^2 + \left(-U\sin 2\theta + \frac{\Gamma}{2\pi}\cos\theta\right)^2 - U^2\right] \\ &= -\frac{\rho}{2}\left[U^2 + \left(\frac{\Gamma}{2\pi}\right)^2 - 2U^2\cos 2\theta - \frac{U\Gamma}{\pi}\sin\theta + \frac{U\Gamma}{\pi}\sin\theta\cos 2\theta - \frac{U\Gamma}{\pi}\cos\theta\sin 2\theta\right] \end{aligned}$$

FMBV7(M)

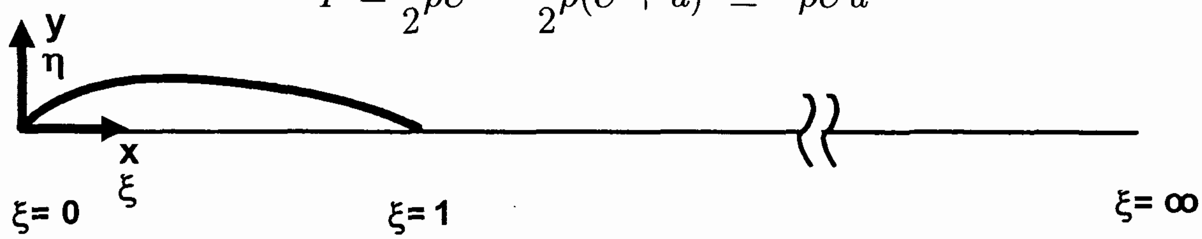
The vertical force, F_w is:

$$F_w = \int_0^{2\pi} P\mathbf{n} \cdot \hat{k} d\theta = \int_0^{2\pi} -P \sin\theta d\theta$$

$$\begin{aligned}
F_w &= \frac{\rho U \Gamma}{2\pi} \int_0^{2\pi} \left(-\sin^2\theta + \sin^2\theta \cos 2\theta - \sin\theta \cos\theta \sin 2\theta \right) d\theta \\
&= \frac{\rho U \Gamma}{2\pi} \int_0^{2\pi} \left(-\sin^2\theta - \frac{1}{2}\cos^2 2\theta - \frac{1}{2}\sin^2 2\theta \right) d\theta \\
&= \frac{\rho U \Gamma}{2\pi} \left(-\pi - \frac{\pi}{2} - \frac{\pi}{2} \right) = -\rho U \Gamma
\end{aligned}$$

Example: Design of 2D Airfoil Mean Line using Dipoles and Vortices

$$P = \frac{1}{2}\rho U^2 - \frac{1}{2}\rho(U+u)^2 \simeq -\rho U u$$



$$\xi = x/c \qquad x = c\xi \qquad dx = c\,d\xi \qquad \eta = y/c$$

Design Condition: $P_{\mathrm{top}} = -1.0\rho U^2 \xi(1-\xi) \qquad P_{\mathrm{bottom}} = 1.0\rho U^2 \xi(1-\xi)$

$$u_t = 1.0U\xi(1-\xi) \qquad u_b = -1.0U\xi(1-\xi)$$

$$\phi_t \simeq \int_0^x u dx' = c\int_0^\xi u d\xi = 1.0Uc\int_0^\xi (\xi - \xi^2) d\xi = 1.0Uc\left(\frac{\xi^2}{2} - \frac{\xi^3}{3}\right)$$

$$\phi_b = -1.0Uc\left(\frac{\xi^2}{2} - \frac{\xi^3}{3}\right)$$

$$\phi_t(\xi = 1) = \frac{1.0}{6}Uc \qquad \phi_b(\xi = 1) = -\frac{1.0}{6}Uc \qquad (\phi_t - \phi_b)_{\xi=1} = \frac{1.0}{3}Uc$$

$$[\text{Dipole Strength}]_{\mathrm{foil}} = \mu = 2.0Uc\left(\frac{\xi^2}{2} - \frac{\xi^3}{3}\right) \qquad \mu_{\mathrm{wake}} = \frac{1.0}{3}Uc$$

$$G = \ln r = \ln\left[(x-x_o)^2 + (y-y_o)^2\right]^{1/2} = \frac{1}{2}\ln\left[(x-x_o)^2 + (y-y_o)^2\right]$$

$$\left(\frac{\partial G}{\partial n}\right)_t = -\frac{\partial G}{\partial y_o} \qquad \left(\frac{\partial G}{\partial n}\right)_b = \frac{\partial G}{\partial y_o}$$

Foil2da(2)

Consider the upper surface:

$$\left(\frac{\partial G}{\partial n}\right)_t = -\frac{\partial G}{\partial y_o} = \frac{y - y_o}{(x - x_o)^2 + (y - y_o)^2}$$

$$\phi(x, y) = \frac{1}{\pi}\int_0^\infty \mu(x_o)\frac{y - y_o}{(x - x_o)^2 + (y - y_o)^2}dx_o$$

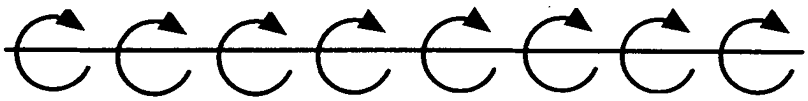
$$\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{1}{\pi}\int_0^c \mu(x_o)\frac{(x - x_o)^2 + (y - y_o)^2 - 2(y - y_o)^2}{[(x - x_o)^2 + (y - y_o)^2]^2}dx_o \\
&\quad + \mu(c)\int_c^\infty \frac{(x - x_o)^2 + (y - y_o)^2 - 2(y - y_o)^2}{[(x - x_o)^2 + (y - y_o)^2]^2}dx_o \\
&= \frac{1}{\pi}\int_0^c \mu(x_o)\frac{(x - x_o)^2 - (y - y_o)^2}{[(x - x_o)^2 + (y - y_o)^2]^2}dx_o + \frac{1}{\pi}\mu(c)\int_c^\infty \frac{(x - x_o)^2 - (y - y_o)^2}{[(x - x_o)^2 + (y - y_o)^2]^2}dx_o
\end{aligned}$$

$$\left[\frac{\partial \phi}{\partial y}\right]_{y = y_o = 0} = \frac{1}{\pi}\int_0^c \mu(x_o)\frac{1}{(x - x_o)^2}dx_o + \frac{1}{\pi}\mu(c)\int_c^\infty \frac{1}{(x - x_o)^2}dx_o$$

The above analysis has an incorrect non-integrable singularity at $x = x_o$ because a careful limiting analysis requiring $\nabla^2\phi = 0$ was not done.

However, another, and simpler, approach exists.

A dipole represents a jump in the potential. Another way to achieve a potential jump is a vortex distribution.



In length dx_o, vortex strength $=\gamma(x_o)dx_o$. $\gamma(x_o)$ is vorticity/unit-length.

$$u_t(x) = U + \frac{\gamma(x)}{2} \qquad u_b(x) = U - \frac{\gamma(x)}{2} \qquad \gamma(x) = u_t(x) - u_b(x)$$

$$v(x) = -\int_0^c \frac{\gamma(x_o)}{2\pi(x - x_o)}dx_o$$

$$\text{slope} = \frac{v(x)}{U} = \int_0^1 \frac{\gamma'(\xi_o)}{2\pi(\xi - \xi_o)}d\xi_o$$

$$\text{where: } \gamma'(\xi_o) = \frac{\gamma(c\xi_o)}{U}$$

Now, we can solve for the mean line shape of the airfoil

For an arbitrarily defined pressure distribution, the integral for the slope can be done numerically. Here, for the particular pressure distribution given, we will solve for the slope analytically.

Then the shape is found by integrating the slope, $s(\xi)$. This will be done numerically.

$$\gamma'(\xi) = 2.0\xi(1.0 - \xi)$$

$$s(\xi) = -\int_0^1 \frac{\gamma'(\xi_o)}{(2\pi(\xi - \xi_o)}d\xi_o = -\frac{1.0}{\pi}\int_0^1 \frac{\xi_o(1-\xi_o)}{\xi - \xi_o}d\xi_o$$

$$s(\xi) = -\frac{1}{\pi}\left\{\frac{1}{2} - (1-\xi)\left[\xi \ln \frac{1-\xi}{\xi} + 1\right]\right\}$$

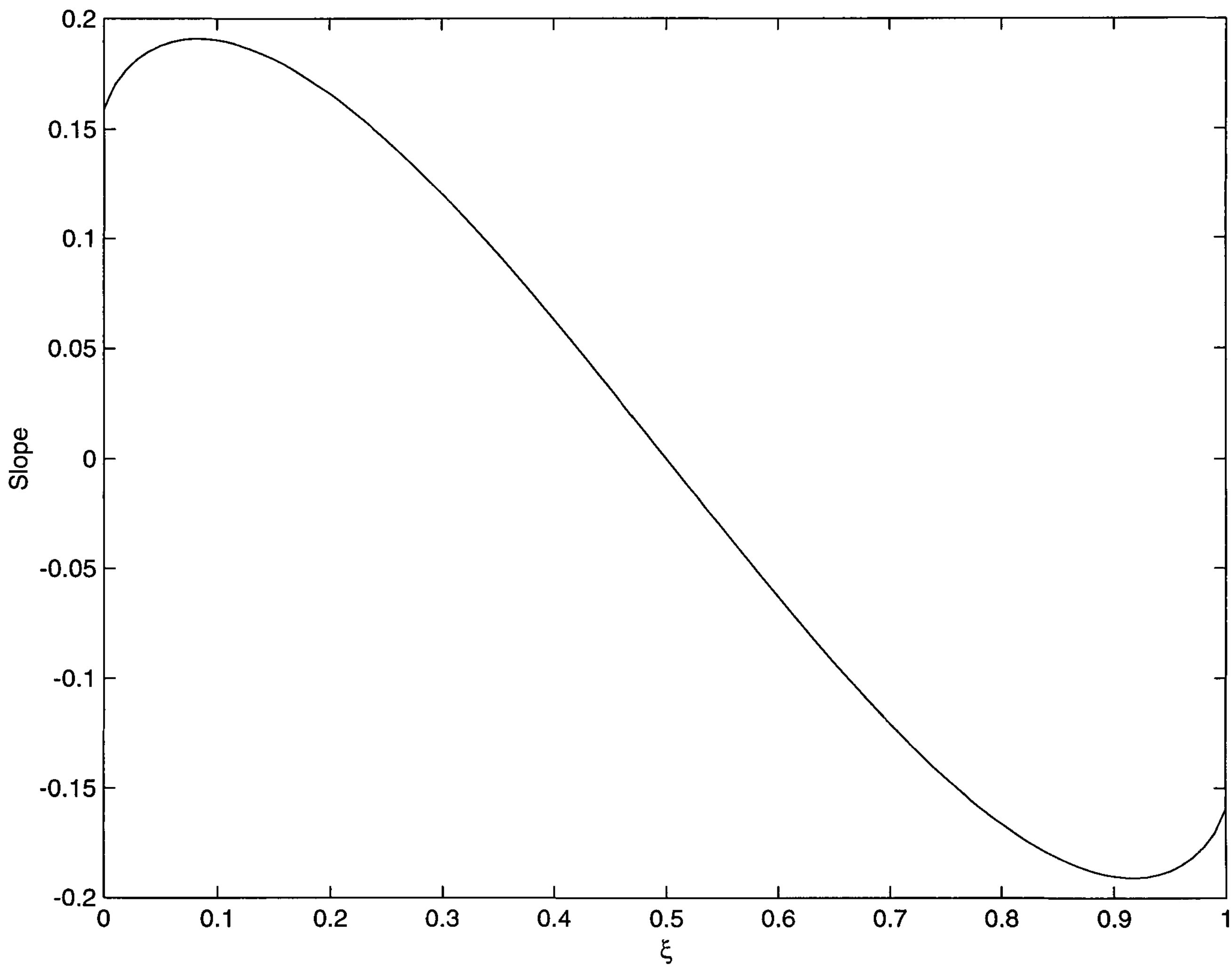
$$\text{Non-Dimensional Height } = \eta(\xi) = \int_0^\xi s(\xi_o)d\xi_o$$

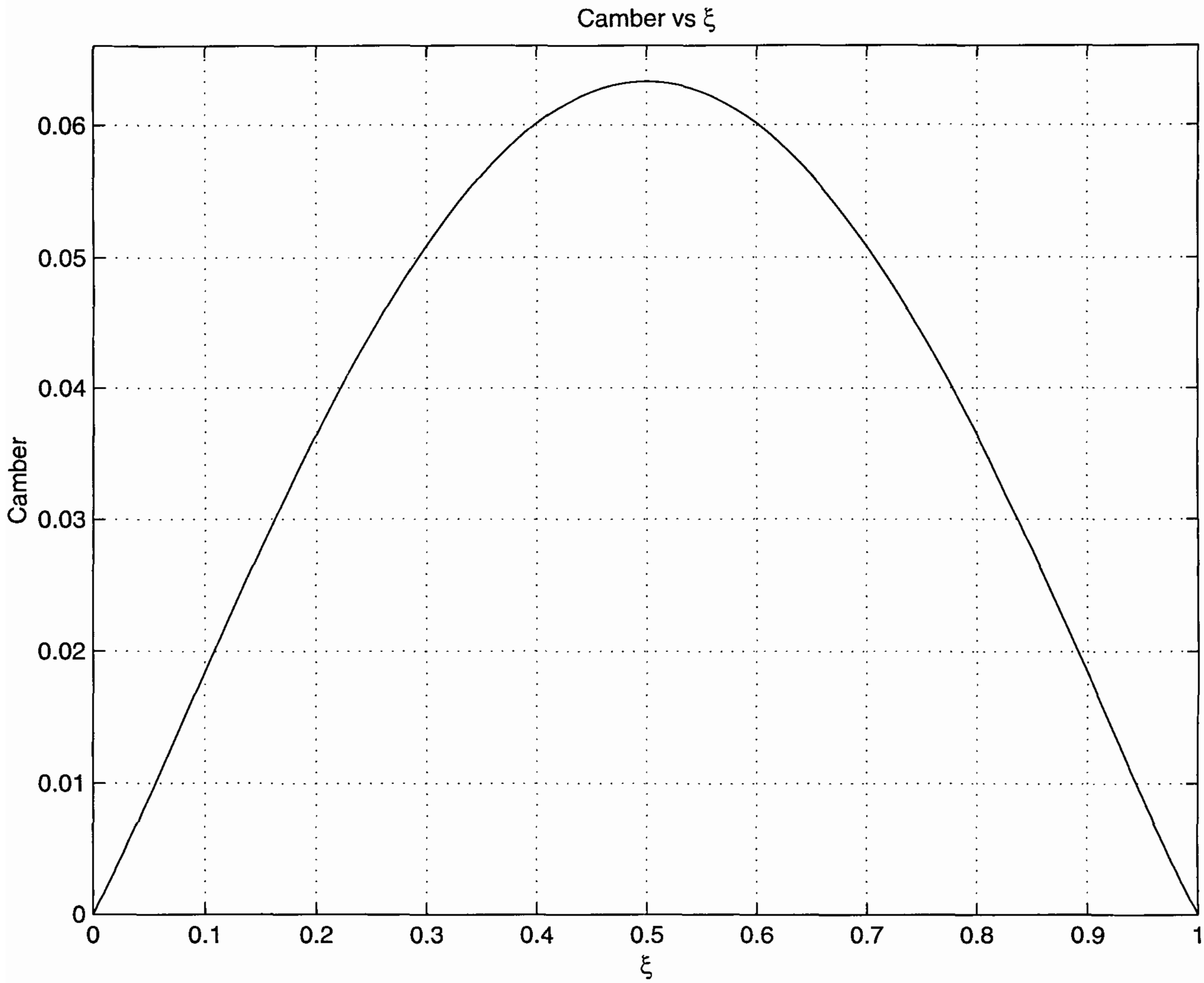
```
format compact
x = 0  : 0.01 : 1.0;
fac = -1.0/pi;
s = fac.*(0.5 - (1.0-x).*( x.*log((1.0-x+eps)./(x+eps))+1.0));
h(1) = 0.0;
for i = 2:101
h(i) = h(i-1) + (x(i)-x(i-1))*0.5*(s(i)+s(i-1));
end;
fid = fopen('ht.dat','w');
for m = 1:101
fprintf(fid,'%6.2f  %7.4f  %7.4f\n',x(m), s(m), h(m));
end;
fclose (fid);
plot(x,s)
ylabel('Slope')
xlabel('\xi')
pause

plot(x,h)
ylabel('Camber')
xlabel('\xi')
title('Camber vs \xi')
axis([0 1 0 0.066])
grid
```

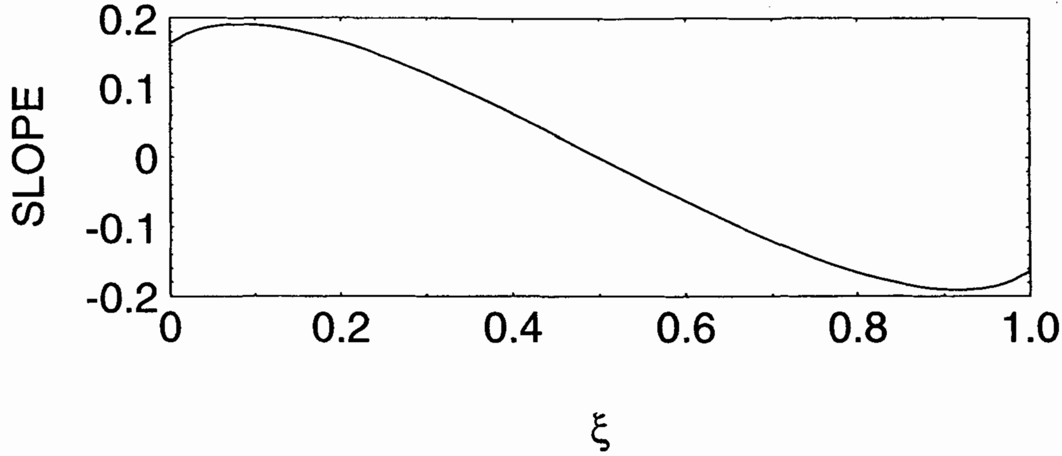
X	SLOPE	HEIGHT
0.00	0.1592	0.0000
0.01	0.1705	0.0016
0.02	0.1771	0.0034
0.03	0.1818	0.0052
0.04	0.1853	0.0070
0.05	0.1878	0.0089
0.06	0.1895	0.0108
0.07	0.1905	0.0127
0.08	0.1909	0.0146
0.09	0.1908	0.0165
0.10	0.1903	0.0184
0.11	0.1893	0.0203
0.12	0.1879	0.0222
0.13	0.1862	0.0240
0.14	0.1842	0.0259
0.15	0.1818	0.0277
0.16	0.1792	0.0295
0.17	0.1763	0.0313
0.18	0.1731	0.0331
0.19	0.1697	0.0348
0.20	0.1661	0.0364
0.21	0.1623	0.0381
0.22	0.1583	0.0397
0.23	0.1541	0.0413
0.24	0.1497	0.0428
0.25	0.1451	0.0442
0.26	0.1405	0.0457
0.27	0.1356	0.0471
0.28	0.1306	0.0484
0.29	0.1255	0.0497
0.30	0.1203	0.0509
0.31	0.1150	0.0521
0.32	0.1095	0.0532
0.33	0.1040	0.0543
0.34	0.0983	0.0553
0.35	0.0926	0.0562
0.36	0.0868	0.0571
0.37	0.0809	0.0580
0.38	0.0749	0.0587
0.39	0.0689	0.0595
0.40	0.0628	0.0601
0.41	0.0567	0.0607
0.42	0.0505	0.0612
0.43	0.0443	0.0617
0.44	0.0380	0.0621
0.45	0.0317	0.0625
0.46	0.0254	0.0628
0.47	0.0191	0.0630
0.48	0.0127	0.0632
0.49	0.0064	0.0632
0.50	0.0000	0.0633
0.51	-0.0064	0.0632
0.52	-0.0127	0.0632
0.53	-0.0191	0.0630
0.54	-0.0254	0.0628
0.55	-0.0317	0.0625
0.56	-0.0380	0.0621

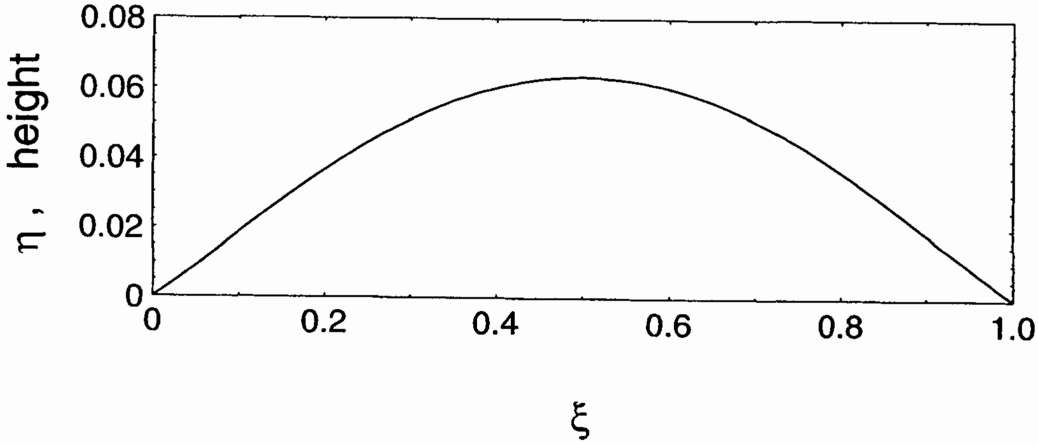
0.57	-0.0443	0.0617
0.58	-0.0505	0.0612
0.59	-0.0567	0.0607
0.60	-0.0628	0.0601
0.61	-0.0689	0.0595
0.62	-0.0749	0.0587
0.63	-0.0809	0.0580
0.64	-0.0868	0.0571
0.65	-0.0926	0.0562
0.66	-0.0983	0.0553
0.67	-0.1040	0.0543
0.68	-0.1095	0.0532
0.69	-0.1150	0.0521
0.70	-0.1203	0.0509
0.71	-0.1255	0.0497
0.72	-0.1306	0.0484
0.73	-0.1356	0.0471
0.74	-0.1405	0.0457
0.75	-0.1451	0.0442
0.76	-0.1497	0.0428
0.77	-0.1541	0.0413
0.78	-0.1583	0.0397
0.79	-0.1623	0.0381
0.80	-0.1661	0.0364
0.81	-0.1697	0.0348
0.82	-0.1731	0.0331
0.83	-0.1763	0.0313
0.84	-0.1792	0.0295
0.85	-0.1818	0.0277
0.86	-0.1842	0.0259
0.87	-0.1862	0.0240
0.88	-0.1879	0.0222
0.89	-0.1893	0.0203
0.90	-0.1903	0.0184
0.91	-0.1908	0.0165
0.92	-0.1909	0.0146
0.93	-0.1905	0.0127
0.94	-0.1895	0.0108
0.95	-0.1878	0.0089
0.96	-0.1853	0.0070
0.97	-0.1818	0.0052
0.98	-0.1771	0.0034
0.99	-0.1705	0.0016
1.00	-0.1592	-0.0000


0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
Slope
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
ξ


Camber vs ξ
Camber
ξ
0.06
0.05
0.04
0.03
0.02
0.01
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

foiltdq - results





foiltd

```
format compact
x = 0  : 0.01 : 1.0;
fac = -1.0/pi;
s = fac.*(0.5 - (1.0-x).*( x.*log((1.0-x+eps)./(x+eps))+1.0));
h(1) = 0.0;
for i = 2:101
h(i) = h(i-1) + (x(i)-x(i-1))*0.5*(s(i)+s(i-1));
end;
fid = fopen('ht.dat','w');
for m = 1:101
fprintf(fid,'%6.2f  %7.4f  %7.4f\n',x(m), s(m), h(m));
end;
fclose (fid);
plot(x,s)
ylabel('Slope')
xlabel('\xi')
pause

plot(x,h)
ylabel('Camber')
xlabel('\xi')
title('Camber vs \xi')
axis([0 1 0 0.066])
grid
```

foiltda

```
% Version of foiltd with one less loop for computing speed improvement

x = 0  : 0.01 : 1.0;
fac = -1.0/pi;
s = fac.*(0.5 - (1.0-x).*( x.*log((1.0-x+eps)./(x+eps))+1.0));
h(1) = 0.0;
for i = 2:101
h(i) = h(i-1) + (x(i)-x(i-1))*0.5*(s(i)+s(i-1));
end;
fid = fopen('hta.dat','w');
q = [x;s;h];
fprintf(fid,'%6.2f  %7.4f  %7.4f\n',q);
fclose (fid);
plot(x,s)
ylabel('Slope')
xlabel('\xi')
pause

plot(x,h)
ylabel('Camber')
xlabel('\xi')
title('Camber vs \xi')
axis([0 1 0 0.066])
grid
```

foiltdb

```
% This version uses even more vectorization and no "for" loops at all.

% Version of foiltd with one less loop for computing speed improvement

x = 0  : 0.01 : 1.0;
fac = -1.0/pi;
s = fac.*(0.5 - (1.0-x).*( x.*log((1.0-x+eps)./(x+eps))+1.0));
h(1) = 0.0;
xd = [0 diff(x)]; % This is [0 x(2)-x(1)  x(3)-x(2) ...]
ss = [0 s(1:end-1) + s(2:end) ] % This is [0 s(2)+s(1)  s(3)+s(2)
h = 0.5*xd .* ss;
h = cumsum(h); % Each element is the sum of the ones bvefore it.
  fid = fopen('htb.dat','w');
q = [x;s;h];
fprintf(fid,'%6.2f  %7.4f  %7.4f\n',q);
fclose (fid);
plot(x,s)
ylabel('Slope')
xlabel('\xi')
pause

plot(x,h)
ylabel('Camber')
xlabel('\xi')
title('Camber vs \xi')
axis([0 1 0 0.066])
grid
```

```
% Version of foiltd with one less loop for computing speed improvement

x = 0  : 0.01 : 1.0;
fac = -1.0/pi;
s = fac.*(0.5 - (1.0-x).*( x.*log((1.0-x+eps)./(x+eps))+1.0));
h(1) = 0.0;
for i = 2:101
h(i) = h(i-1) + (x(i)-x(i-1))*0.5*(s(i)+s(i-1));
end;
fid = fopen('hta.dat','w');
q = [x;s;h];
fprintf(fid,'%6.2f  %7.4f  %7.4f\n',q);
fclose (fid);
plot(x,s)
ylabel('Slope')
xlabel('\xi')
pause

plot(x,h)
ylabel('Camber')
xlabel('\xi')
title('Camber vs \xi')
axis([0 1 0 0.066])
grid
```

```
% This version uses even more vectorization and no "for" loops at all.
% Version of foiltd with one less loop for computing speed improvement

x = 0  : 0.01 : 1.0;
fac = -1.0/pi;
s = fac.*(0.5 - (1.0-x).*( x.*log((1.0-x+eps)./(x+eps))+1.0));
h(1) = 0.0;
xd = [0 diff(x)]; % This is [0 x(2)-x(1)  x(3)-x(2) ...]
ss = [0 s(1:end-1) + s(2:end) ] % This is [0 s(2)+s(1)  s(3)+s(2)
h = 0.5*xd .* ss;
h = cumsum(h); % Each element is the sum of the ones bvefore it.
  fid = fopen('htb.dat','w');
q = [x;s;h];
fprintf(fid,'%6.2f  %7.4f  %7.4f\n',q);
fclose (fid);
plot(x,s)
ylabel('Slope')
xlabel('\xi')
pause

plot(x,h)
ylabel('Camber')
xlabel('\xi')
title('Camber vs \xi')
axis([0 1 0 0.066])
grid
```

Usecalctitle

Some Useful Results from Calculus

Derivation of Gauss' Theorem

Let $f(x, y, z)$ be a differentiable scalar function of (x, y, z).

$$\vec{f}_1 \equiv \hat{\imath} f$$

By the divergence theorem,

$$\int_V \nabla \cdot \vec{f}_1 \, dv = \int_S \vec{f}_1 \cdot \vec{n} \, ds = \int_S n_x f \, ds$$

$$\nabla \cdot \vec{f}_1 = \frac{\partial f}{\partial x}$$

$$\int_V \hat{\imath} \, \nabla \cdot \vec{f}_1 \, dv = \int_V \hat{\imath} \, \frac{\partial f}{\partial x} \, dv = \int_S \hat{\imath} \, n_x f \, ds$$

Similarly,

$$\int_V \hat{\jmath} \, \nabla \cdot \vec{f}_2 \, dv = \int_V \hat{\jmath} \, \frac{\partial f}{\partial y} \, dv = \int_S \hat{\jmath} \, n_y f \, ds$$

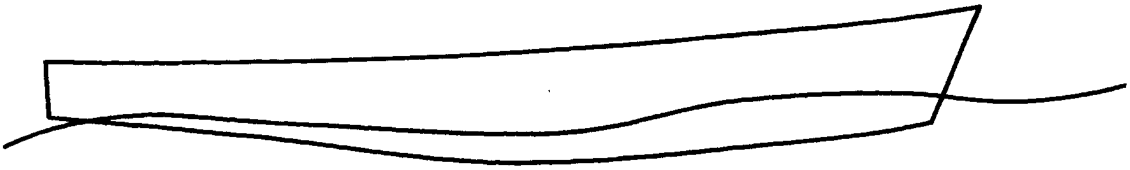
$$\int_V \hat{k} \, \nabla \cdot \vec{f}_3 \, dv = \int_V \hat{k} \, \frac{\partial f}{\partial z} \, dv = \int_S \hat{k} \, n_z f \, ds$$

Now, add the last three equations together,

$$\int_V \nabla f \, dv = \int_S \vec{n} f \, ds$$

gaussexp (1)

Example of Use of Gauss Theorem: Froude Krylov Surge Force on a Ship



$$P = \rho g A e^{-kz} \cos(kx - \omega t) \qquad \qquad \vec{F} = \int\int_S -p\vec{n}\, dS$$

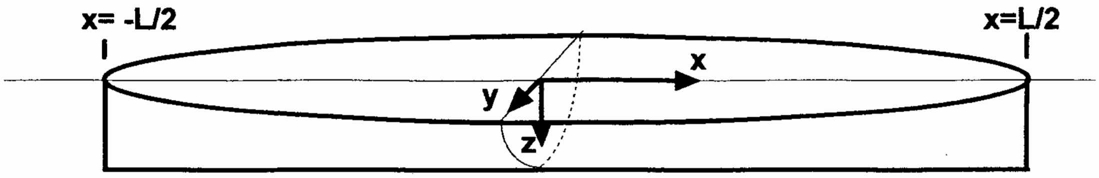
$$F_x = \hat{i} \cdot \int\int_S -P\vec{n}\, ds = -\hat{i} \int\int\int_V \nabla P\, dV$$

$$F_x = -\int\int\int_V \frac{\partial P}{\partial x} dV = \rho g A k \int\int\int_V e^{-kz} \sin(kx - \omega t) dV$$

$$\begin{aligned} F_x &\simeq \rho g A k \int\int\int_V (1 - kz) \sin(kx - \omega t) dV \\ &= \rho g A k \int_L \left[\int\int_{\text{section}} dy\, dz\right] \sin(kx - \omega t) dx \\ &\quad -\rho g A k^2 \int_L \left[\int\int_{\text{section}} z\, dy\, dz\right] \sin(kx - \omega t) dx \end{aligned}$$

$$F_x = \rho g A k \int_L S(x) \sin(kx - \omega t) dx - \rho g A k^2 \int_L z_{ca} S(x) \sin(kx - \omega t) dx$$

Example with Given Ship Shape



$$y = \frac{2W}{L^2D^2}\left(\frac{L}{2} - x\right)^2 (D - z)^2$$

For this shape:

$$S(x) = \frac{2}{3}\frac{WD}{L^2}\left(\frac{L}{2} - x\right)^2 \qquad z_{ca} = \frac{D}{4}$$

Using these values:

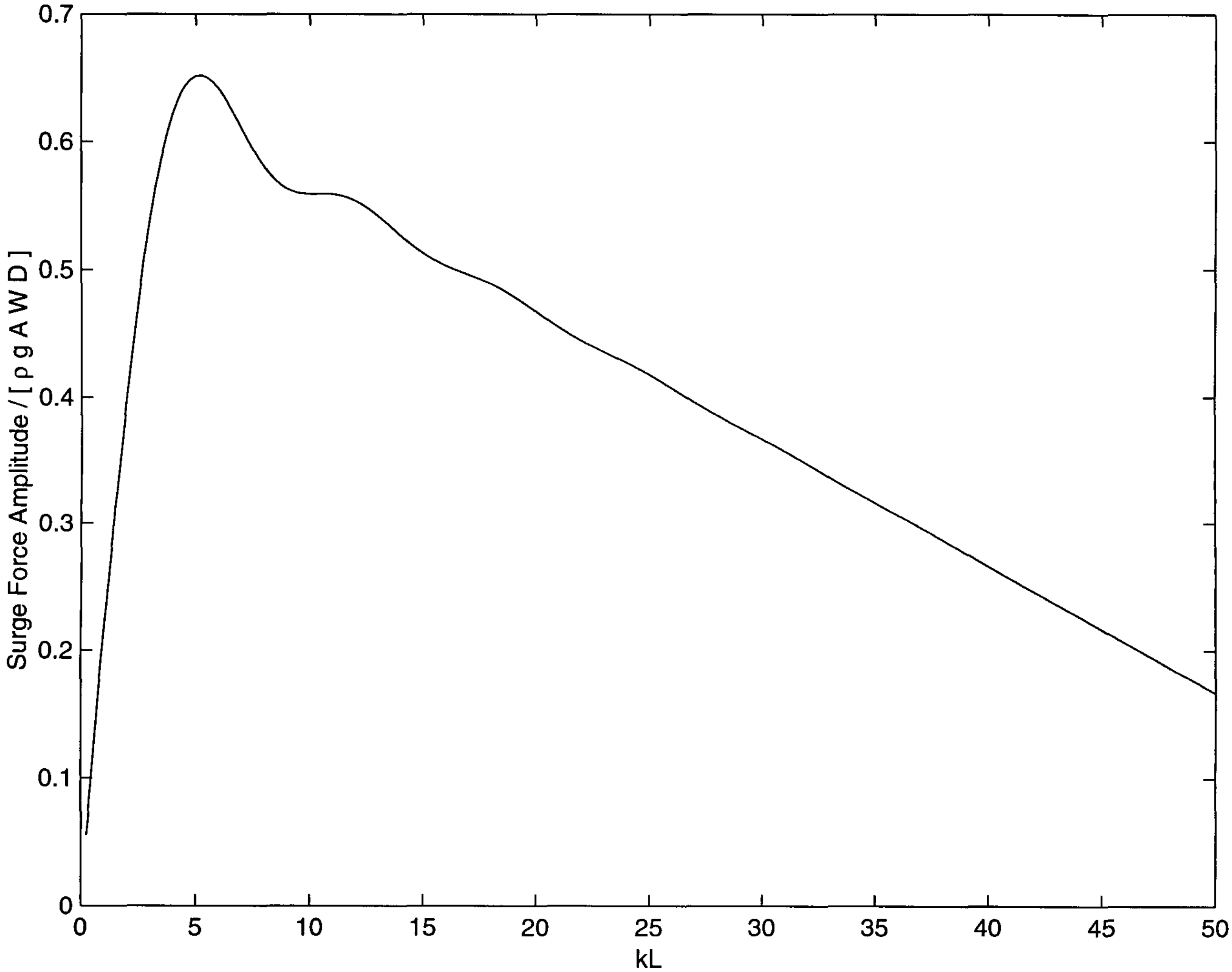
$$\begin{aligned} F_x &= \frac{2}{3}\rho g A k \frac{WD}{L^2}\left(1 - \frac{kD}{4}\right)\left\{\cos\omega t\left[-\frac{2L}{k^2}\sin\frac{kL}{2} + \frac{L^2}{k}\cos\frac{kL}{2}\right]\right. \\ &\qquad \left. -\sin\omega t\left[\left(\frac{L^2}{k} - \frac{4}{k^3}\right)\sin\frac{kL}{2} + \frac{2L}{k^2}\cos\frac{kL}{2}\right]\right\} \end{aligned}$$

$$\begin{aligned} \frac{F_x}{\rho g A W D} &= \frac{2}{3}kL\left(1 - \frac{kL}{4}\frac{D}{L}\right)\left\{\cos\omega t\left[-\frac{2}{(kL)^2}\sin\frac{kL}{2} + \frac{1}{kL}\cos\frac{kL}{2}\right]\right. \\ &\qquad \left. -\sin\omega t\left[\left(\frac{1}{kL} - \frac{4}{(kL)^3}\right)\sin\frac{kL}{2} + \frac{2}{(kL)^2}\cos\frac{kL}{2}\right]\right\} \end{aligned}$$

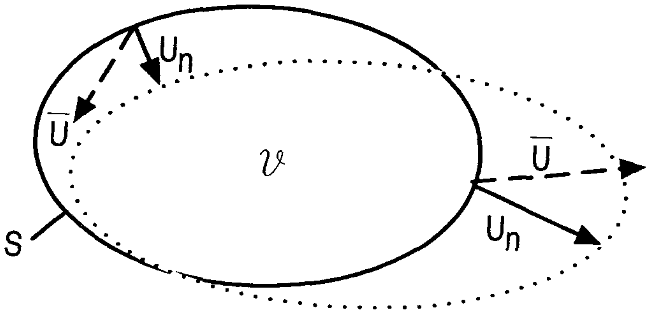
$$\begin{aligned} \left[\frac{F_x}{\rho g A W D}\right]_{\max} &= \frac{2}{3}kL\left(1 - \frac{kL}{4}\frac{D}{L}\right)\left\{\left[-\frac{2}{(kL)^2}\sin\frac{kL}{2} + \frac{1}{kL}\cos\frac{kL}{2}\right]^2\right. \\ &\qquad \left. + \left[\left(\frac{1}{kL} - \frac{4}{(kL)^3}\right)\sin\frac{kL}{2} + \frac{2}{(kL)^2}\cos\frac{kL}{2}\right]^2\right\}^{1/2} \end{aligned}$$

```
% m-file script for gaussexp
tt = 2.0/3.0;
DoL = 0.06;
DoLf = DoL/4.;
m = 1:1:200;
kL = 0.25.*m;
kLi = 1.0 ./ kL;
sn = sin(kL ./ 2.0);
cs = cos(kL ./ 2.0);
fnd = tt .* kL .*(1.0 - kL .* DoLf) .* ((( -2 ./ (kL .^ 2)) .* sn + kLi .* ↙
cs) .^ 2 ...
  + ( ( kLi -4.0 .* (kLi .^3)) .* sn + (2.0 ./ (kL.^2)).* cs) .^2) .^ 0.5;
q = [kL;fnd];
fid = fopen('surge.dat','w');
fprintf(fid,'%8.3f, %9.4f\n',q);
fclose(fid);
plot(kL,fnd)
xlabel('kL')
ylabel('Surge Force Amplitude / [ \rho g A W D ]')
```

gaussexp(%)
SURGE.dat



The Transport Theorem



Let $f(\mathbf{x}, t)$ be a differentiable scalar function of $\mathbf{x}$ and t.

Consider the integral,

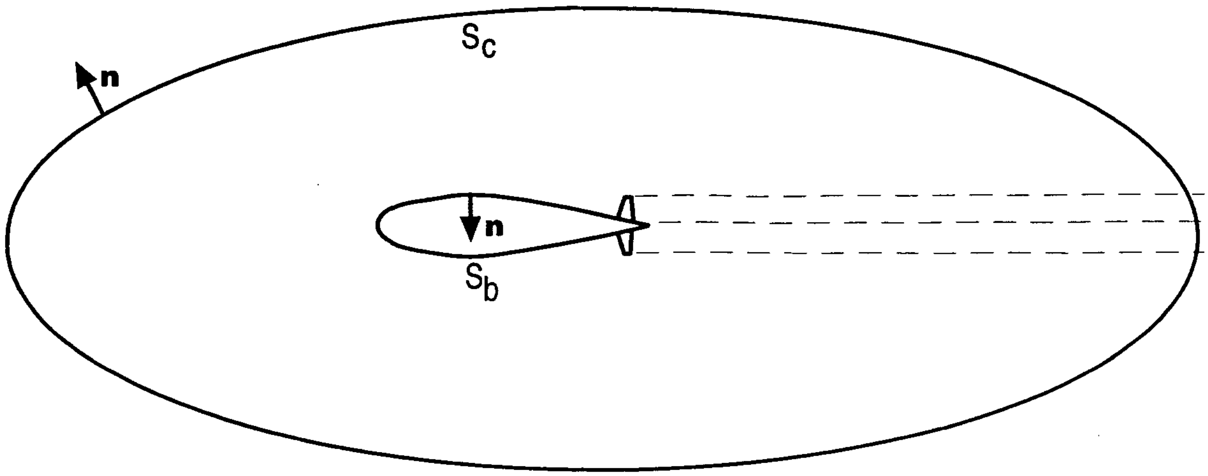
$$I(t) = \int\int\int_{\mathcal{V}(t)} f(\mathbf{x}, t) d\mathcal{V}$$

f is changing with time and $\mathcal{V}$ is changing with time. The normal component of the velocity of any point on the surface, S of $\mathcal{V}$ is called U_n.

$$\frac{dI}{dt} = \int\int\int_{\mathcal{V}} \frac{\partial f}{\partial t} d\mathcal{V} + \int\int_S f\, U_n dS = \int\int\int_{\mathcal{V}} \left\{ \frac{\partial f}{\partial t} + \nabla \cdot \left(f\, \vec{U} \right) \right\} d\mathcal{V}$$

Note that if $\vec{U}$ is the fluid velocity, the surface S is a material surface and the Transport Theorem is simply the integral form of the Substantial Derivative.

Pressure Forces and Moments on an Object



$$\mathbf{F} = \int\int_{S_b} p\,\mathbf{n}\,dS \qquad\qquad \mathbf{M} = \int\int_{S_b} p\,(\mathbf{r}\times\mathbf{n})\,dS$$

Now use the (unsteady) Bernoulli equation:

$$\mathbf{F} = -\rho\int\int_{S_b}\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi\right]\mathbf{n}\,dS$$

$$\mathbf{M} = -\rho\int\int_{S_b}\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi\right](\mathbf{r}\times\mathbf{n})\,dS$$

The following results from applying Gauss, Transport and divergence theorems and boundary conditions:

$$\mathbf{F} = -\rho\frac{d}{dt}\int\int_{S_b}\phi\,\mathbf{n}\,dS - \rho\int\int_{S_c}\left[\frac{\partial\phi}{\partial n}\nabla\phi - \mathbf{n}\frac{1}{2}\nabla\phi\cdot\nabla\phi\right]\,dS$$

$$\mathbf{M} = -\rho\frac{d}{dt}\int\int_{S_b}\phi\,(\mathbf{r}\times\mathbf{n})\,dS - \rho\int\int_{S_c}\mathbf{r}\times\left[\frac{\partial\phi}{\partial n}\nabla\phi - \mathbf{n}\frac{1}{2}\nabla\phi\cdot\nabla\phi\right]\,dS$$

An Application Using Complex Numbers

Cvla

Example of Programming with Complex Numbers

Conformal Mapping of a Circle into an Airfoil

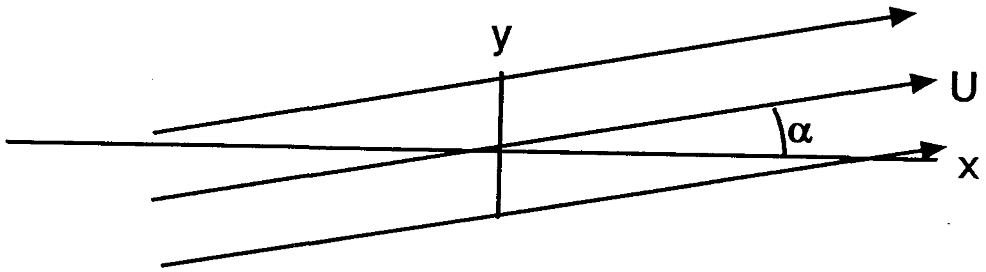
2D Flow: ϕ is velocity potential, ψ is stream function.

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Complex Numbers

$$z = x + iy \qquad \Phi = \phi + i\psi \qquad \frac{d\Phi}{dz} = u - iv$$

Simple Example



$$u = U\cos\alpha \qquad v = U\sin\alpha$$

$$\phi = Ux\cos\alpha + Uy\sin\alpha \qquad \psi = Uy\cos\alpha - Ux\sin\alpha$$

$$\Phi = \phi + i\psi = Ux\cos\alpha + Uy\sin\alpha + iUy\cos\alpha - iUx\sin\alpha$$

$$\frac{\partial \Phi}{\partial x} = U\cos\alpha - iU\sin\alpha = u - iv$$

$$\frac{\partial \Phi}{\partial (iy)} = \frac{1}{i}\frac{\partial \Phi}{\partial y} = -i\frac{\partial \Phi}{\partial y} = -iU\sin\alpha + U\cos\alpha = u - iv$$

Now we map a circle in the z-plane to an airfoil in the ζ-plane.

Streamlines in z-plane map into streamlines in ζ-plane.

The circle is a streamline in the z-plane and the airfoil is a streamline in the ζ-plane.

$$(u - iv)_\zeta = \frac{d\Phi}{d\zeta} = \frac{d\Phi/dz}{d\zeta/dz} = \frac{(u-iv)_z}{d\zeta/dz}$$

The Karman-Trefftz mapping function is:

$$\zeta = \lambda a \frac{(z+a)^\lambda + (z-a)^\lambda}{(z+a)^\lambda - (z-a)^\lambda}$$

λ and a are real numbers and $\lambda > 1$.

$$\frac{d\zeta}{dz} = 4\lambda^2 a^2 \frac{(z-a)^{\lambda-1}(z+a)^{\lambda-1}}{[(z+a)^\lambda - (z-a)^\lambda]^2}$$

For large z,

$$\zeta = \lambda a \frac{(z^\lambda + a\lambda z^{\lambda-1} + ...) + (z^\lambda - a\lambda z^{\lambda-1} + ...)}{(z^\lambda + a\lambda z^{\lambda-1} + ...) - (z^\lambda - a\lambda z^{\lambda-1} + ...)}$$

$$\zeta = \frac{\lambda a 2 z^\lambda + ...}{2\lambda z^{\lambda-1} a + ...} = z + ...$$

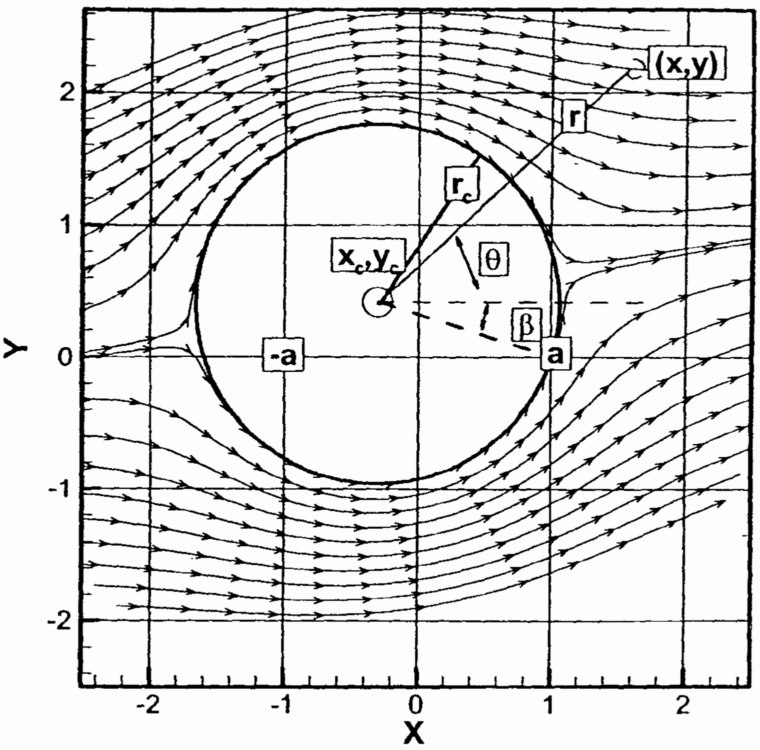
Far field flow in z-plane is equal to far field flow in ζ-plane.

$d\zeta/dz = 0$ at $z = a$ and at $z = -a$. If either of these points are in the flow field, $u - iv$ must equal zero there to avoid infinite velocity in ζ-plane.

Approach

Locate circle so that $z = -a$ is inside it.
Locate circle so that $z = a$ is on circle and $u - iv$ there is zero. $z = a$ maps into the trailing edge of the airfoil and since $d\zeta/dz = 0$ there it can be sharp.

cvic



Flow around a circle with zero circulation. The center of the circle is located at $x = -.3, y = 0.4$. The circle passes through $x = a = 1.0$. The flow angle of attack is 10 degrees.

The inflow angle is $\alpha = 10$ degrees, the circle radius is $r_c = \sqrt{1.3^2 + 0.4^2} = 1.3602$ and the flow is:

$$u = U\cos\alpha - U\left(\frac{r_c}{r}\right)^2 \cos(2\theta - \alpha)$$

$$v = U\sin\alpha - U\left(\frac{r_c}{r}\right)^2 \sin(2\theta - \alpha)$$

This flow is not zero at $z = a$.

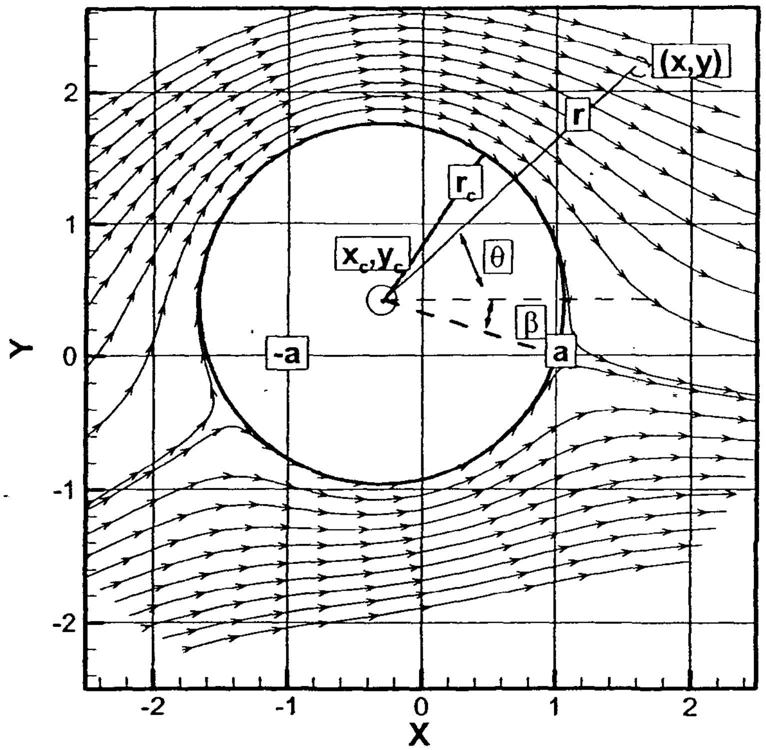
To make the flow zero at $z = a$ add circulation Γ

$$\Gamma = 4\pi r_c U \sin(-\beta - \alpha) \qquad\qquad \beta = \sin^{-1}\frac{y_c}{r_c}$$

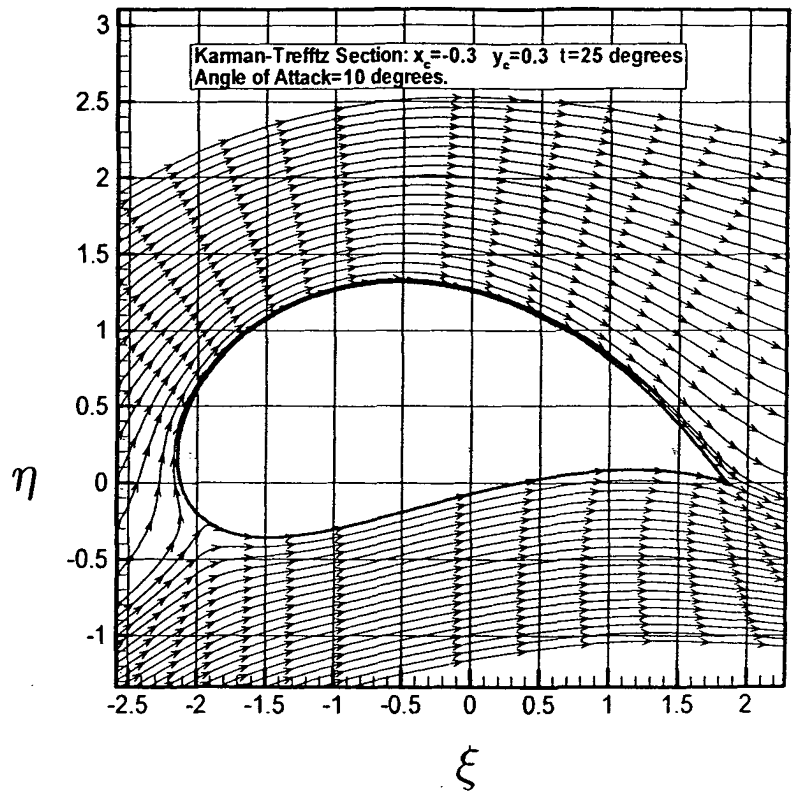
Then:

$$u = U\cos\alpha - U\left(\frac{r_c}{r}\right)^2 \cos(2\theta - \alpha) - \frac{\Gamma}{2\pi r}\sin\theta$$

$$v = U\sin\alpha - U\left(\frac{r_c}{r}\right)^2 \sin(2\theta - \alpha) + \frac{\Gamma}{2\pi r}\cos\theta$$



Flow around a circle with circulation. The center of the circle is located at $x = -.3, y = 0.4$. The circle passes through $x = a = 1.0$. Note that the rear stagnation point has moved to $x = a$.



The circle maps into an airfoil shape. The included angle , τ (in degrees) at the tail is:

$$\tau = 180(2 - \lambda)$$

The Pressure Distribution

$$P - P_\infty = \frac{1}{2}\rho U^2 - \frac{1}{2}\rho q^2 \qquad q^2 = u^2 + v^2$$

$$C_p = \frac{P - P_\infty}{1/2\rho U^2} = 1 - \left(\frac{q}{U}\right)^2$$

Procedure to Compute Pressure Coefficient

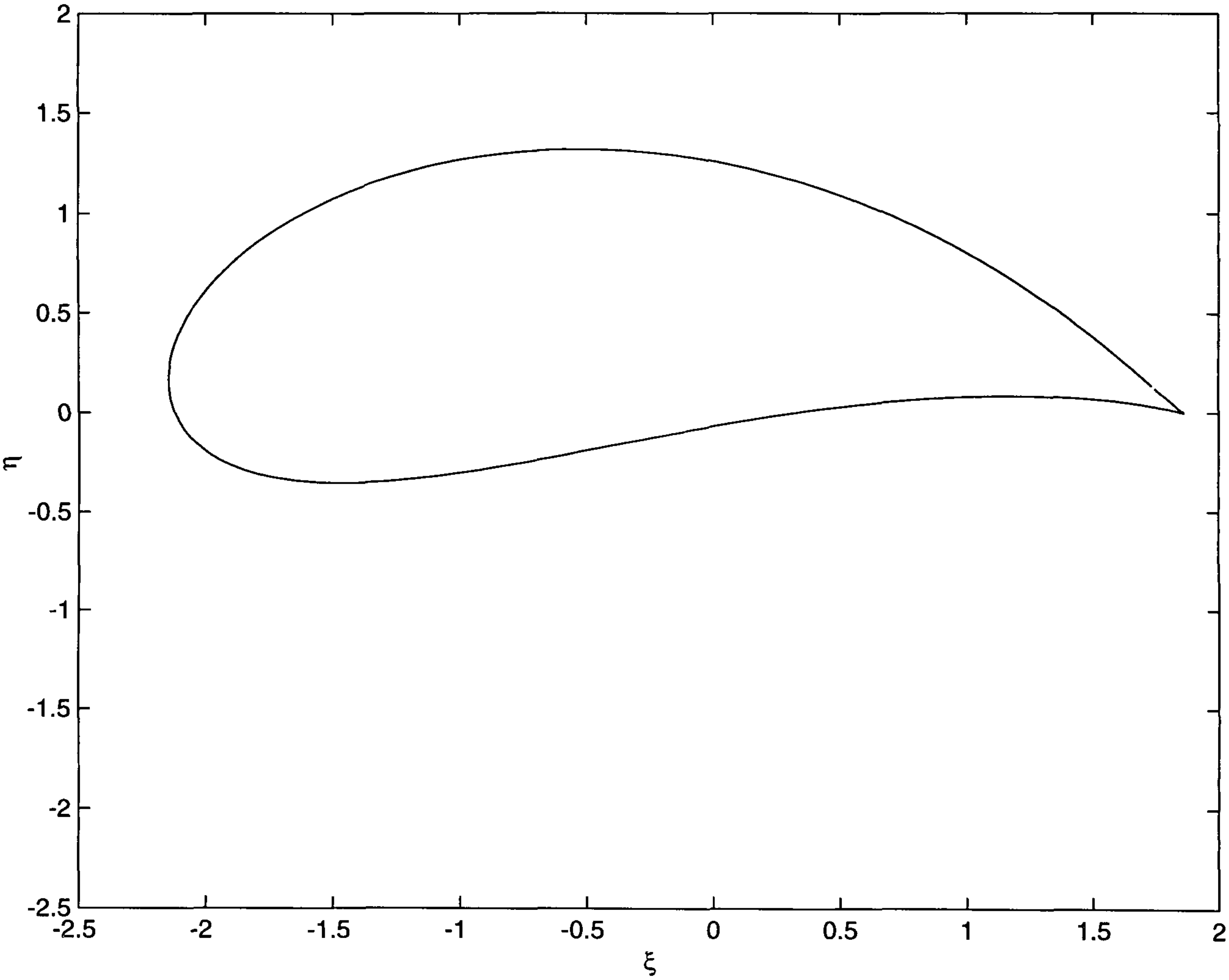
1. Make a sequence of points on the circle.
2. Determine value of z for each point.
3. Use complex number programming to determine the value of z and $d\zeta/dz$ for each point.
4. $(u - iv)_\zeta = (u - iv)_z / \frac{d\zeta}{dz}$.
5. $q^2 = (u - iv)_\zeta (u + iv)_\zeta$.
6. $C_p = 1 - (q/U)^2$.

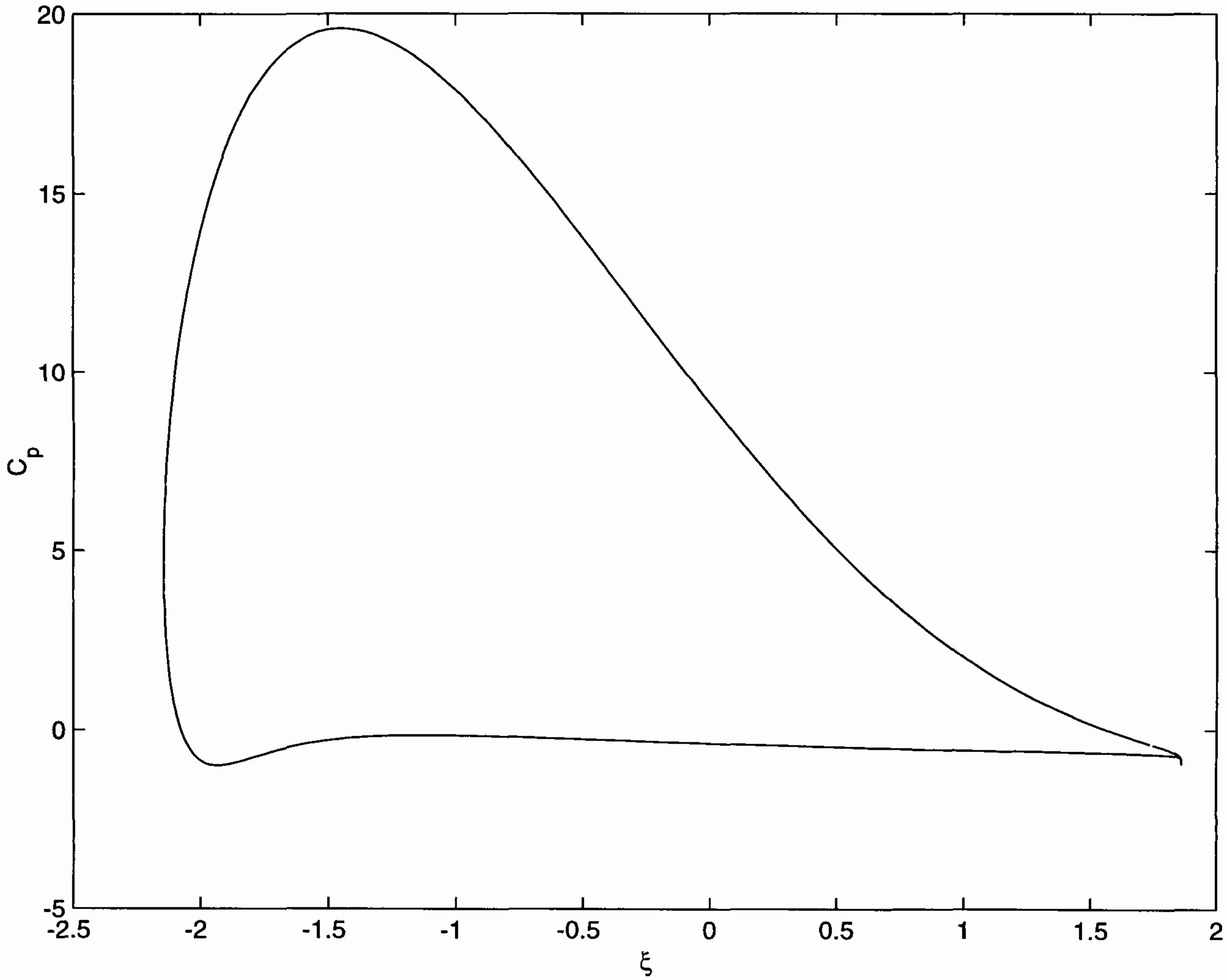
KTPROG(


```
% cp1 in matlab

a=1.0;
alpha=0.1745;
lambda=1.8611;
xc = -0.3;
yc =0.4;
UU=1.0;
gamma=-7.779695;
dpr=180./pi;
rc = sqrt((1.0-xc).^2 + yc .^2);
fid = fopen('cpm.dat','w');
degv = (1:1:360);
angv=degv ./dpr;
xv = xc + ( rc .* cos(angv));
yv = yc + ( rc .* sin(angv));
zv = xv + i*yv;
zetav=lambda*a*((zv + a) .^ lambda + (zv-a) .^lambda) ./ ...
    ((zv+a) .^ lambda - (zv-a) .^ lambda);
lm = lambda - 1.0;
dzetadzv = 4.0 * lambda ^2 * a ^2 * (zv-a) .^ lm .*(zv+a) .^lm ./ ...
    (((zv + a) * lambda - (zv -a) .^ lambda) .^ 2);
uv = (UU*cos(alpha)) - UU*cos(2.0 .* angv - alpha) - ...
    (gamma / (2.0*pi*rc)) * sin(angv);
vv = (UU * sin(alpha)) - UU*sin(2.0*angv - alpha) + ...
    (gamma/(2.0*pi*rc)) .* cos(angv);
wz = uv -i*vv;
wzeta = wz ./(dzetadzv + eps);
q = wzeta.*(conj(wzeta));
cp = 1.0 - q / (UU .^2);
cpm = -cp;
for m = 1:360
fprintf(fid,'%7.3f  %7.3f  %7.3f  %7.3f %7.3f  %7.3f  %7.3f\n', ...
  real(zetav(m)), imag(zetav(m)), cpm(m), real(zv(m)),imag(zv(m)),...
    real(wz(m)),imag(wz(m)));
end,
fclose(fid) ,
```

VT PROG'2




20
15
10
5
0
-5
C_p
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
ξ

rootFINDcov

Root Finding

Root Finding

Suppose you wish to find the wave number, k, of gravity water waves with a frequency, f, of 0.2 Hz. in water that is 5 meters deep. The circular frequency of 0.2 Hz. waves is $\omega = 2\pi f = 1.2566$ radians/second. The dispersion relation for gravity water waves is:

$$kg \tanh kh = \omega^2$$

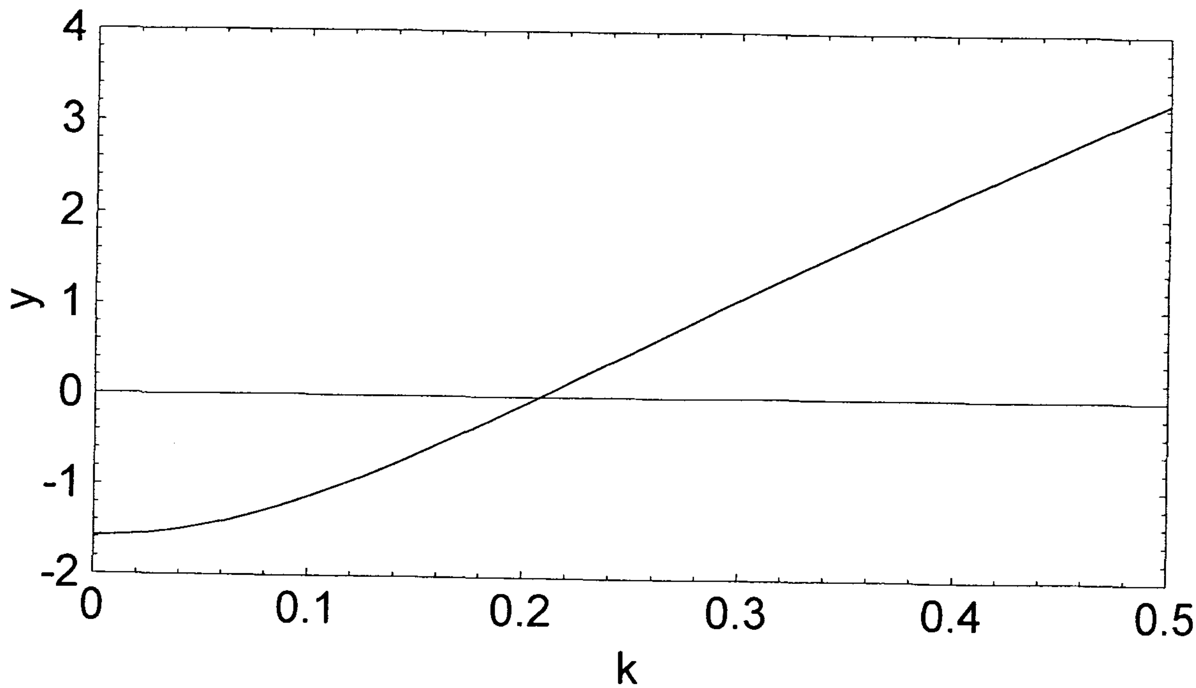
g is the acceleration of gravity, 9.81 m/s^2 and h is the water depth, 5 m.

This equation can be written as:

$$kg \tanh kh - \omega^2 = 0$$

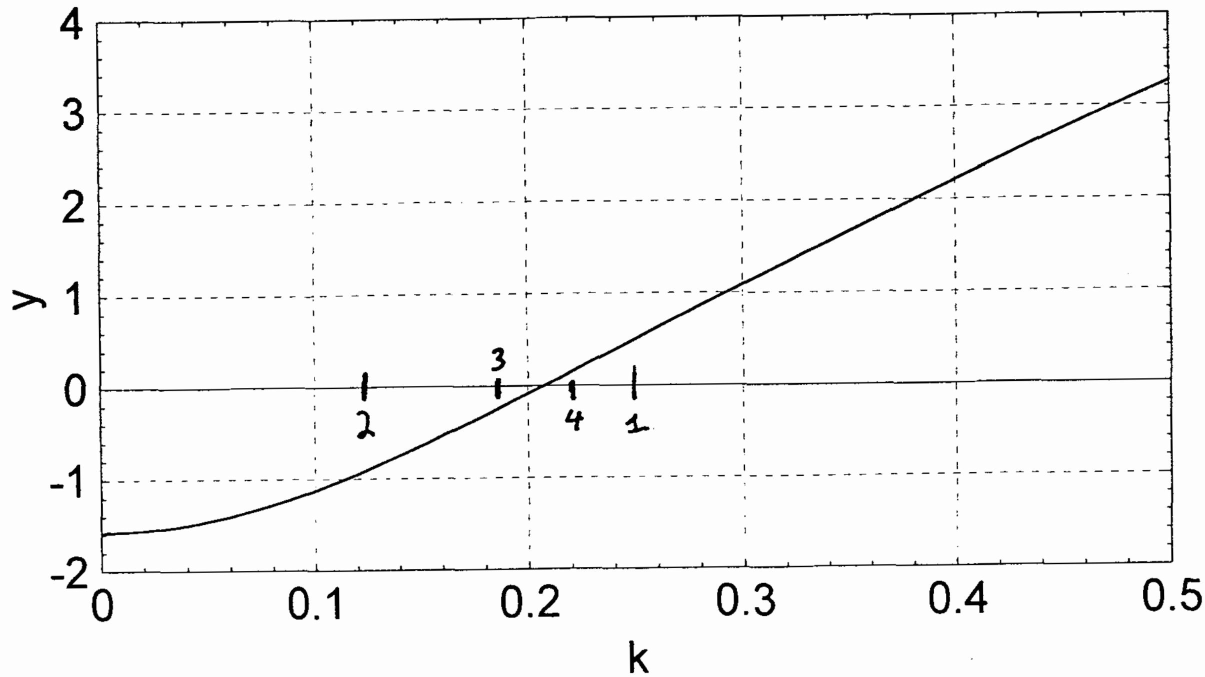
If we write an equation: $y(k) = kg \tanh kh - \omega^2$,

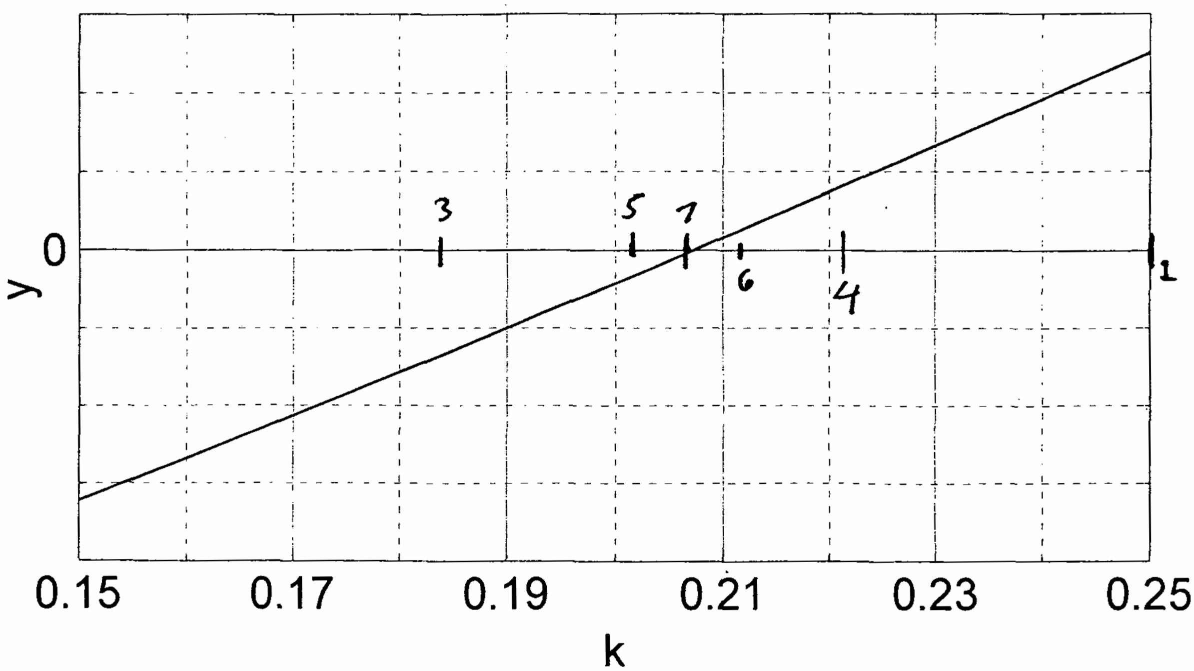
The problem at hand is the same as asking: "What is the value of k such that $y(k) = 0$? The value of a quantity that makes another equal to zero is called a root and the question above is called *Root Finding*.



rf2

Bisection Method



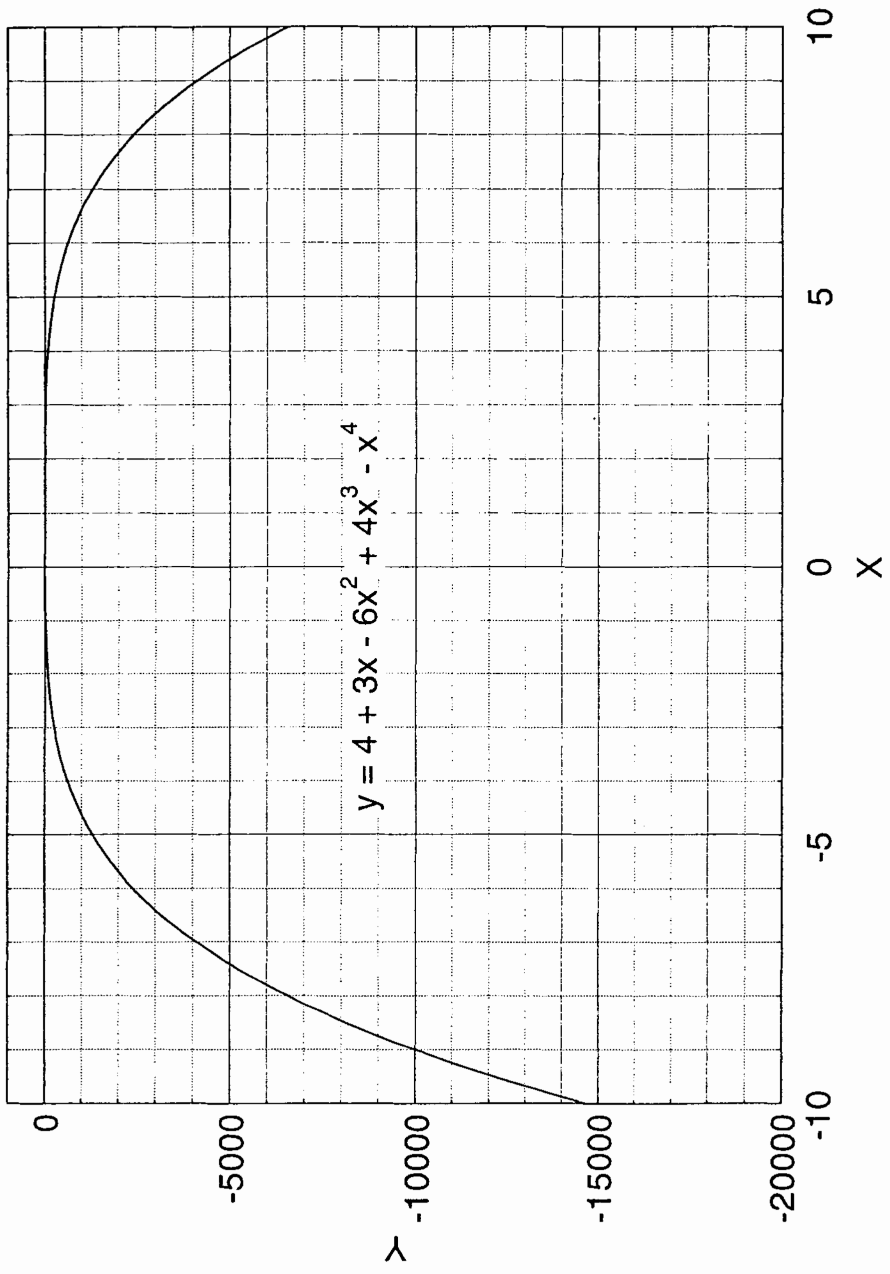


rf2-cont

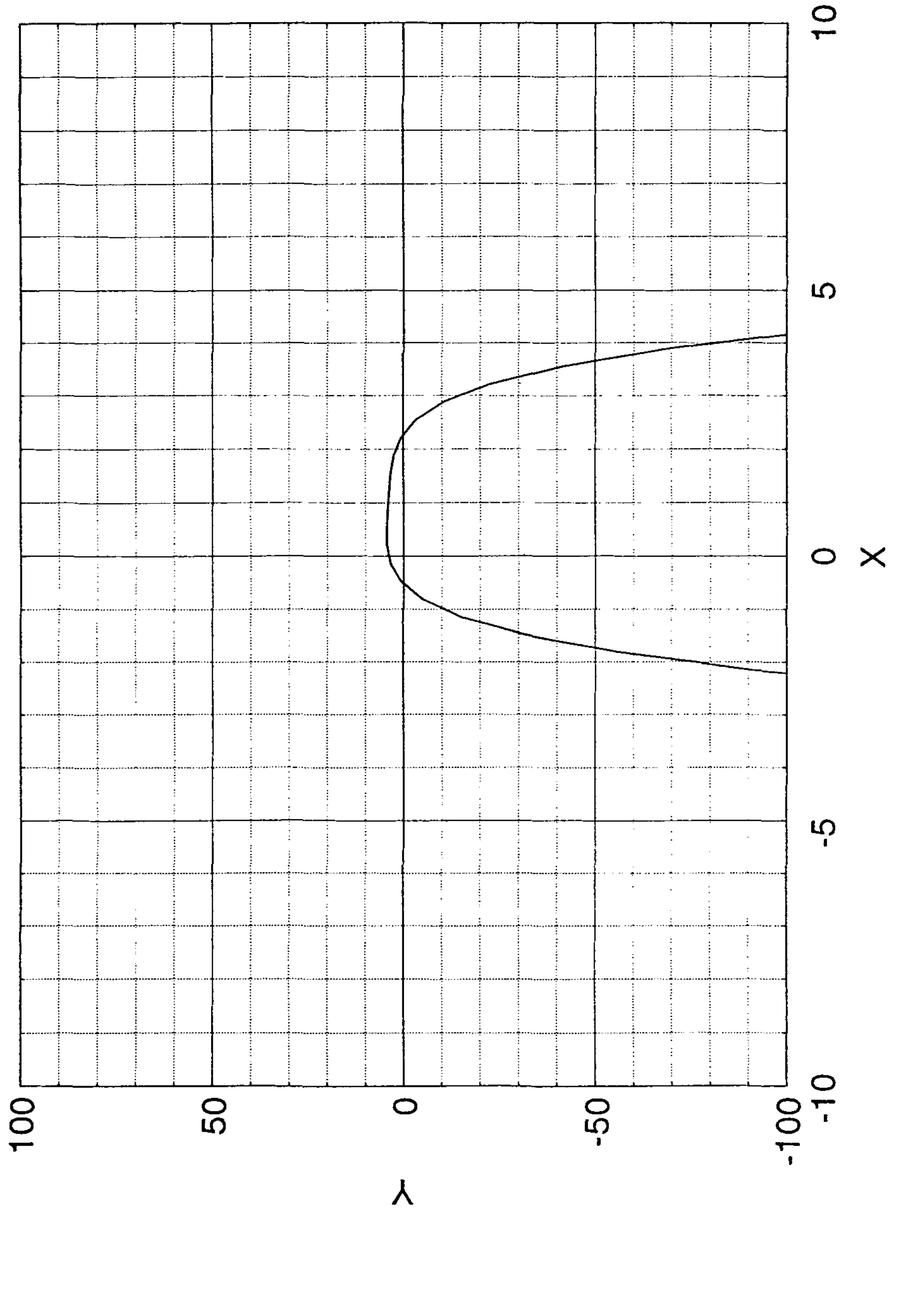
```
% biseck program to find k given omega
% using the bisection root finding method
om = 1.2566;
g = 9.81;
h = 5.0;
k1 = 0.0;
k2 = 0.5;
k3 = 0.25;
y = k3*g*tanh(k3*h) - om^2;
for m = 1:50
   y = k3*g*tanh(k3*h) - om^2;
  if (y*y < 1.0e-8);
      break
   end
   if(y >= 0.0);
      k2 = k3;
      k3 = 0.5*(k1+k2);
   elseif (y <= 0.0) ;
      k1 = k3;
      k3 = 0.5*(k1  + k2);
   else;
      fprintf(1,'there was no root')
   end;
end ;
fprintf(1,' k = %8.4f\n',k3);
fprintf(1,' Number of iterations =%3.0f\n',m);
```

```
>> biseck
 k =   0.2073
 Number of iterations = 15
>>
```

RF2a



RF2b



RF7C

$y=4+3x-6x^2+4x^3-x^4$

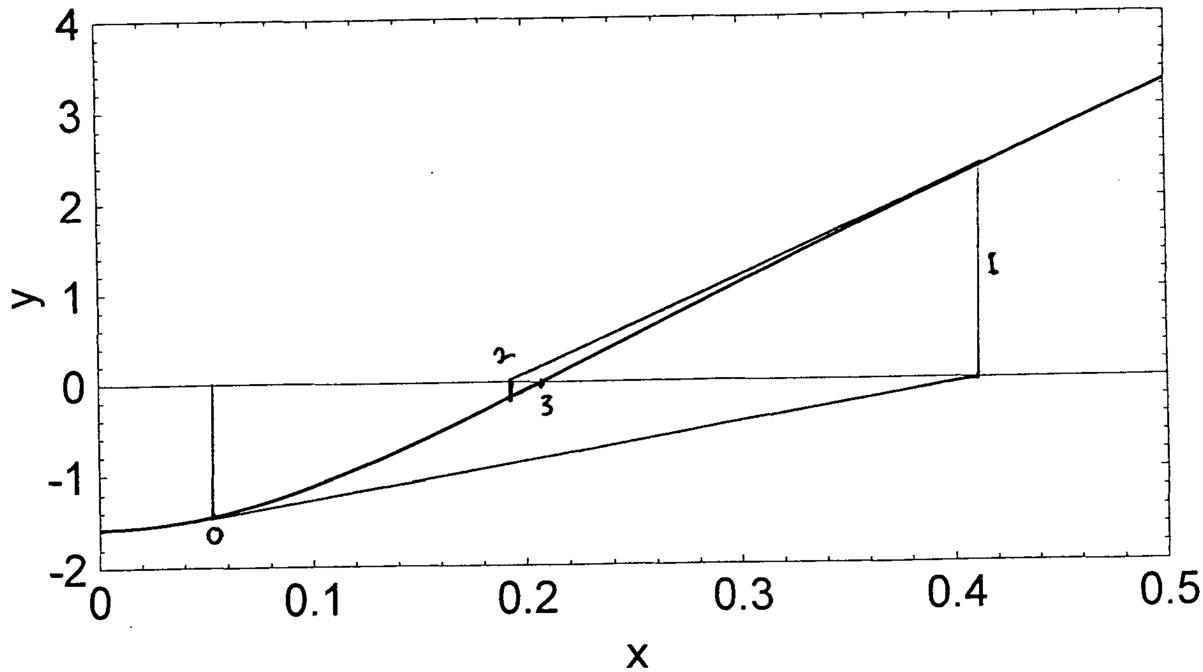
1.00000	4.000000000	0.00000	4.000000000
2.00000	2.000000000	-1.00000	-10.000000000
3.00000	-14.000000000	-0.50000	0.437500000
2.50000	-2.562500000	-0.75000	-3.628906250
2.25000	0.308593750	-0.62500	-1.347900391
2.37500	-0.949462891	-0.56250	-0.397964478
2.31250	-0.280044556	-0.53125	0.033507347
2.28130	0.023423625	-0.54688	-0.178792423
2.29690	-0.125854491	-0.53906	-0.071706616
2.28910	-0.050608813	-0.53516	-0.018951368
2.28520	-0.013441856	-0.53320	0.007398979
2.28325	0.005028455	-0.53418	-0.005762632
2.28423	-0.004244654	-0.53369	0.000821562
2.28374	0.000394274	-0.53345	0.004044001
2.28398	-0.001877260	-0.53381	-0.000790267
2.28386	-0.000741351	-0.53363	0.001627324
2.28380	-0.000173503	-0.53354	0.002835777
2.28377	0.000110395	-0.53368	0.000955863
2.28379	-0.000078868	-0.53374	0.000150016
2.28378	0.000015764	-0.53378	-0.000387271
		-0.53376	-0.000118622
		-0.53375	0.000015699

rf3

Newton's Method for Finding Roots of $y(x)$

The approach taken in Newton's method is to take an estimate of the location of a root of $y(x)$ and then improve upon it. Thus, it is iterative, starting with a "guess" for x with each successive iteration being the result of the last iteration. The basic formula for each iteration is:

$$x_i = x_{i-1} - \frac{y_{i-1}}{y'_{i-1}} \qquad \text{where} \quad y'_{i-1} = \left|\frac{dy}{dx}\right|_{x=x_{i-1}}$$



RF3a

$y = 4 + 3x - 6x^2 + 4x^3 - x^4$ $y' = 3 - 12x + 12x^2 - 4x^3$

x	y	dy/dx	(y)/(dy/dx)
1.00000000	4.000000000	-1.000000000	-4
5.00000000	-256.000000000	-257.000000000	0.9961089
4.00389105	-80.424942796	-109.420778749	0.7350061
3.26888492	-24.769129200	-47.719415341	0.5190577
2.74982724	-7.125030491	-22.431151690	0.31764
2.43218726	-1.639446076	-12.750582959	0.1285781
2.30360914	-0.191558554	-9.861396677	0.0194251
2.28418404	-0.003809366	-9.471138816	0.0004022
2.28378183	-0.000001600	-9.463181800	1.691E-07
2.28378167	0.000000000	-9.463178456	3.003E-14

Review of Matrix Algebra

An m x n matrix (m rows, n columns) ia said to be of order $m \times n$ and is written symbolically as:

$$\mathbf{A} = \underline{a} = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix}$$

Each a_{ij} represents a numerical value. If the a_{ij}'s are real numbers, the matrix is called a *real matrix.* If the a_{ij}'s are complex numbers, the matrix is called a *complex matrix.*

The matrix is called *square* if $m = n$. Matrices **A** and **B** are called *equal* if $a_{ij} = b_{ij}$ for all i and j and they have the same number of rows and the same number of columns.

If **A** and **B** are both order $m \times n$ matrices, the matrix $\mathbf{C} = \mathbf{A} + \mathbf{B}$ is defined by the relations:

$$c_{ij} = a_{ij} + b_{ij}$$

The matrix $\mathbf{D} = \gamma\mathbf{A}$ is defined by the relations:

$$d_{ij} = \gamma a_{ij}$$

An important relation is the the *matrix product*, of two matrices **A** $(m \times n)$ and **B** $(n \times p)$ which is denoted by $\mathbf{C} = \mathbf{AB}$ whose elements are defined by:

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}, \qquad i = 1, 2, \cdots, m, \qquad j = 1, 2, \cdots, p$$

When **A** and **B** are square and of the same order, both **AB** and **BA** are defined, but except under special circumstances, $\mathbf{AB} \neq \mathbf{BA}$.

Mab2

An $m \times 1$ matrix (m rows, 1 column) is called a *column vector* or a *vector* and is written symbolically as:

$$\mathbf{x} = \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_m \end{bmatrix}$$

Each x_i represents a numerical value.

The standard form for a set of linear equations is:

$$\mathbf{Ax} = \mathbf{b}$$

where $\mathbf{A}$ is an $m \times n$ matrix called the *coefficient matrix* and $\mathbf{x}$ is an *unknown vector* of length n. $\mathbf{b}$ is a known vector of length n.

Let $\mathbf{a}_\mathrm{j}$ denote the j^{th} column of $\mathbf{A}$. Then the set of equations can be written as:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdot\cdot\cdot + x_n\mathbf{a}_n = \mathbf{b}$$

The vectors $\mathbf{a}_j$, $j = 1, 2, ..., n$ are *linearly dependent* if there is a set of numbers $x_1, x_2, ...x_n$, with at least one x_j being non zero such that:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdot\cdot\cdot + x_n\mathbf{a}_n = 0$$

In this instance, at least one $\mathbf{a}_j$ is a linear combination of the remaining $\mathbf{a}_j$'s.

The vectors $\mathbf{a}_j$, $j = 1, 2, ..., n$ are *linearly independent* if they are not linearly dependent.

If each of the m linear equations is *independent* then there is an exact solution if $m = n$. If $m > n$, there are more equations than unknowns and there is no exact solution. Rather, there is an approximate solution for $\mathbf{x}$ which is usually chosen to achieve minimum sum of the squared errors from each equation.

Example:

$$3x_1 + x_2 = 5$$
$$x_1 + 4x_2 = -3$$
$$-2x_1 + 3x_2 = -6$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -6 \end{bmatrix}$$

Approximate Solution is: $x_1 = y_1$, $x_2 = y_2$: $\underline{x} = \underline{y}$

Need to find $\underline{y}$. Error Vector is $\underline{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$

$$e_1 = 5 - 3y_1 - y_2$$
$$e_2 = -3 - y_1 - 4y_2$$
$$e_3 = -6 + 2y_1 - 3y_2$$

sum of squares of errors

$$E = e_1^2 + e_2^2 + e_3^2$$

$$E = (5 - 3y_1 - y_2)^2 + (-3 - y_1 - 4y_2)^2 + (-6 + 2y_1 - 3y_2)^2$$

We seek $\underline{y}$ such that $\frac{\partial E}{\partial y_1} = 0$ and $\frac{\partial E}{\partial y_2} = 0$

which is the same as; $\frac{1}{2}\frac{\partial E}{\partial y_1} = 0$ and $\frac{1}{2}\frac{\partial E}{\partial y_2} = 0$

$$\frac{1}{2}\frac{\partial E}{\partial y_1} = -3(5 - 3y_1 - y_2) - 1(-3 - y_1 - 4y_2) + 2(-6 + 2y_1 - 3y_2)$$

$$\begin{aligned} = \; & -15 + 9y_1 + 3y_2 \\ & 3 + y_1 + 4y_2 \\ & -12 + 4y_1 - 6y_2 \\ \hline = \; & -24 + 14y_1 + y_2 \end{aligned}$$

$$\frac{1}{2}\frac{\partial E}{\partial y_2} = 25 - y_1 + 26y_2$$

$$0 = -24 + 14y_1 + y_2 \qquad 14y_1 + y_2 = 24$$

$$0 = 25 + y_1 + 26y_2 \qquad y_1 + 26y_2 = -25$$

$$\begin{bmatrix} 14 & 1 \\ 1 & 26 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 24 \\ -25 \end{bmatrix}$$

$$y_1 = 1.7879, \qquad y_2 = -1.0303$$

$$e_1 = 5 - 5.3637 + 1.0303 = 0.6667$$

$$e_2 = -3 - 1.7879 + 4.1212 = -0.6667$$

$$e_3 = -6 + 3.5758 - 3.0909 = 0.6667$$

Consider a square $n \times n$ matrix, $\mathbf{A}$. It is called the *identity matrix of order* n, $\mathbf{I}_n$ (or simply $\mathbf{I}$) if

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

This is often written as: $a_{ij} = \delta_{ij}$

If $\mathbf{B}$ is an $n \times p$ matrix,

$$\mathbf{I_n B = B} \qquad \textbf{and} \qquad \mathbf{B I_n = B}$$

The j^{th} column of $\mathbf{I}$ is called the j^{th} *unit vector* and denoted by $\mathbf{e}_j$. Any n-vector $\mathbf{b}$ can be written as:

$$\mathbf{b} = \sum_{j=1}^{n} b_j \mathbf{e}_j$$

The j^{th} column of a matrix $\mathbf{B}$ is given by $\mathbf{Be}_j$. Therefore, if $\mathbf{C = AB}$, the j^{th} column of $\mathbf{C}$, called $\mathbf{c}_j$ is obtained as:

$$\mathbf{c}_j = \mathbf{Ce}_j = (\mathbf{AB})\mathbf{e}_j = \mathbf{A}(\mathbf{Be}_j) = \mathbf{Ab}_j$$

A collection V of linearly independent vectors in R^n (this means that the vectors are $1 \times n$) is called a *basis* for R^n if every n-vector can be written as a linear combination of the vectors in V. Obviously, the columns of $\mathbf{I}_n$, which are the $\mathbf{e}_j$'s form a basis for R^n. However, this is not the only set V of basis vectors. Any basis in R^n contains exactly n vectors.

THEOREM: The linear system $\mathbf{Ax = b}$ has a unique solution if and only if the only solution to $\mathbf{Ax = 0}$ is $\mathbf{x = 0}$.

THEOREM: If the homogeneous linear system $\mathbf{Ax = 0}$ has fewer equations than unknowns, it has nonzero (non trivial) solutions.

THEOREM: $\mathbf{A}$ is an $m \times n$ matrix. If the linear system $\mathbf{Ax = b}$ has a solution for every m-vector $\mathbf{b}$, then $m \leq n$.

Consider a square $n \times n$ matrix, $\mathbf{A}$. If there is a square $n \times n$ matrix, $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$, $\mathbf{B}$ is called the *inverse* of $\mathbf{A}$ and denoted by $\mathbf{A}^{-1}$. If $\mathbf{A}$ has an inverse $\mathbf{A}$ is called *nonsingular* and if it does not have an inverse, $\mathbf{A}$ is called *singular.*

Fact: If $\mathbf{A}$ and $\mathbf{B}$ are invertible, then, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1}$

Suppose there is a linear system of equations $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is invertible and $\mathbf{b}$ is a known vector and $\mathbf{x}$ is an unknown vector to be determined. Pre-multiplying the equation by $\mathbf{A}^{-1}$ gives:

$$\mathbf{A}^{-1}\,\mathbf{A}\,\mathbf{x} = \mathbf{A}^{-1}\,\mathbf{b}$$

$$\mathbf{x} = \mathbf{A}^{-1}\,\mathbf{b}$$

Thus, if the inverse of $\mathbf{A}$ is determined, the solution to the system of equations can be obtained by straightforward matrix multiplication. This is not a numerically efficient way to solve sets of linear equations, but it demonstrates theoretically that a solution exists of the *coefficient matrix* is nonsingular.

The principle of obtaining the inverse of a matrix can be demonstrated as follows: An $n \times n$ matrix $\mathbf{A}$ exists and the goal to to determine its inverse $\mathbf{B} \equiv \mathbf{A}^{-1}$ whose j^{th} column is called $\mathbf{b}_j$. $\mathbf{A}\,\mathbf{b}_j = \mathbf{e}_j$ is a system of n linear equations for the n elements of $\mathbf{b}_j$. Finding $\mathbf{A}^{-1}$ in this way requires solving a set of n equations for each of the n column vectors $\mathbf{b}_j$.

A more computationally efficient way to find the inverse of a nonsingular matrix will be shown subsequently.

Determinant of a Matrix

A matrix **A** has a *determinant* which is denoted by det(**A**) or |**A**|. The determinant of a matrix plays a large theoretical role in linear algebra and a practical role in determining the inverse of a matrix. The determinant is defined as the sum of all signed elementary products from the matrix. An elementary product is the product of n elements all of which are from different rows and different columns. The sign is + if the number of inversions of the indices is even and − if the number of invertsions is odd.

$$|A| = \sum_{j,k,l,\dots,q=1}^{n} (-1)^i a_{1j}\, a_{2k}\, a_{3l} \dots a_{nq}$$

where $j, k, l, ..., q$ are all different and i is the number of inversions in the sequence $j, j, l, ..., q$.

A recursive definition of $|A|$ which includes a "prescription" of how to calculate it is as follows:

1. If **R** is a 1×1 matrix, $\mathbf{R} = [r]$, $\det(\mathbf{R}) \equiv r$.
2. $\mathbf{A}_{ij}$ is the *submatrix* of **A** obtained by deleting the i^{th} row and j^{th} column of **A**. The *minor* m_{ij} (associated with the matrix **A**) is the determinant of $\mathbf{A}_{ij}$
$$m_{ij} = \det(\mathbf{A}_{ij})$$
3. The *cofactor* c_{ij} is defined by $c_{ij} \equiv (-1)^{i+j}\, m_{ij}$
4. $$\det(\mathbf{A}) \equiv \sum_{j=1}^{n} a_{ij}\, c_{ij} \qquad \text{for any } i = 1, 2, ..., n, \text{ or}$$
$$\det(\mathbf{A}) \equiv \sum_{i=1}^{n} a_{ij}\, c_{ij} \qquad \text{for any } j = 1, 2, ..., n$$

- Adding a constant times one row of a matrix to another row does not change the determinant.
- Adding a constant times one column of a matrix to another column does not does not change the determinant.

Transpose of a Matrix

The transpose of a matrix $\mathbf{A}$ is called $\mathbf{A}^T$ and is obtained by making each row of $\mathbf{A}^T$ the corresponding column of $\mathbf{A}$. For example, if we define $\mathbf{C} \equiv \mathbf{A}^T$,

$$c_{ij} = a_{ji}$$

Calculating the Inverse of a Matrix

Consider a matrix $\mathbf{A}$. It's cofactors are c_{ij}. The matrix $\mathbf{C}$ is the matrix whose ij element is c_{ij}.

The matrix $\mathbf{C}^T$ is called the *adjugate* or the *adjoint* of $\mathbf{A}$ and is denoted by $\mathrm{adj}(\mathbf{A})$.

The inverse a $\mathbf{A}$ is given by:

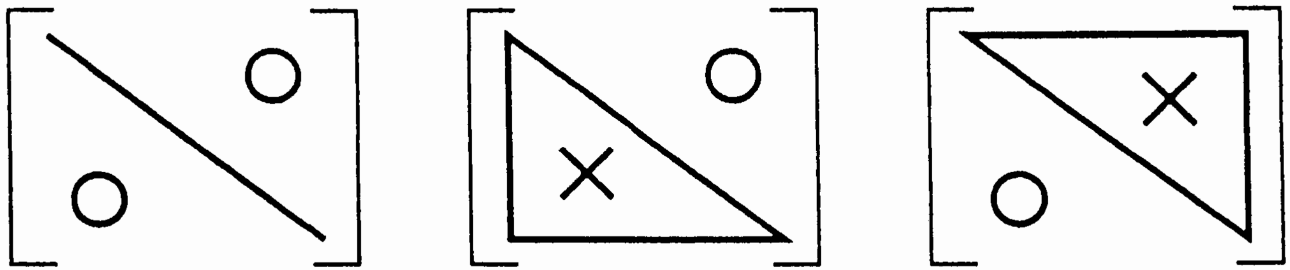
$$A^{-1} = \frac{\mathrm{adj}(\mathbf{A})}{\det(\mathbf{A})}$$

Cramer's Rule

Consider the system of linear equations, $\mathbf{A}\,\mathbf{x} = \mathbf{b}$. Define $\mathbf{A}^j$ as the matrix formed by replacing the j^{th} column of $\mathbf{A}$ by the column vector $\mathbf{b}$.

$$\text{Cramer's Rule is:} \quad x_j = \frac{|\mathbf{A}^j|}{|\mathbf{A}|}$$

mab 6



Diagonal | Lower triangular | Upper triangular

The X's and the straight lines denote nonzero elements
and the O's denote zero elements.

Special matrices.

m 96 >

Matrix Norms

First we define vector norms $N(\mathbf{x})$ of the vector $\mathbf{x}$ in n dimensional space..
A vector norm has the following properties:

1. $N(\mathbf{x}) \geq 0$ for all n-vectors $\mathbf{x}$.
2. $N(\mathbf{x}) = 0$ if and only if $\mathbf{x} = 0$.
3. $N(\alpha\mathbf{x}) = |\alpha| N(\mathbf{x})$ for all real α and n-vectors $\mathbf{x}$.
4. $N(\mathbf{x}+\mathbf{y}) \leq N(\mathbf{x}) + N(\mathbf{y})$ for all n-vectors $\mathbf{x}$ and $\mathbf{y}$.

Norms are denoted by $||\cdot||$.

$$||\mathbf{x}||_1 = \sum_{i=1}^{n} |x_i| \, , \qquad ||\mathbf{x}||_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \, , \qquad ||x||_\infty = \max_{1 \leq i \leq n} |x_i|$$

An *operator norm* $N(\mathbf{A})$ of a real valued $n \times n$ matrix $\mathbf{A}$ is a real valued function having the following properties:

1. $N(\mathbf{A}) \geq 0$.
2. $N(\mathbf{A}) = 0$ if and only if all the elements of $\mathbf{A}$ are zero.
3. $N(\alpha\mathbf{A}) = |\alpha| \, N(\mathbf{A})$ for all real α.
4. $N(\mathbf{A}+\mathbf{B}) \leq N(\mathbf{A}) + N(\mathbf{B})$.
5. $N(\mathbf{AB}) \leq N(\mathbf{A}) \, N(\mathbf{B})$.

There are a number of equivalent definitions for the v-norm of $\mathbf{A}$, $N(\mathbf{A}) = ||\mathbf{A}||_v$. One of them is:

For all n-vectors $\mathbf{z}$ such that $||\mathbf{z}||_v \leq 1$,

$$||\mathbf{A}||_v = \max_{||\mathbf{z} \leq 1||} ||\mathbf{A}\mathbf{z}||_v$$

MAB8

The Condition Number of A Matrix

Consider a set of linear equations, $\mathbf{Ax} = \mathbf{b}$ which is to be solved numerically. The exact solution is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. How sensitive is the accuracy of the solution to numerical errors? After obtaining an approximate numerical solution $\hat{\mathbf{x}}$, we can always compute the residual $\mathbf{r}$ as:

$$\mathbf{r} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}}$$

We would like to find some relationship between the computable residual $\mathbf{r}$ and the error $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$.

$$\mathbf{r} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}} = \mathbf{Ax} - \mathbf{A}\hat{\mathbf{x}} = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) = \mathbf{Ae}$$

$$\text{Since } ||\mathbf{r}|| \leq ||\mathbf{A}||\,||\mathbf{e}||, \qquad ||\mathbf{e}|| \geq \frac{||\mathbf{r}||}{||\mathbf{A}||}$$

$$\text{Then, since } ||\mathbf{x}|| \leq ||\mathbf{A}^{-1}||\,||\mathbf{b}|| \qquad \frac{||\mathbf{e}||}{||\mathbf{x}||} \geq \frac{1}{||\mathbf{A}||\,||\mathbf{A}^{-1}||}\frac{||\mathbf{r}||}{||\mathbf{b}||}$$

$$\text{Now we use: } ||\mathbf{e}|| \leq ||\mathbf{A}^{-1}||\,||\mathbf{r}|| \quad \text{and} \quad ||\mathbf{b}|| \leq ||\mathbf{A}||\,||\mathbf{x}||$$

$$\text{These give: } \frac{||\mathbf{e}||}{||\mathbf{x}||} \leq ||\mathbf{A}||\,||\mathbf{A}^{-1}||\,\frac{||\mathbf{r}||}{||\mathbf{b}||}$$

$$\text{Hence: } \frac{1}{||\mathbf{A}||\,||\mathbf{A}^{-1}||}\frac{||\mathbf{r}||}{||\mathbf{b}||} \leq \frac{||\mathbf{e}||}{||\mathbf{x}||} \leq ||\mathbf{A}||\,||\mathbf{A}^{-1}||\,\frac{||\mathbf{r}||}{||\mathbf{b}||}$$

$$\frac{1}{||\mathbf{A}||\,||\mathbf{A}^{-1}||}\frac{||\mathbf{r}||}{||\mathbf{b}||} \leq \frac{||\mathbf{e}||}{||\mathbf{x}||} \leq ||\mathbf{A}||\,||\mathbf{A}^{-1}||\,\frac{||\mathbf{r}||}{||\mathbf{b}||}$$

The *condition number* of an $n \times n$ matrix, $\mathbf{A}$, with respect to the operator norm $||\cdot||$ is called $\kappa(\mathbf{A})$ and defined by:

$$\kappa(\mathbf{A}) = ||\mathbf{A}||\ ||\mathbf{A}^{-1}||$$

The condition number multiplied by the norm of the relative residual is an upper bound on the norm of the relative error. $1/\kappa(\mathbf{A})$ multiplied by the relative residual is a lower bound on the norm of the relative error.

For a large condition number, the relative residual is a poor indicator of the relative error. For $\kappa \approx 1$, the relative residual is a good measure of the relative error.

GAUSSIAN ELIMINATION

Triangular Systems

$$
\begin{aligned}
u_{11}x_1 + u_{12}x_2 + \cdots + u_{1,n-1}x_{n-1} + u_{1n}x_n &= f_1 \\
u_{22}x_2 + \cdots + u_{2,n-1}x_{n-1} + u_{2n}x_n &= f_2 \\
\ddots \qquad\qquad &\vdots \\
u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n &= f_{n-1} \\
u_{nn}x_n &= f_n
\end{aligned}
$$

back substitution

Solve the following upper triangular system by using back substitution.

$$
\begin{aligned}
2x_1 - 5x_2 + x_3 &= 3 \\
3x_2 - x_3 &= 7 \\
4x_3 &= 8
\end{aligned}
$$

Gaussian Elimination. This general solution technique is based on the following basic properties of linear systems:

1. Multiplying an equation by a constant does not alter the solution to the system.
2. Replacing an equation by a linear combination of itself with some other equations (one or more) in the system does not alter the solution to the system.
3. Interchanging the order of equations in the system does not affect the solution to the system.

These *Elementary Row Operations* are used to form an equivalent triangular system of equations from an original system of equations to be solved.

ge2

$$
\begin{aligned}
3x_1 - x_2 + 2x_3 &= -3 \\
x_1 + x_2 + x_3 &= -4 \\
2x_1 + x_2 - x_3 &= -3
\end{aligned}
$$

Use 1^{st} equation to eliminate x_1 from 2^{nd} and 3^{rd} equation

$$
\begin{aligned}
3x_1 - x_2 + 2x_3 &= -3 \\
\frac{4}{3}x_2 + \frac{1}{3}x_3 &= -3 \\
\frac{5}{3}x_2 - \frac{7}{3}x_3 &= -1
\end{aligned}
$$

Use 2^{nd} equation to eliminate x_2 from 3^{rd} equation

$$
\begin{aligned}
3x_1 - x_2 + 2x_3 &= -3 \\
\frac{4}{3}x_2 + \frac{1}{3}x_3 &= -3 \\
-\frac{11}{4}x_3 &= \frac{11}{4}
\end{aligned}
$$

Gaussian Elimination Operation Count for n Equations

The number of multiplications and divisions is called $\mathcal{M}$.

The number of additions and subtractions is called $\mathcal{A}$.

$$\mathcal{M} = \frac{n^3}{3} + n^2 - \frac{n}{3}$$

$$\mathcal{A} = \frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}$$

Errors in Numerical Solutions of Sets of Linear Equations

When terms that are subtracted from each other in a solution method have nearly the same magnitude, computational round off errors can result in large relative errors in the solution.

Computational errors are generally reduced if each equation is scaled as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

Let $$s_i = \max_j |a_{ij}|, \quad i = 1, 2, ..., n$$

Divide the i^{th} row of the **A** and **b** matrices by s_i.
This is called *scaling.*

In a Gaussian Elimination the element that is used to eliminate its column in the equations that do not contain it is called the *pivot element*

Errors in numerical solutions are generally reduced if pairs of equations are interchanged so the magnitude of the pivot element is the largest one possible.

Scaled Partial Pivoting Rule

Both of the above error-reduction steps can be incorporated in what is called the *Scaled Partial Pivoting Rule.*

1. Start by determining s_i for each row as explained above.
2. At the start of the k';th elimination step, scan the k^{th} column of **A** and determine the integer p such that:

$$\frac{|a_{pk}|}{s_p} \geq \frac{|a_{lk}|}{s_l}, \qquad l = k, k+1, ...n$$

3. If $p \neq k$, then interchange rows p and k.

This procedure removes the need to do the scaling explicitly which can be another source of round off error.

Scaling

$$\begin{bmatrix} 1 & 3 & 4 & 6 \\ 7 & 2 & 3 & 2 \\ 4 & 3 & 8 & 1 \\ 5 & 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 3 \\ 2 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1/6 & \frac{1}{2} & \frac{2}{3} & 1 \\ 1 & 2/7 & 3/7 & 2/7 \\ \frac{1}{2} & 3/8 & 1 & 1/8 \\ 1 & 1/5 & -2/5 & 3/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 8/7 \\ 3/8 \\ 2/5 \end{bmatrix}$$

Pivoting

$$\begin{bmatrix} 1 & 3 & 4 & 6 \\ 7 & 2 & 3 & 2 \\ 4 & 3 & 8 & 1 \\ 5 & 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 3 \\ 7 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 7 & 2 & 3 & 2 \\ 1 & 3 & 4 & 6 \\ 4 & 3 & 8 & 1 \\ 5 & 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 3 \\ 7 \end{bmatrix}$$

Scaled Partial Pivoting

$$\begin{array}{cc} a_{p1}/S_p & S_i \\ 1/6 & 6 \\ 1/3 & 6 \\ 2/3 & 6 \\ 1 & 4 \end{array} \quad \begin{bmatrix} 1 & 3 & 4 & 6 \\ 2 & 6 & 3 & 2 \\ 4 & 3 & 6 & 1 \\ 4 & 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Interchange 1^{st} and 4^{th} rows (equations)

SLES

Solution of Linear Equations by LU Decomposition

Equation to Solve: $\mathbf{Ax} = \mathbf{b}$

$\mathbf{A}$ is presumed to be non-singular. Suppose we can decompose $\mathbf{A}$ into $\mathbf{A} = \mathbf{LU}$ where $\mathbf{L}$ is lower triangular with diagonal elements equal to 1 and $\mathbf{U}$ is upper triangular. Then the solution is straightforward.

$$\mathbf{LUx} = \mathbf{b}$$

Define: $\mathbf{y} \equiv \mathbf{Ux}$ $\qquad \mathbf{Ly} = \mathbf{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$y_1 = b_1$$

$$L_{21}y_1 + y_2 = b_2 \qquad y_2 = b_2 - L_{21}b_1$$

$$L_{31}y_1 + L_{32}y_2 + y_3 = b_3 \qquad y_3 = b_3 - L_{31}y_1 - L_{32}y_2$$

$$\mathbf{Ux} = \mathbf{y}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_3 = y_3/u_{33}$$

$$x_2 = (y_2 - u_{23}x_3)/u_{22}$$

$$x_1 = (y_1 - u_{12}x_2 - u_{13}x_3)/u_{11}$$

Procedure for Factorization of A

$\mathbf{A}$ is a nonsingular $n \times n$ matrix. $\mathbf{A} = \mathbf{LU}$. As an example, suppose:

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 9 \\ -12 & 0 & -10 \end{bmatrix}$$

From the second row we subtract $m_2^{(1)} = 2$ times the first row.
From the third row we subtract $m_3^{(1)} = -4$ times the first row.
The result is":

$$\mathbf{U}' = \begin{bmatrix} 3 & -1 & 3 \\ 0 & 2 & 3 \\ 0 & -4 & 2 \end{bmatrix}$$

From the third row we subtract $m_3^{(2)} = -2$ times the second row.
The result is the desired upper triangular matrix:

$$\mathbf{U} = \begin{bmatrix} 3 & -1 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 8 \end{bmatrix}$$

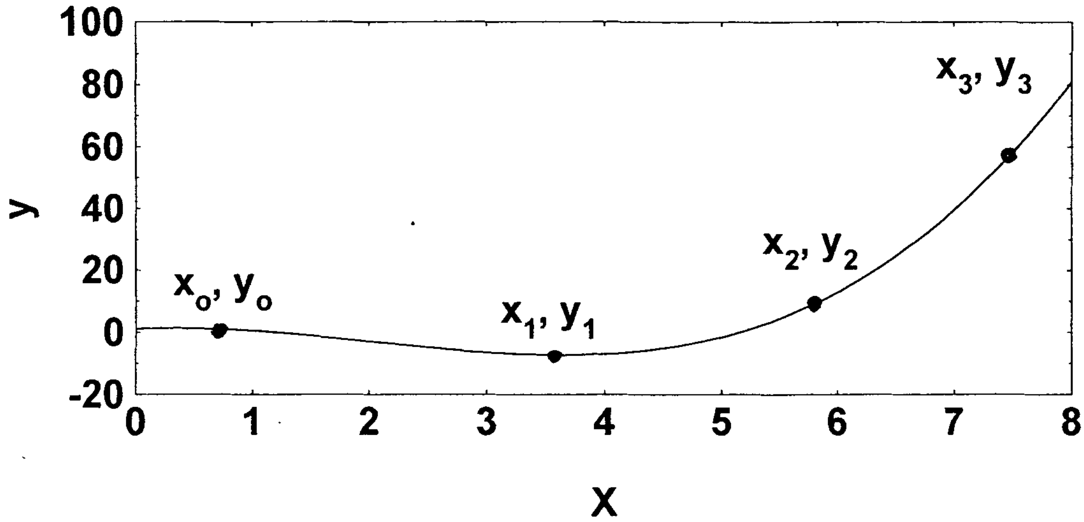
The lower triangular matrix with 1's on the diagonal is given by the formula $l_{ij} = m_i^{(j)}$ for $i > j$.

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -2 & 1 \end{bmatrix}$$

This results in $\mathbf{A} = \mathbf{LU}$

Curve Fitting and Interpolation

Polynomial Approximation to a Function



Suppose $y = f(x)$ where $f(x)$ is an unknown function. However, Suppose we have $N+1$ pairs of values (x_k, y_k), $k = 0, 2, ..., N$. An approximation to $f(x)$ is the N^{th} order polynomial, $p_N(x)$ that passes through the $N+1$ points. This can be very useful. For example derivatives or integrals of f can be approximated by the corresponding derivatives or integrals of p_N. Also, p_N is an interpolating function for f.

One obvious way to determine the required $N+1$ coefficients, c_i for p_N is to write the $N+1$ equations:

$$\sum_{i=0}^{N} x_k^i c_i = y_k, \qquad k = 0, 1, ..., N$$

This is equivalent to the matrix equation, $\qquad \mathbf{X}_p \mathbf{c} = \mathbf{y}$

There is another way to determine an approximating (interpolating) polynomial that does not require solution of a matrix equation. To introduce it, suppose we seek the polynomial that passes through just two points (x_0, y_0), (x_1, y_1). It is easy to show that the polynomial is given by:

$$p_1(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) y_0 + \left(\frac{x - x_0}{x_1 - x_0}\right) y_1$$

p is the linear combination of two order-1 polynomials L and can be written as:

$$p_1(x) = L_{1,0}(x) y_0 + L_{1,1}(x) y_1$$

The polynomials $L_{N,k}(x)$ are called *Lagrange Polynomials.* The polynomial representation can be extended to the case of $N+1$ points as:

$$p_N(x) = \sum_{k=0}^{N} L_{N,k} f(x_k)$$

The Lagrange Polynomials, $L_{N,k}(x)$ are polynomials of order N and have the following properties:

$$L_{N,k}(x_j) = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{for } j \neq k \end{cases}$$

The polynomials that have these properties are:

$$L_{N,k}(x) = \prod_{j=0, j\neq k}^{N} \frac{x - x_j}{x_k - x_j}$$

Lagrange Polynomials - Example

$y = f(x)$

x	y	k
0	1	0
1	2	1
3	1	2

$$P_2(x) = \sum_{k=0}^{2} L_{2k}\, y_k = L_{20}\, y_0 + L_{21}\, y_1 + L_{22}\, y_2$$

$$L_{20} = \frac{x - x_1}{x_0 - x_1}\,\frac{x - x_2}{x_0 - x_2} \qquad L_{21} = \frac{x - x_0}{x_1 - x_0}\,\frac{x - x_2}{x_1 - x_2}$$

$$L_{22} = \frac{x - x_0}{x_2 - x_0}\,\frac{x - x_1}{x_2 - x_1}$$

$$P_2(x) = 1\,\frac{x - x_1}{x_0 - x_1}\,\frac{x - x_2}{x_0 - x_2} + 2\,\frac{x - x_0}{x_1 - x_0}\,\frac{x - x_2}{x_1 - x_2}$$
$$+ 3\,\frac{x - x_0}{x_2 - x_0}\,\frac{x - x_1}{x_2 - x_1}$$

$$P_2(x) = \frac{x-1}{-1}\,\frac{x-3}{-3} + 2\,\frac{x}{1}\,\frac{x-3}{-2} + 1\,\frac{x}{3}\,\frac{x-1}{2}$$

$$P_2(x) = \frac{1}{3}\left(x^2 - 4x + 3\right) - x^2 + 3x + \frac{1}{6}\left(x^2 - x\right)$$

at $x = 0$, $P_2(x) = 1$

at $x = 1$, $P_2(x) = \frac{1}{3}(1 - 4 + 3) - 1 + 3 + \frac{1}{6}(1 - 1) = 2$

at $x = 3$, $P_2(x) = \frac{1}{3}(9 - 12 + 3) - 9 + 9 + \frac{1}{6}(9 - 3) = 1$

Numerical Differentiation

Numerical Differentiation is used when:

1. A functional form is so complicated that it is more convenient to do numerical integration,

2. when we have a table of values of $[x_i, f(x_i)]$ and we wish to find df/dx for some given value(s) of x.

Examples of situations for which derivatives are needed include:

1. Quantities given in terms of derivatives: $v = \frac{dx}{dt}$, $u = \frac{\partial \phi}{\partial x}$.
2. Mathematical procedures requiring derivatives:
   - A function $y = f(x)$ is to be approximated by $\hat{y} = \hat{f}(x)$ and $\hat{f}$ contains constants to be determined which minimize the error in the fit of the function to N points at x_i. Error $= \Sigma_1^N \left[\hat{f}(x_i) - f(x_i)\right]^2$
   - Finding the roots of $y = f(x)$. In other words, find the values of x such that $f(x) = 0$.

Two principal methods for obtaining numerical estimates of $f'(x_j)$ when we have a set (table) of pairs of values $[x_i,\ f(x_i) \equiv f_i]$, $i = 1, 2, ..., N$ are:

1. Develop relatively simple formulae that provide estimates of the derivative in terms of values of f_i and x_i,

2. Determine an analytic function $g(x)$ which is a good approximation to $f(x)$ and differentiate $g(x)$ analytically.

We will consider the first method here. The second is in the category of functional estimation or approximation.

Numerical Differentiation

Finite Difference Differentiation

Formal Definition of the Derivative $f'(x_o) = \lim_{h \to 0} \dfrac{f(x_o + h) - f(x_o)}{h}$

If we simply let h be "small", we have an approximation to the derivative,

$$f'(x_o) \approx \frac{f(x_o + h) - f(x_o)}{h}$$

If $h > 0$ this is a *forward-difference* formula and
if $h < 0$ this is a *backward-difference* formula.

It has a unique relation to the Taylor series for $f(x_o + h)$

$$f(x_o + h) = f(x_o) + hf'(x_o) + \frac{1}{2}h^2 f''(\xi)\,, \qquad \xi \in (x_o, x_o + h)$$

$$f'(x_o) = \frac{f(x_o + h) - f(x_o)}{h} + \frac{h}{2} f''(\xi)$$

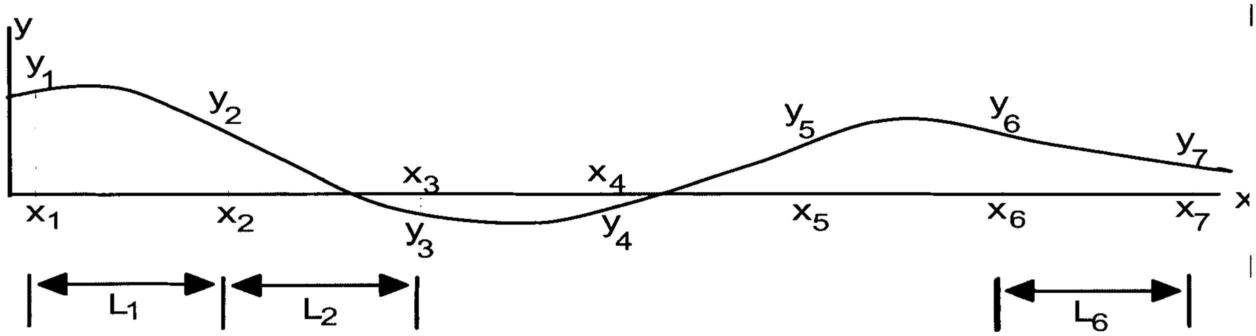
Our approximation to the derivative is obtained by dropping the last term which introduces an error of $O(h)$. The smaller the value of h, the smaller the mathematical error. However, very small h results in numerical subtraction of two "nearly identical" numbers so it introduces round-off error.

A *centered difference* formula for the derivative is:

$$f'(x_o) \approx \frac{f(x_o + h) - f(x_o - h)}{2h}$$

The error in this formula is $O(h^2)$

Sometimes the points are not equally spaced so numerical implementation of the centered difference formula is impossible. Consider the case of $y = f(x)$ with values at specific points known as sketched below:



For all interior points (x_2 to x_6) in the figure, interpolation of derivatives at the center of adjacent points can be used to generate the equivalent of a centered difference formula at each of the x-points, even when the distnces, L, are not equal.

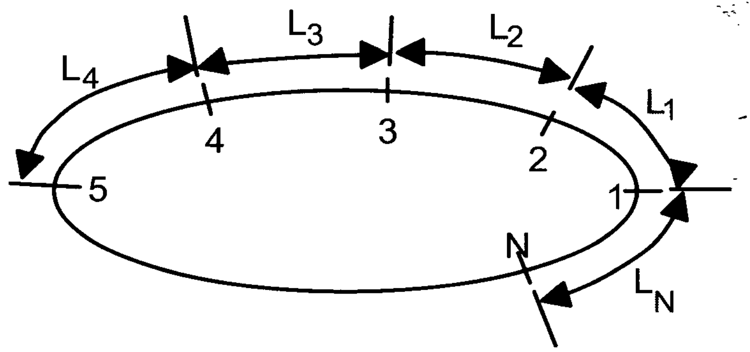
The resulting formula is:

$$f'(x_m) = y_m \frac{q_m - q_{m-1}}{2\, q_m\, q_{m-1}} + \frac{1}{2}\left[\frac{y_{m+1}}{q_m} - \frac{y_{m-1}}{q_{m-1}}\right] \tag{1}$$

$$\text{where: } q_m = \frac{1}{2}(L_{m+1} + L_m)$$

For estiating the derivatives at the end points, extapolation can be used from the numerical derivative half way between the two endmost points using the forward or backward difference formula and the the derivative at the nearest interior point given by the above formula.

Sometimes values of a function, y, are given at unequally spaced points around the periphery of a plane curve as sketched below.



The lengths L are arc lengths (s) between points on the curve and values of y are known at points (1, 2, ..., N) on the curve. To obtain the numerical approximation of the tangential derivative $y'(s)$ at points 1,2,3, ... N, equation (1) can be used. However, for point m = 1, special values for some of the y's and some of the L's must be used. In particular $y_{m-1} = y_N$ and $L_{m-1} = L_N$.

Likewise for for point m = N, $y_{m+1} = y_1$ and $L_{m+1} = L_1$.

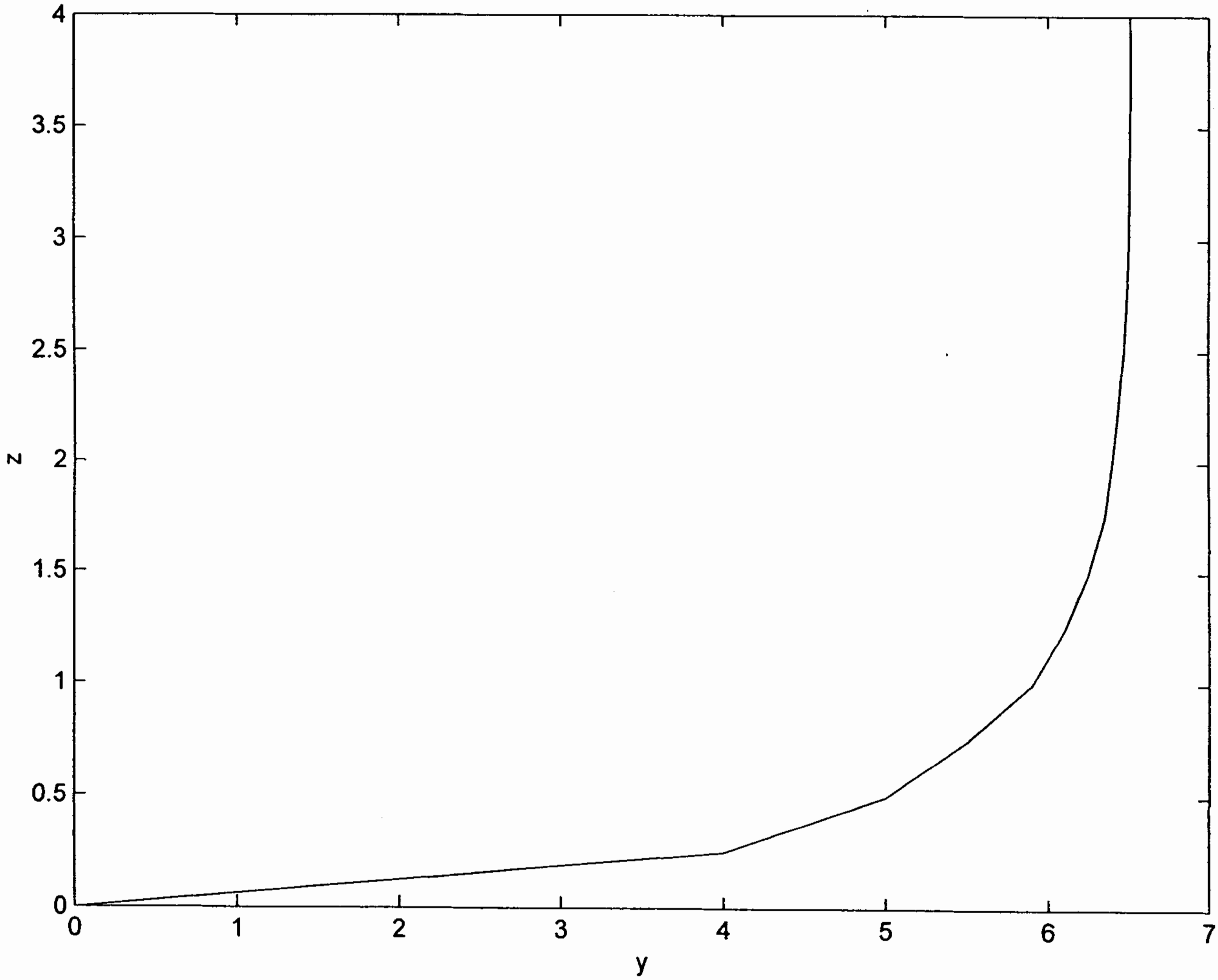
To estimate the error in Simpson's rule, the function over four successive points can be expanded in a Taylor series up to order 3. With equally spaced points, the error in the integral from the cubic term vanishes and the dominant term in the error is proportional to the fourth derivative d^4f/dx^4. For equally spaced points with $h = \Delta x$, the total error, E_T takes the form:

$$E_T = -\frac{h^5}{90}\sum_{i=1}^{n-1}\frac{d^4f(\eta_i)}{dx^4}$$

η_i is some value of x in the i^th interval.

```
%SCRIPT TO DO NUMERICAL INTEGRATION WITH EQUALLY SPACED POINTS
fil = input('Enter input file name: ','s');
fid = fopen(fil,'r');
n = fscanf(fid,'%d',1);
for k = 1:n;
    z(k) = fscanf(fid, '%f',1);
    y(k) = fscanf(fid, '%f',1);
end;
h = z(2) - z(1);
Ir = 0;
It = 0;
Is = 0;
for k = 2:n-1
    Ir = Ir + y(k);
    It = It + y(k);
end;
Ir = h*(Ir+y(1));
It = h*(It + 0.5*y(1)+ 0.5*y(n));
for k = 1:2:n-2;
    Is = Is + y(k) +4.0*y(k+1) +y(k+2);
end;
Is =Is*h/3.0 ;
fprintf(1,'%s \n','Integrals from Rectangular, Trapezoidal and Simpson''s Rules');
fprintf(1,'%9.5f    %9.5f     %9.5f\n', Ir, It,Is);
```

```
17
0.00     0.000
0.25     4.000
0.50     5.000
0.75     5.500
1.00     5.900
1.25     6.100
1.50     6.250
1.75     6.350
2.00     6.400
2.25     6.440
2.50     6.470
2.75     6.490
3.00     6.500
3.25     6.504
3.50     6.507
3.75     6.509
4.00     6.510
```



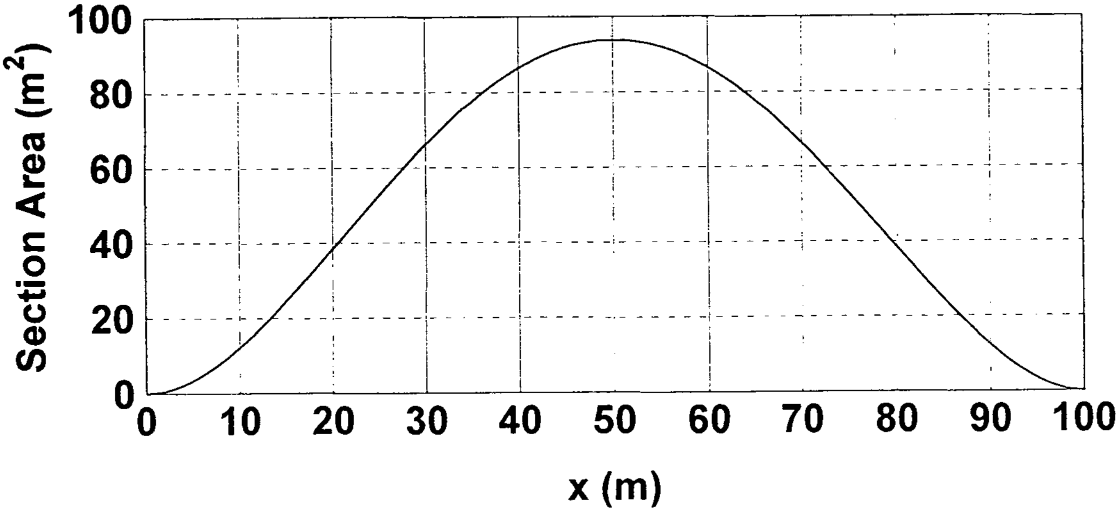
z
y
0
0.5
1
1.5
2
2.5
3
3.5
4
0
1
2
3
4
5
6
7

```
>> numint
Enter input file name: shipsec.txt
Integrals from Rectangular, Trapezoidal and Simpson's Rules
 22.73000     23.54375      23.67800
>>
```

Numerical Integration

Numerical Integration

1. Used to integrate a function that we do not know how it integrate by quadrature. For example, suppose we seek $I = \int_{2.2}^{5.3} \exp(\sqrt{x} + x^3 dx$
2. Used when we want to determine $\int_a^b f(x)dx$ and we have a set of pairs of values of $[x,\ f(x)]$

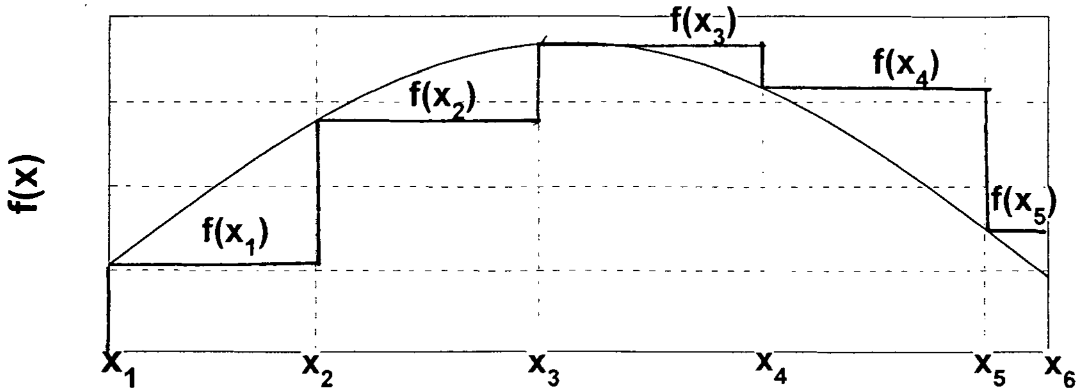


One approach, which we will not consider in detail here, is to fit a polynomial to the integrand and then to integrate the polynomial analytically.

The approach we follow here is to consider *integration rules* which provide a numerical approximation to the integral in terms of discrete values of the integrand for a set of values of x.

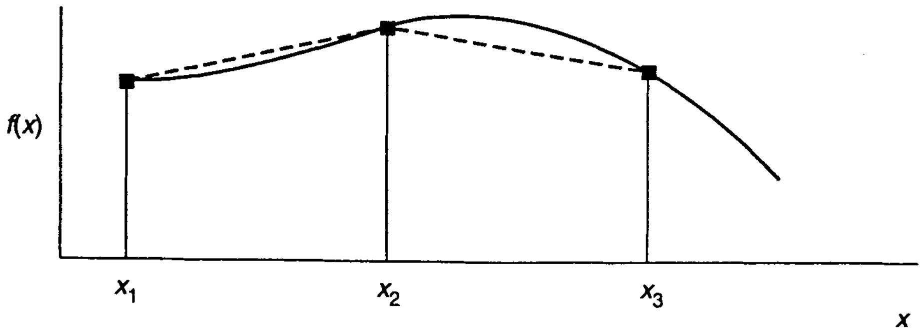
Rectangular Rule approximates the function by a set of rectangles and estimates the integral as the sum of the areas of the rectangles.

$$\int_{x_1}^{x_n} \approx \sum_{i=1}^{n-1} (x_{i+1} - x_i) f(x_i)$$

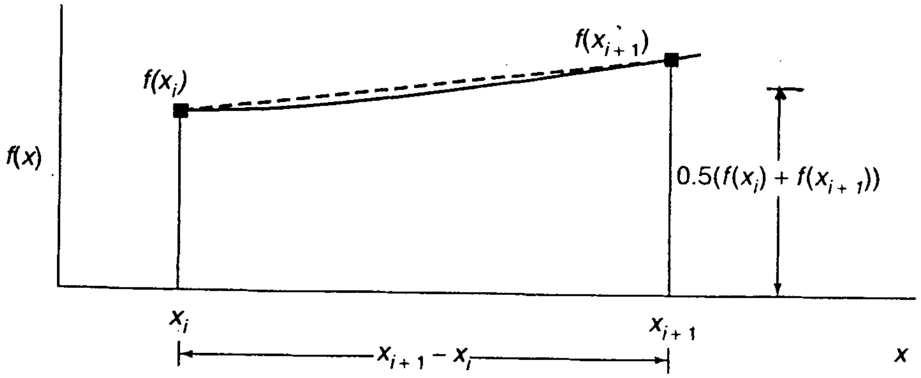


Trapezoidal Rule

The trapezoidal rule fits a trapezoid to each successive pair of values of $[x, f(x)]$ and estimates the integral as the sum of the areas of the trapezoids.



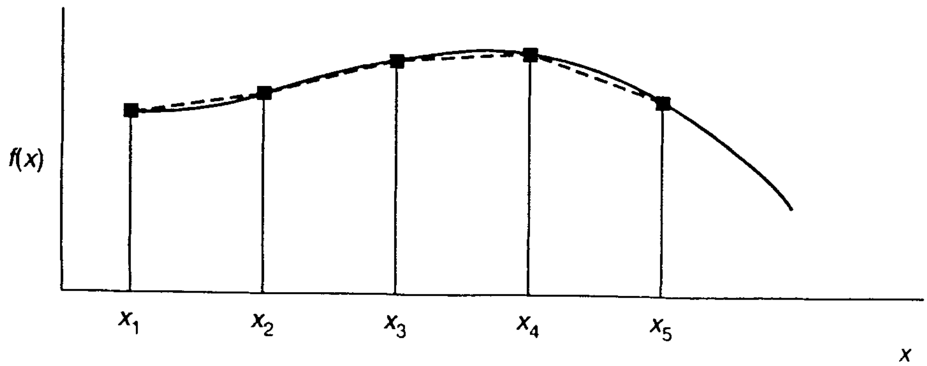
$$\int_{x_1}^{x_n} f(x)dx \approx \sum_{i=1}^{n-1}(x_{i-1} - x_i)\frac{1}{2}[f(x_{i+1} + f(x_i)]$$



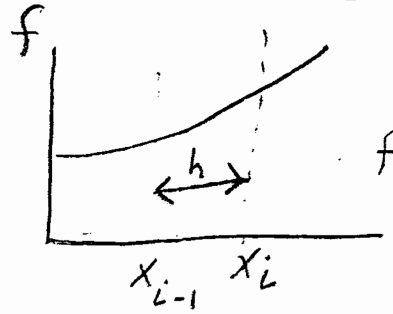
By expanding $f(x)$ in each interval as a Taylor series we find the error in the approximate integral, E_T is given by:

$$E_T = -\frac{h^3}{12}\sum_{i=1}^{n-1} f''(\eta_i)$$

where η_i is some value of x in the i^{th} interval. Since the number of terms in the sum is proportional to $1/h$, smaller intervals result in less error.

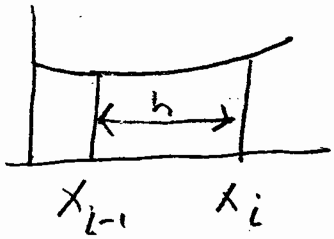


TRAPEZOIDAL Rule Error



$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) + \cdots$$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + \frac{h}{2} f''(x_i) + \cdots$$



$$I = \int_{x_{i-1}}^{x_i} f(x)\,dx = F(x_i) - F(x_{i-1}) \quad \text{where } \frac{dF}{dx} = F'(x) = f(x)$$

$$F(x_{i-1}) = F(x_i) - hF'(x_i) + \frac{h^2}{2} F''(x_i) - \frac{h^3}{6} F'''(x_i) + \cdots$$

$$I = F(x_i) - F(x_{i-1}) = hF'(x_i) - \frac{h^2}{2} F''(x_i) + \frac{h^3}{6} F'''(x_i) + \cdots$$

Since $F'(x_i) = f(x_i)$,

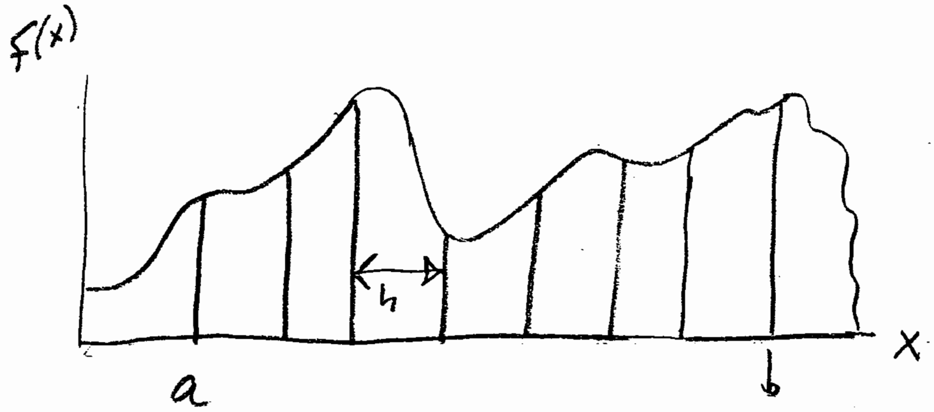
$$I = hf(x_i) - \frac{h^2}{2} f'(x_i) + \frac{h^3}{3} f''(x_i) + \cdots$$

$$I = hf(x_i) - \frac{h^2}{2}\left[\frac{f(x_i) - f(x_{i-1})}{h} + \frac{h}{2} f''(x_i) + \cdots\right] + \frac{h^3}{3} f''(x_i) + \cdots$$

$$I = hf(x_i) - \frac{h}{2} f(x_i) + \frac{h}{2} f(x_{i-1}) - \frac{h^3}{4} f''(x_i) + \frac{h^3}{3} f''(x_i) + \cdots$$

$$I = h\left[\frac{f(x_i) + f(x_{i-1})}{2}\right] + \frac{h^3}{12} f''(x_i) + \cdots$$

Error in interval x_{i-1} to x_i is $\frac{h^3}{12} f''(x_i) + O(h^4)$



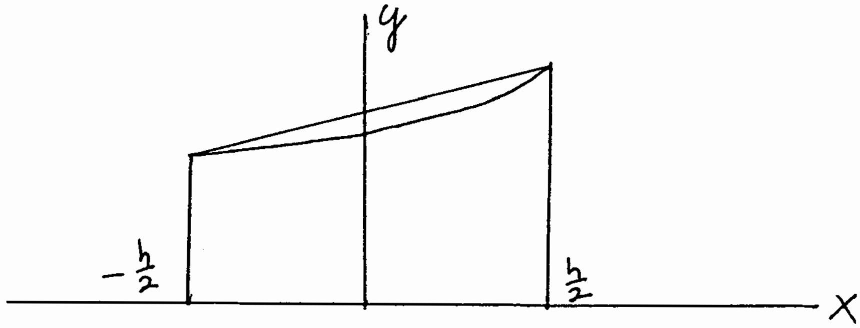
Number of intervals is $\frac{b-a}{h}$

If $|f''(x)| \leq M$ in $a < x < b$

Total error $\leq \frac{b-a}{h}\left[\frac{h^3}{12} M + O(h^4)\right]$

Total error $\leq (b-a)\left[\frac{h^2}{12} M + O(h^3)\right]$

Estimate 3 - Usual Trapezoidal Rule



$$y = f(x) = a + bx + cx^2 + \cdots$$

$$a = \frac{1}{2}\left[f\left(-\frac{h}{2}\right) + f\left(\frac{h}{2}\right)\right] \qquad b = \frac{f\left(\frac{h}{2}\right) - f\left(-\frac{h}{2}\right)}{h}$$

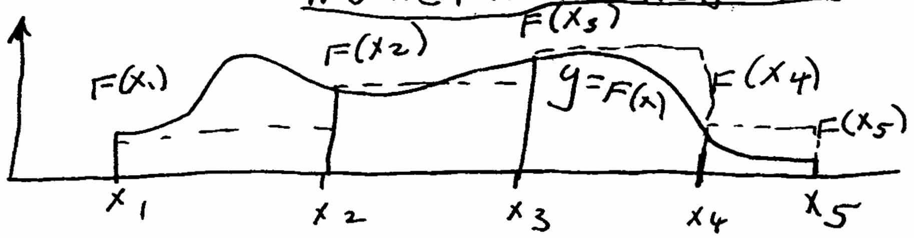
$$c = \frac{1}{2}f''(\beta) \qquad -\frac{h}{2} \leq \beta \leq \frac{h}{2}$$

$$I = ah + c\frac{h^3}{12} + \cdots = \frac{h}{2}\left[f\left(-\frac{h}{2}\right) + f\left(\frac{h}{2}\right)\right] + \frac{1}{2}f''(\beta)\frac{h^3}{12} + \cdots$$

$$I_{\text{trap}} = \frac{h}{2}\left[f\left(-\frac{h}{2}\right) + f\left(\frac{h}{2}\right)\right]$$

$$\text{Error} = \frac{h^3}{24}f''(\beta) + O(h^4)$$

Numerical Integration



$$I = \int_{x_1}^{x_5} f(x)\,dx$$

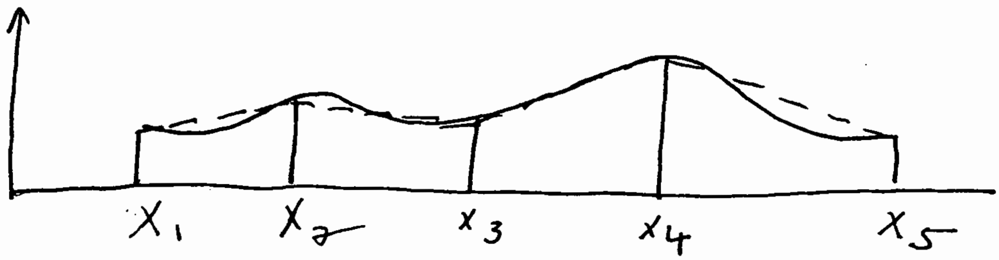
Rectangular Rule

Abbreviation $F(x_i) \equiv F_i$

$$I = F(x_1)(x_2 - x_1) + F(x_2)(x_3 - x_2) + F(x_3)(x_4 - x_3) + F(x_4)(x_5 - x_4)$$

$$= x_2 F_1 - x_1 F_1 + x_3 F_2 - x_2 F_2 + x_4 F_3 - x_3 F_3 + x_5 F_4 - x_4 F_4$$

Trapezoidal Rule



$$I = \left(\frac{F_1 + F_2}{2}\right)(x_2 - x_1) + \left(\frac{F_2 + F_3}{2}\right)(x_3 - x_2) + \left(\frac{F_3 + F_4}{2}\right)(x_4 - x_3) + \left(\frac{F_4 + F_5}{2}\right)(x_5 - x_4)$$

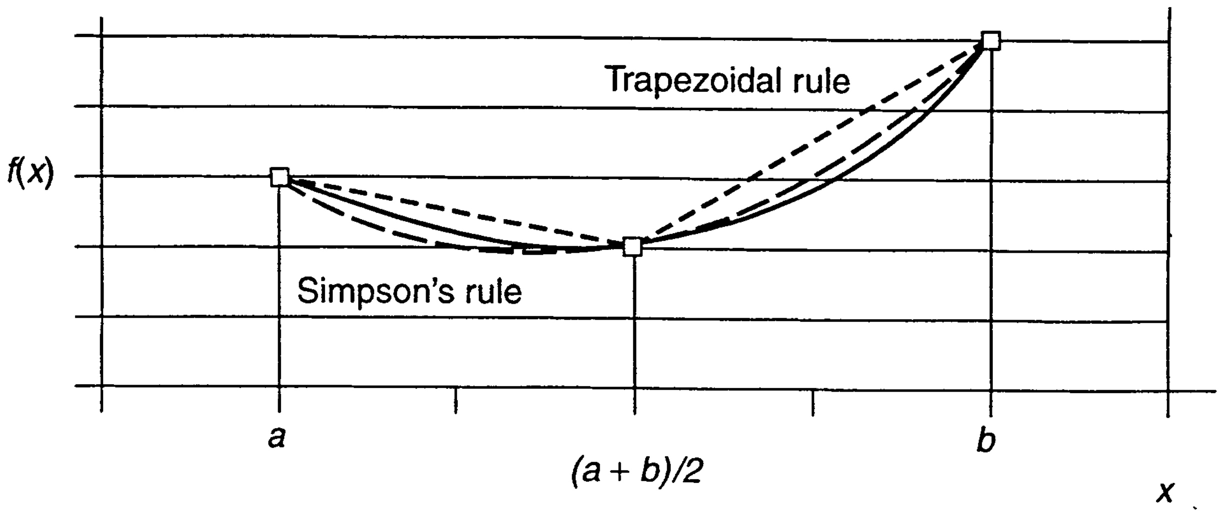
$$= \frac{1}{2}\left\{ F_1 x_2 - F_1 x_1 + \cancel{F_2 x_2} - F_2 x_1 + F_2 x_3 - \cancel{F_2 x_2} + \cancel{F_3 x_3} - F_3 x_2 + F_3 x_4 - \cancel{F_3 x_3} + \cancel{F_4 x_4} - F_4 x_3 + F_4 x_5 - \cancel{F_4 x_4} + F_5 x_5 - F_5 x_4 \right\}$$

$$= \frac{1}{2}\left\{ F_1(x_2 - x_1) + F_2(x_3 - x_1) + F_3(x_4 - x_2) + F_4(x_5 - x_3) + F_5(x_5 - x_4) \right\}$$

If $x_{n+2} - x_n = 2(x_{n+2} - x_{n+1}) \Leftrightarrow$ equal spacing

$$I = \tfrac{1}{2} F_1(x_2 - x_1) + F_2(x_3 - x_2) + F_3(x_4 - x_3) + F_4(x_5 - x_4) + \tfrac{1}{2} F_5(x_5 - x_4)$$

Simpson's Rule
Simpson's rule fits a parabola (2^{nd} order polynomial) to each interval between $[x_i, f(x_i)]$ and $[x_{i+2}, f(x_{i+2})]$ and estimates the integral as the sum of the areas under the parabolas.



For Simpson's Rule, the intervals $[x_{i+1} - x_i]$ and $[x_{i+2} - x_{i+1}]$ must be equal and the x-distance interval length is called Δx. For three values of x_i; $a, a + \Delta x, b$ the integral is

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}[f(a) + 4f(a + \Delta x) + f(b)]$$

Simpson's Rule requires that there be an even number of intervals which means that there are an odd number of data pairs $[x_i, f(x_i)]$ Then the integral is approximated by the sum of the areas under the approximating parabolas as

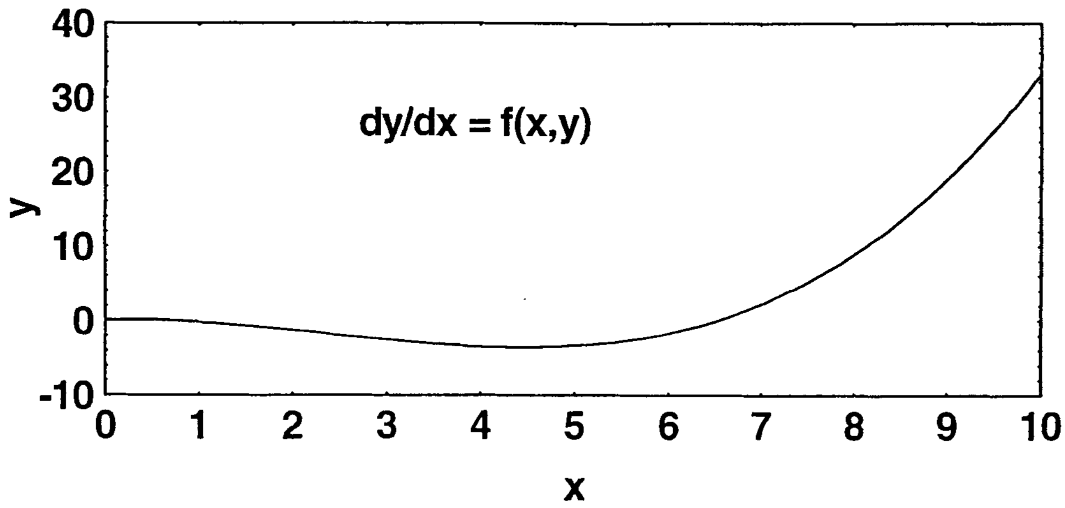
$$\int_{x_1}^{x_n} f(x)dx \approx \sum_{i=1,3,5,\ldots}^{n-2} \frac{x_{i+1} - x_i}{3}[f(x_i) + 4f(x_{i+1}) + f(x_{i+2})]$$

For the usual case of equal spacing, Δx, of all the x_i's, Simpson's Rule can be expressed as:

$$\int_{x_1}^{x_n} f(x)dx \approx \frac{\Delta x}{3}\left[f(x_1) + f(x_n) + 4\sum_{i=2,4,6,\ldots}^{n-1} f(x_i) + 2\sum_{i=3,5,7,\ldots}^{n-2} f(x_i)\right]$$

Numerical Integration of Differential Equations

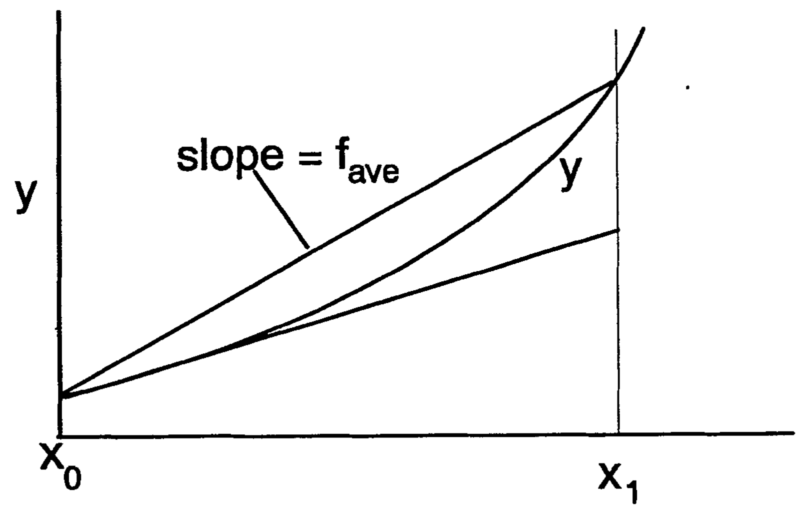
Numerical Integration of Differential equations



Consider the differential equation: $\frac{dy}{dx} = f(x,y)$ subject to y(a) = b

The condition $y(a) = b$ is called an *initial condition.* a and b are known numbers.

The approach to integrate the differential equation numerically is to start at $x = a$ where we know the solution, $y = b$. Then we move to $x = a + \Delta x$. The solution at that point is: $y = a + f_{ave}\Delta x$, where f_{ave} is the average value of $\frac{dy}{dx}$ over the interval from a to $a + \Delta x$. Then we proceed in the same way to obtain $y(a + 2\Delta x)$. The accuracy of the numerical solution depends of the accuracy of the estimate of f_{ave}. The various ways of making this estimate are called *rules* or *methods.* We will index the points at which y is determined by i so the computed values are $y_0, y_1, y_2, ...$ at values of x: $x_0, x_1, x_2, ...$ The simplest method is Euler's Method.



Euler's Method In Euler's method, to estimate y_{i+1}, f_{ave} is approximated by its value at (x_i, y_i) or its value at (x_{i+1}, y_i) The former case is called the forward Euler Method and the latter case is called the backward Euler Method.

For the forward Euler Method: $\quad y_{i+1} \approx y_i + f(x_i, y_i)(x_{i+1} - x_i)$

Expanding y in a Taylor series about x_o gives:

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \ldots$$

The error,ϵ, for one step in the solution by the Euler Method can be obtained from the first omitted term in the above Taylor series:

$$\epsilon = \frac{(x - x_0)^2}{2} \frac{d^2y}{dx^2}$$

The meaning of the error depends on the value of x for which $\frac{d^2y}{dx^2}$ is evaluated. There is one point in the interval for which the computed error is exact. However, the location of that point is not known. If we take the absolute value of the maximum value of $\frac{d^2y}{dx^2}$ in the interval, the result is an upper bound on the error. ϵ is called the local error for the interval. The total error for the entire interval is called the global error. The sum of the values of ϵ for all the intervals is an upper bound on the global error. For demonstration purposes, if we have a differential equation that can be solved analytically, the exact global error can be calculated. It can be shown that the global error diminishes as the step (interval) size, Δx, is made smaller.

Modified Euler's Method

The forward Euler method uses the derivative at the initial point of the interval. The backward Euler method uses the derivative close to the end point of the interval. A numerical solution with improved accuracy is obtained by using the average of the derivative at the initially computed end points.

$$y_{i+1} \approx y_i + \frac{x_{i+1} - x_i}{2} \{y'(x_i, y_i) + y'[x_{i+1}, y_i + (x_{i+1} - x_i)\, y'(x_i, y_i)]\}$$

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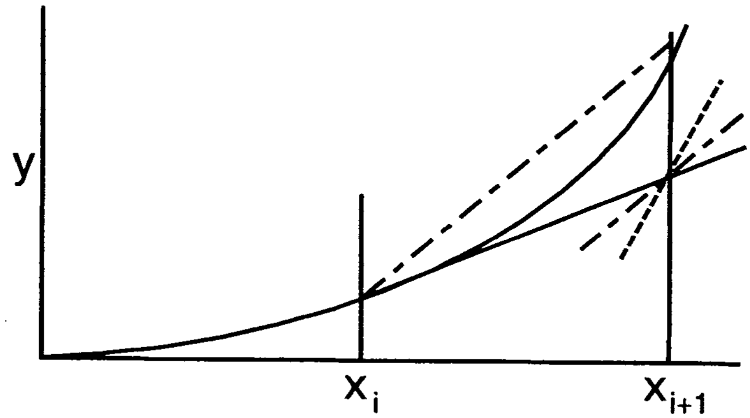
Modified Euler's Method (continued)

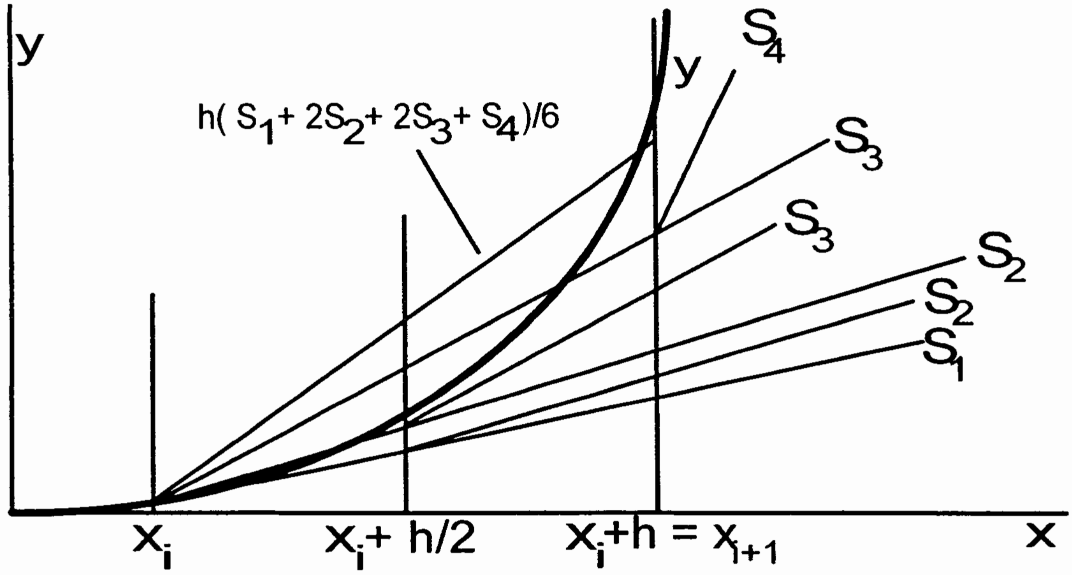


For each successive interval, the steps are:

1. Evaluate the slope $[y'(x_i, y_i)]$ at the start of the interval.
2. Estimate y_{i+1} at the end of the interval using Euler's Method.
3. Evaluate the slope, $y'[x_{i+1}, y_i+(x_{i+1}-x_i)y'(x_i, y_i)]$, at the end of the interval.
4. Calculate the average of the two slopes,y'_{ave} from steps 1 and 3.
5. Calculate a revised value of y_{i+1} using the average slope,
   $y_{i+1} = y_i + (x_{i+1} - x_i)y'_{ave}$.

DE14
RK46

Fourth Order Runge Kutta Method



The fourth order Runge Kutta method estimates the average value of $f = \frac{dy}{dx}$ in an interval of length h in terms of values of f at four locations in (x, y) space. If the initial point of the interval is (x_i, y_i), one derivative is evaluated there, one is at $x = x_i + h$ and two are at $x = x_i + h/2$, for two different values of y. If $\frac{dy}{dx}$ depends only on x and not on y, the two intermediate values of f are the same.

The four slopes are:

$$s_1 = f(x_i, y_i)$$

$$s_2 = f(x_i + 0.5h, y_i + 0.5hs_1)$$

$$s_3 = f(x_i + 0.5h, y_i + 0.5hs_2)$$

$$s_4 = f(x_i + h, y_i + hs_3)$$

Then the value of y at the end of the interval is estimated as:

$$y_{i+1} = y_i + \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4)$$

DELS

Predictor-Corrector Methods

Two difficulties with high order Runge Kutta methods are the computation time required to compute several values of the derivative at each time step and the exact error is unknown. Predictor-corrector methods allow fewer computations if the number of corrector steps is limited and allow iterations of the corrector step converging to a very accurate solution if the required computation time is available.

The approach with a predictor-corrector method involves two steps. The first is a prediction step using any integration method and the second is the correction step which improves upon the prediction. The process is explained here using the simplest rules: the Forward Euler Method for the prediction step and the trapezoidal rule for the correction step. The equation to be solved is: $\frac{dy}{dx} = f(x,y)$. Consider the interval from index i to index $i+1$ with $x_{i+1} - x_i = h$. The prediction step is:

$$y_{i+1,0} = y_{i,*} + hf_{i,*}$$

The second subscript on the left hand side, 0 indicates that there have been zero correction steps (or trials). The second subscripts, $*$, on the right hand side means that the subscripted quantities are for the final step; in this case for the i^{th} integration step.

Now we use the trapezoidal rule to correct the estimate of y_{i+1}.

$$y_{i+1,1} = y_{i,*} + \frac{h}{2}\left[f_{i,*} + f_{i+1,0}\right]$$

The rightmost term can be evaluated since we know $y_{i+1,0}$ from the previous prediction step. We can refine the estimate, using the value of y_{i+1} from the prior correction step.

$$y_{i+1,j} = y_{i,*} + \frac{h}{2}\left[f_{i,*} + f_{i+1,j-1}\right]$$

This process can be continued until y converges. The converged result is the exact solution to the numerical problem as posed, but is not necessarily the solution to the actual continuous mathematics solution. In this case, the numerical problem is based on the average derivative over the interval being the average of its value at the two ends. This is not exactly correct for differential equations in general.

D E I 6

Higher Order Differential Equations

$$\text{Consider the equation: } \frac{d^2y}{dx^2} + f_1(x,y)\frac{dy}{dx} + f_o(x,y)\,y = g(x,y)$$

$$\text{Let: } \frac{dy}{dx} = z$$

$$\text{Then: } \frac{dz}{dx} = -f_1(x,y)\,z - f_o(x,y)\,y + g(x,y)$$

When we use an integration rule, at each step it is applied to the two equations above and everything else proceeds as it does for a first order differential equation.

The same approach can be used for equations of higher order. For example:

$$\frac{d^3y}{dx^3} + f_2(x,y)\frac{d^2y}{dx^2} + f_1(x,y)\frac{dy}{dx} + f_o(x,y)\,y = g(x,y)$$

Make the following definitions:

$$\frac{dy}{dx} = z$$

$$\frac{dz}{dx} = \frac{d^2y}{dx^2} = w$$

$$\text{Then: } \frac{dw}{dx} = -f_2(x,y)\,w - f_1(x,y)\,z - f_0(x,y)\,y + g(x,y)$$

In this case, the integration rule is applied to three equations at each step.

Review and Extension

Simple Example

$$\text{Suppose: } \frac{d^2y}{dt^2} + f(y,t)\frac{dy}{dt} + g(y,t)\,y = h(y,t)$$

$$\frac{d^2y}{dt^2} = -f(y,t)\frac{dy}{dt} - g(y,t)\,y + h(y,t)$$

$$\text{Let: } q(y,t) = \frac{dy}{dt}$$

$$\frac{dq}{dt} = -f(y,t)\,q - g(y,t)\,y + h(y,t)$$

Consider the simplest integration rule: Forward Euler Integration

$$q(t+\delta t) = q(t) + [-f(y(t),t)\,q(t) - g(y(t),t)\,y(t) + h(y(t),t)]\,\delta t$$

$$y(t+\delta t) = y(t) + q(t)\,\delta t$$

Slightly More Complicated Example

$$\frac{\partial^2 y(x,t)}{\partial t^2} + f(x,y,t)\frac{\partial y(x,t)}{\partial t} + g(x,y,t)y(x,t)m = h(x,y,t)$$

Follow the above procedure for each (fixed) value of x.

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -f(x,y,t)\frac{\partial y(x,t)}{\partial t} - g(x,y,t)\,y + h(x,y,t)$$

$$\text{Let: } q(x,y,t) = \frac{\partial y(x,t)}{\partial t}$$

$$\frac{\partial q(x,y,t)}{\partial t} = -f(x,y(x,t),t)\,q(x,y,t) - g(x,y,t)\,y(x,y,t) + h(x,y,t)$$

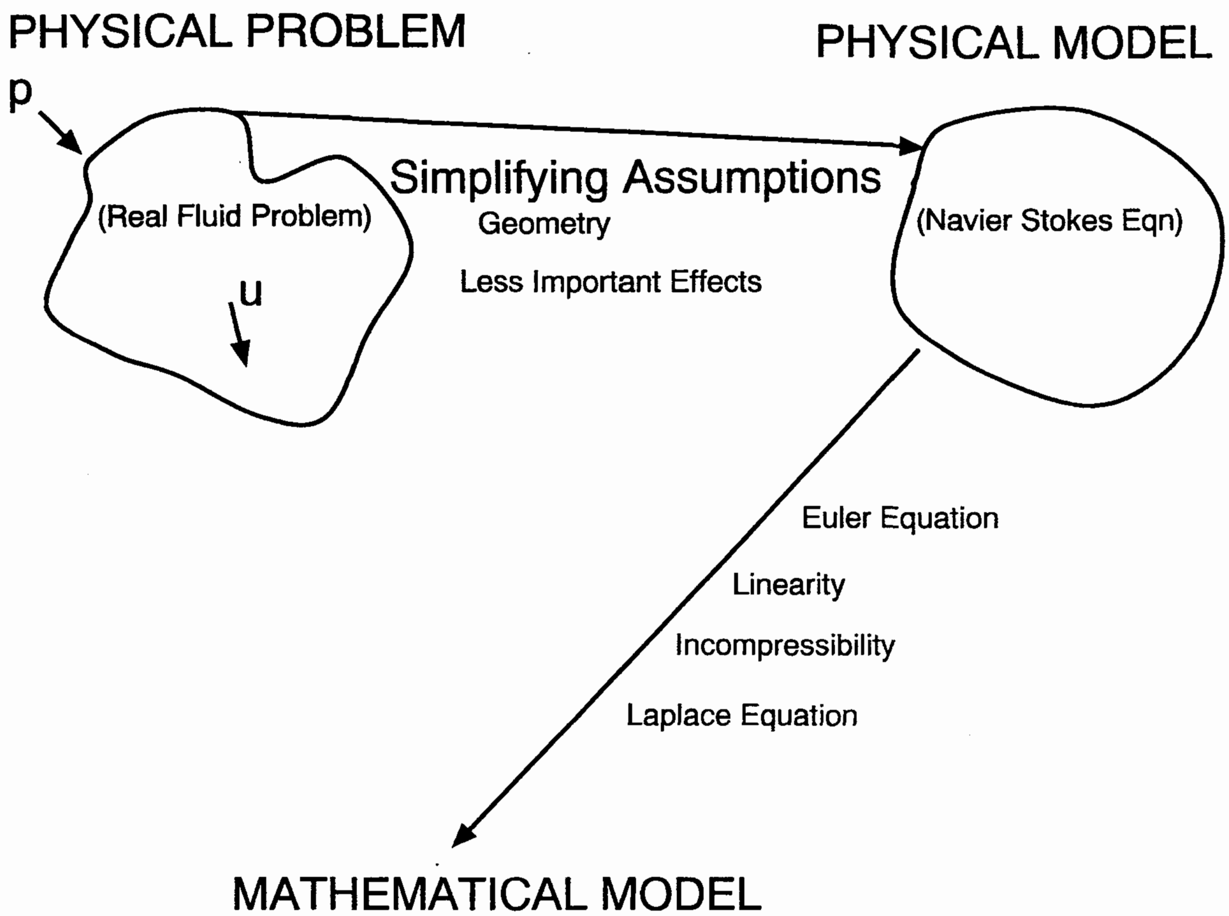
Now we do the numerical integration by the Euler Method. Any other method can also be used.

$$q(x,y,t+\delta t) = q(x,y,t) + [-f(x,y(x,t),t)\,q(x,y(x,t),t) - g(x,y(x,t),t)\,y(x,t) + h(x,y(x,t),t)]\,\delta t$$

$$y(x,t+\delta t) = y(x,t) + q(x,y,t)\,\delta t$$

examples_error.pdv

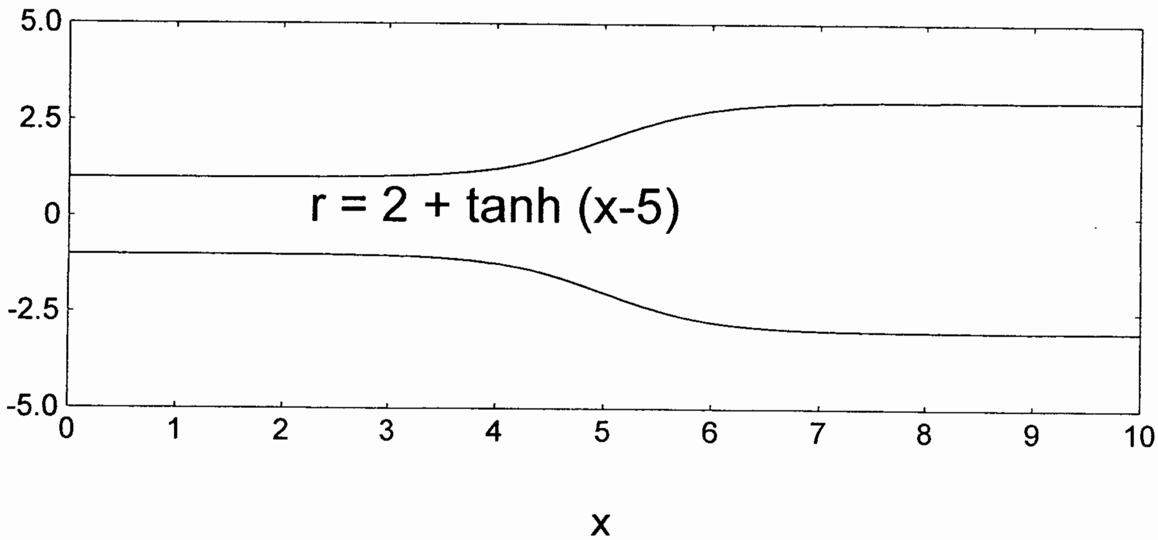
Some Examples and Numerical Errors


PHYSICAL PROBLEM
p
(Real Fluid Problem)
u
Simplifying Assumptions
Geometry
Less Important Effects
PHYSICAL MODEL
(Navier Stokes Eqn)
Euler Equation
Linearity
Incompressibility
Laplace Equation
MATHEMATICAL MODEL

Types of Numerical Hydrodynamics Problems

1. Evaluation of Mathematical Functions
2. Simulation
3. Direct Solution of Differential or Integral Equations

Example of Function Evaluation



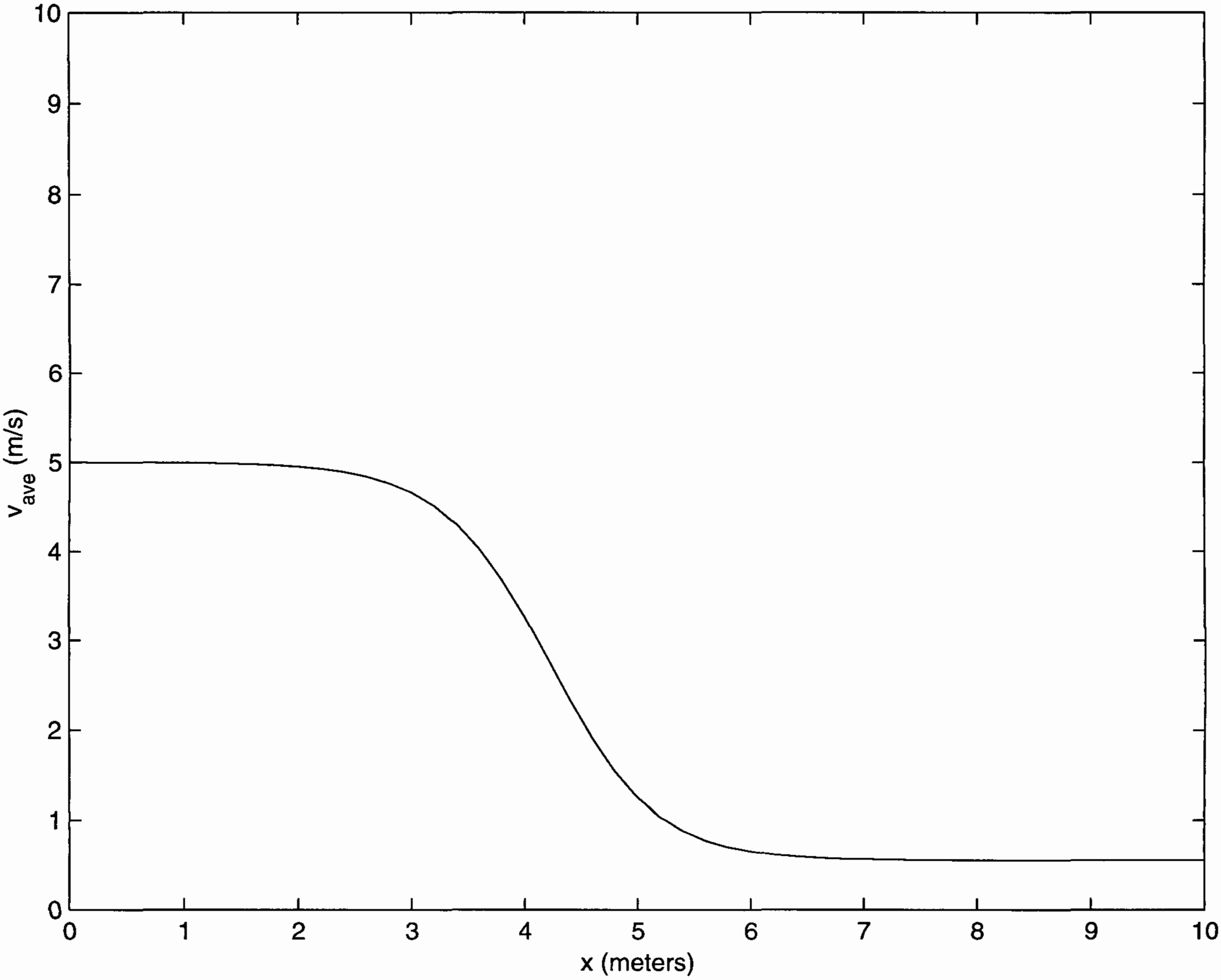
Consider a diffuser with circular crossections and radius vs length as shown. Units are meters. The average velocity of an incompressible fluid across the inlet at $x = 0$ is 5 m/s. Determine the average velocity across all cross sections.

$$Q = 5\pi[2 + \tanh(-5)]^2$$

$$V_{ave}(x) = \frac{Q}{\pi r^2} = \frac{Q}{\pi[2 + \tanh(x-5)]^2}$$

numet 3

```
% MATLAB Program diffu
fname = input(' Type name for output file: ','s');
fid = fopen(fname,'w');
q = 5.0 * pi * (2.0 + tanh(-5.0))^2;
x = 0 : 0.2 : 10.0 ;
v = q ./ (pi * (2.0 + tanh(x-5.0)) .^2);
for j = 1:51;
    fprintf(fid,'        %10.4f  %10.4f  \n', x(j), v(j) )
end;
fclose(fid);
plot (x,v);
xlabel('x (meters)');
ylabel( 'v_{ave} (m/s)');
axis([0 10 0 10]);
```

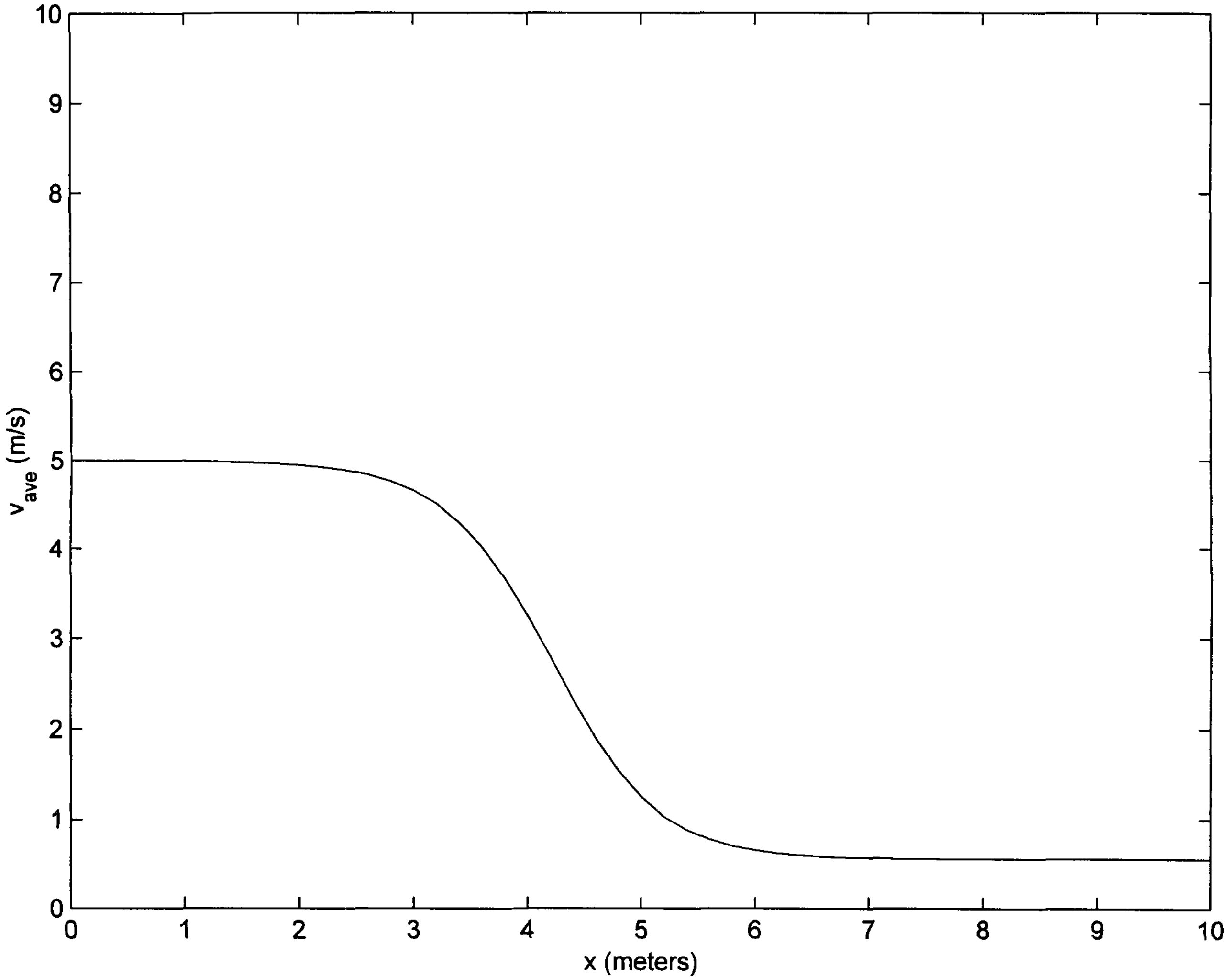


10
9
8
7
6
5
4
3
2
1
0
v_{ave} (m/s)
0
1
2
3
4
5
6
7
8
9
10
x (meters)

numef4

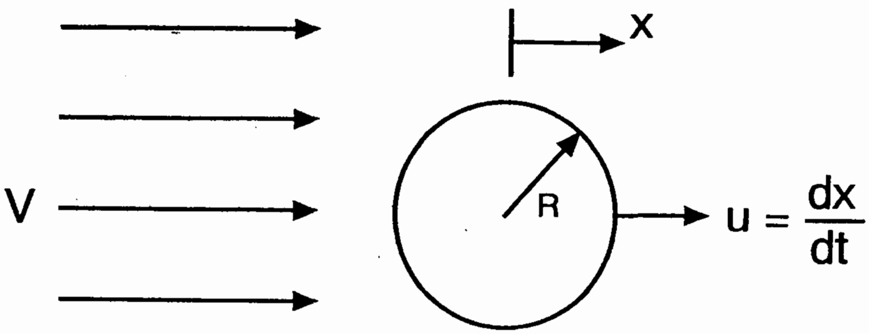
x	v
0.0000	5.0000
0.2000	4.9996
0.4000	4.9989
0.6000	4.9979
0.8000	4.9964
1.0000	4.9942
1.2000	4.9909
1.4000	4.9860
1.6000	4.9787
1.8000	4.9679
2.0000	4.9518
2.2000	4.9280
2.4000	4.8929
2.6000	4.8415
2.8000	4.7668
3.0000	4.6596
3.2000	4.5085
3.4000	4.3008
3.6000	4.0251
3.8000	3.6762
4.0000	3.2608
4.2000	2.8019
4.4000	2.3366
4.6000	1.9054
4.8000	1.5390
5.0000	1.2502
5.2000	1.0357
5.4000	0.8829
5.6000	0.7769
5.8000	0.7046
6.0000	0.6557
6.2000	0.6228
6.4000	0.6007
6.6000	0.5859
6.8000	0.5759
7.0000	0.5692
7.2000	0.5648
7.4000	0.5618
7.6000	0.5597
7.8000	0.5584
8.0000	0.5575
8.2000	0.5569
8.4000	0.5565
8.6000	0.5562
8.8000	0.5560
9.0000	0.5559
9.2000	0.5558
9.4000	0.5558
9.6000	0.5557
9.8000	0.5557
10.0000	0.5557

numer 4a



Example of Solution of Ordinary Differential Equation

Motion of a Sphere Due to Drag



$$M\frac{d^2x}{dt^2} = D \qquad\qquad D = \frac{1}{2}\rho C_d \pi R^2 \left(V - \frac{dx}{dt}\right)^2$$

$$\frac{dx}{dt} = u \qquad\qquad \frac{du}{dt} = \frac{\rho C_d \pi R^2}{2M}\left(V^2 - 2uV + u^2\right)$$

Discretize and use forward Euler Integration:

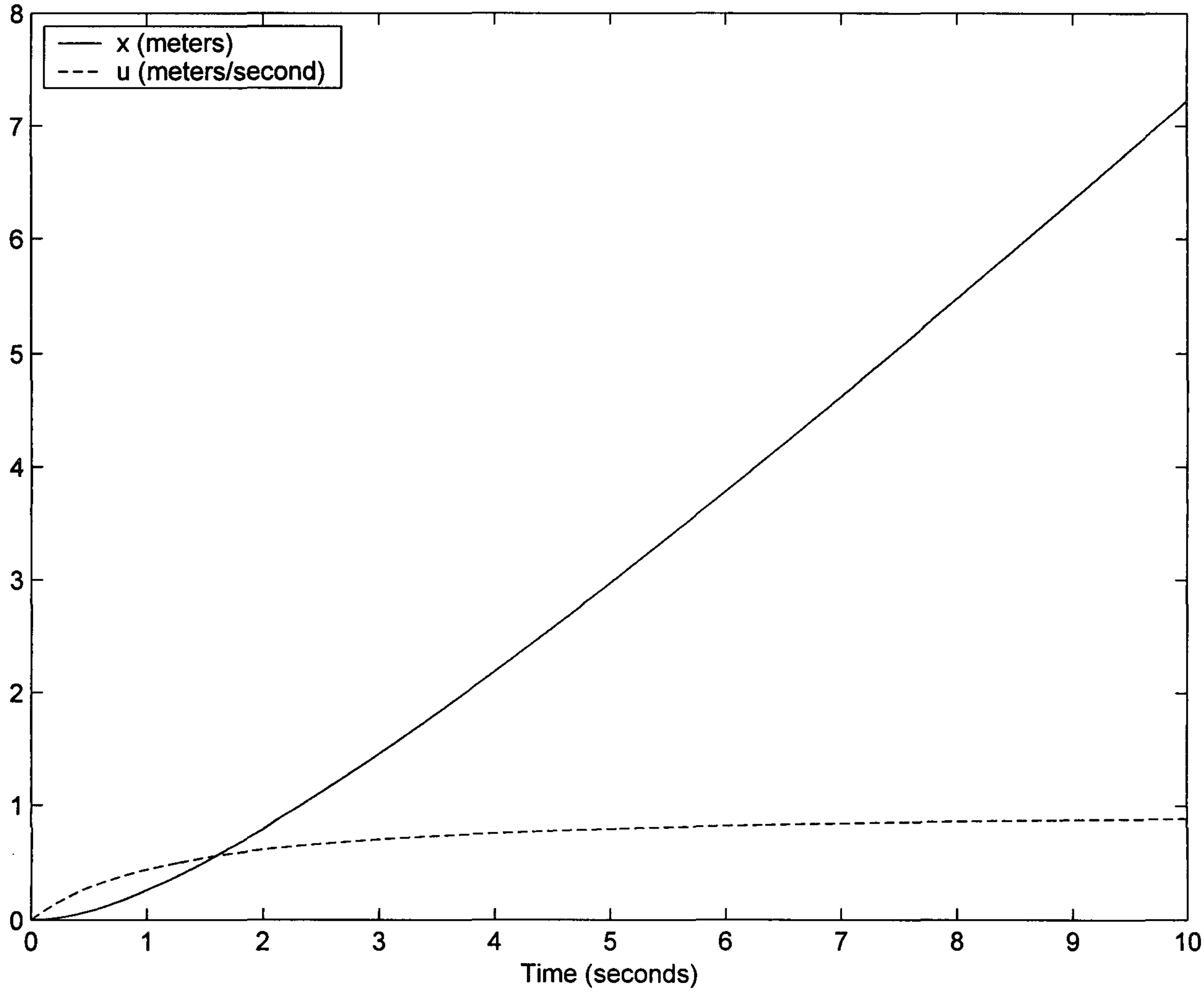
$$u_{i+1} = u_i + \left(\frac{du}{dt}\right)_i \Delta t \qquad\qquad x_{i+1} = x_i + u_i \Delta t$$

Initial Conditions: $x_0 = 0$ $\qquad u_0 = 0$

```
% MATLAB program spheredg
fid = fopen('sphere.out','w');
rho = 1000.0;
cd = 1.0;
v =1.0;
r = 0.05;
m = 5.0;
fac = rho  * cd *pi *r *r /(2.0 * m);
tt = 10.0;
dt = 0.01;
n = tt/dt + 1;

%initialize

t(1) = 0.0;
x(1) = 0.0;
u(1) = 0.0;
fprintf(fid,'%15.7f   %15.7f   %15.7f  \n', t(1), x(1),
u(1));
for i=2:n;
  j = i-1;
  t(i) = t(j) + dt;
  x(i)= x(j) + u(j)*dt;
  u(i) = u(j) + fac*(v*v- 2.0* u(j)*v + u(j)*u(j))*dt;
  fprintf(fid,'%15.7f   %15.7f   %15.7f \n',t(i),x(i),u(i));
end;
plot(t,x,'-',t,u,'--')
xlabel('Time (seconds)');
h=legend( 'x (meters)' , 'u (meters/second)' , 2);
```


x (meters)
u (meters/second)
Time (seconds)
0
1
2
3
4
5
6
7
8
9
10
0
1
2
3
4
5
6
7
8

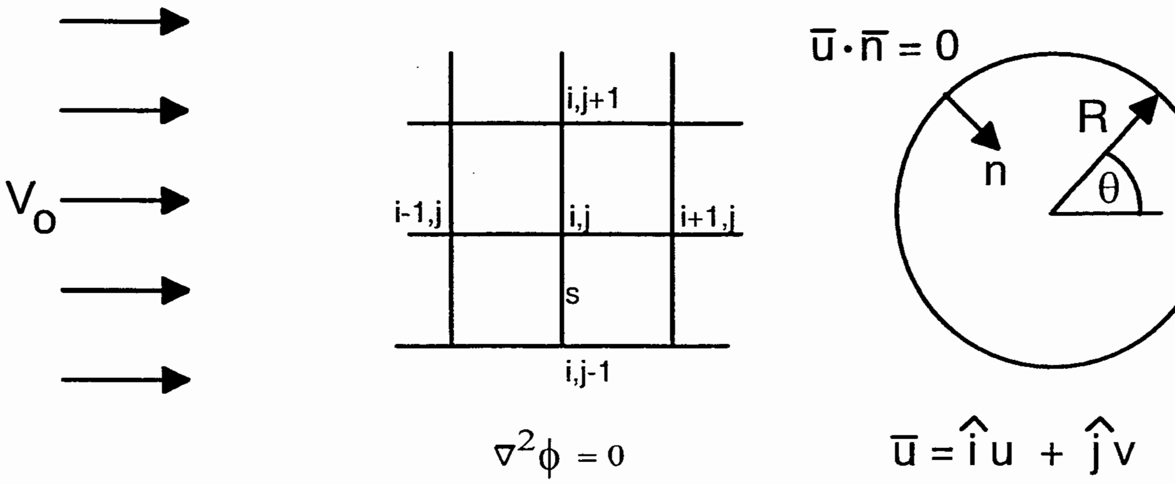
NUmeT8B

t	x	u
0.0000000	0.0000000	0.0000000
0.0100000	0.0000000	0.0078540
0.0200000	0.0000785	0.0155851
0.0300000	0.0002344	0.0231962
0.0400000	0.0004664	0.0306900
0.0500000	0.0007733	0.0380693
0.0600000	0.0011539	0.0453367
0.0700000	0.0016073	0.0524947
0.0800000	0.0021323	0.0595457
0.0900000	0.0027277	0.0664922
0.1000000	0.0033926	0.0733364
0.1100000	0.0041260	0.0800807
0.1200000	0.0049268	0.0867271
0.1300000	0.0057941	0.0932779
0.1400000	0.0067269	0.0997350
0.1500000	0.0077242	0.1061005
0.1600000	0.0087852	0.1123762
0.1700000	0.0099090	0.1185642
0.1800000	0.0110946	0.1246662
0.1900000	0.0123413	0.1306840
0.2000000	0.0136481	0.1366193
0.2100000	0.0150143	0.1424739
0.2200000	0.0164391	0.1482493
0.2300000	0.0179215	0.1539472
0.2400000	0.0194610	0.1595691
0.2500000	0.0210567	0.1651166
0.2600000	0.0227079	0.1705911
0.2700000	0.0244138	0.1759940
0.2800000	0.0261737	0.1813267
0.2900000	0.0279870	0.1865906
0.3000000	0.0298529	0.1917871
0.3100000	0.0317708	0.1969174
0.3200000	0.0337399	0.2019828
0.3300000	0.0357598	0.2069844
0.3400000	0.0378296	0.2119236
0.3500000	0.0399489	0.2168014
0.3600000	0.0421169	0.2216190
0.3700000	0.0443331	0.2263776
0.3800000	0.0465968	0.2310781
0.3900000	0.0489076	0.2357217
0.4000000	0.0512648	0.2403094
0.4100000	0.0536679	0.2448422
0.4200000	0.0561163	0.2493210
0.4300000	0.0586096	0.2537469
0.4400000	0.0611470	0.2581207
0.4500000	0.0637282	0.2624434
0.4600000	0.0663527	0.2667159
0.4700000	0.0690198	0.2709390
0.4800000	0.0717292	0.2751137
0.4900000	0.0744804	0.2792406
0.5000000	0.0772728	0.2833207
0.5100000	0.0801060	0.2873548
0.5200000	0.0829795	0.2913435
0.5300000	0.0858929	0.2952877
0.5400000	0.0888458	0.2991882
0.5500000	0.0918377	0.3030455
0.5600000	0.0948682	0.3068606
---	---	---

t	x	u
9.6900000	6.9506656	0.8840910
9.7000000	6.9595065	0.8841966
9.7100000	6.9683485	0.8843019
9.7200000	6.9771915	0.8844070
9.7300000	6.9860356	0.8845120
9.7400000	6.9948807	0.8846167
9.7500000	7.0037269	0.8847213
9.7600000	7.0125741	0.8848256
9.7700000	7.0214224	0.8849298
9.7800000	7.0302717	0.8850338
9.7900000	7.0391220	0.8851376
9.8000000	7.0479734	0.8852413
9.8100000	7.0568258	0.8853447
9.8200000	7.0656792	0.8854479
9.8300000	7.0745337	0.8855510
9.8400000	7.0833892	0.8856539
9.8500000	7.0922458	0.8857566
9.8600000	7.1011033	0.8858591
9.8700000	7.1099619	0.8859614
9.8800000	7.1188215	0.8860635
9.8900000	7.1276822	0.8861655
9.9000000	7.1365438	0.8862673
9.9100000	7.1454065	0.8863689
9.9200000	7.1542702	0.8864703
9.9300000	7.1631349	0.8865715
9.9400000	7.1720006	0.8866725
9.9500000	7.1808673	0.8867734
9.9600000	7.1897351	0.8868741
9.9700000	7.1986038	0.8869746
9.9800000	7.2074735	0.8870749
9.9900000	7.2163443	0.8871751
10.0000000	7.2252160	0.8872751

Example of Solution of Partial Differential Equation

Potential Streaming Flow about a Circular Cylinder



There is an equation at each grid point:

$$\frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{s^2} + \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{s^2} = 0$$

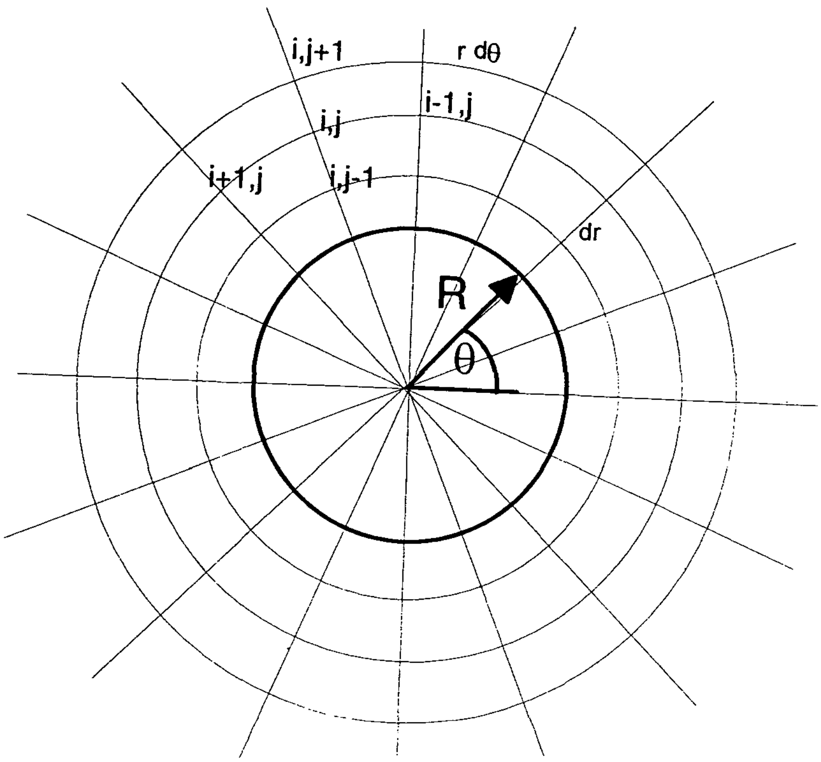
Boundary Condition far (many diameters) from the cylinder:

$$\frac{\phi_{i+1,j} - \phi_{i,j}}{s} = V_o \qquad\qquad \frac{\phi_{i,j+1} - \phi_{i,j}}{s} = 0$$

Boundary Condition on the cylinder:

$$\frac{\phi_{i+1,j} - \phi_{i,j}}{s}\cos\theta + \frac{\phi_{i,j+1} - \phi_{i,j}}{s}\sin\theta = 0$$

cylindrical coordinates



Equation at each grid point:

$$\frac{\phi_{i+1,j}-\phi_{i-1,j}}{2rdr}+\frac{\phi_{i+1,j}-2\phi_{i,j}+\phi_{i-1,j}}{(dr)^2}+\frac{\phi_{i,j+1}-2\phi_{i,j}+\phi_{i,j-1}}{r^2d\theta^2}=0$$

Boundary Condition far (many diameters) from the cylinder:

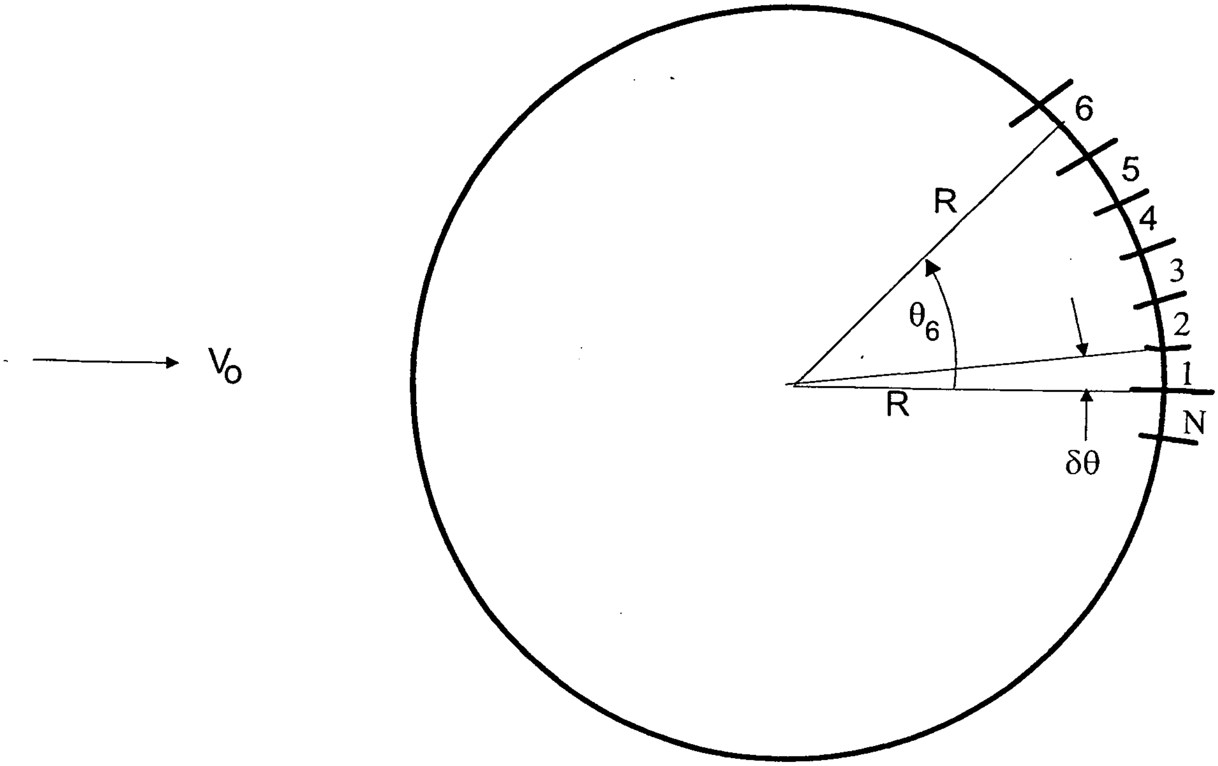
$$\frac{\phi_{i,j+1}-\phi_{i,j}}{dr}=V_o\cos\theta \qquad \frac{\phi_{i+1,j}-\phi_{i,j}}{rd\theta}=-V_o\sin\theta$$

Boundary Condition on the cylinder:

$$\frac{\phi_{i,j+1}-\phi_{i,j}}{dr}=0$$

Example of Discretized Integral Equation

Potential Streaming Flow about a Circular Cylinder Here we seek the perturbation velocity that exists in addition to the uniform flow of speed V_o.



Exterior flow is represented as a source distribution of strength $\sigma(\theta)$ on the surface of the cylinder.

$$\vec{U}(r) = \int_0^{2\pi} \frac{\sigma(\theta)[r - \vec{r}'(\theta)]}{2\pi \left| r - \vec{r}'(\theta) \right|^2} R d\theta$$

Just outside the surface of the cylinder, $\vec{U} \cdot \vec{n} = -V_o \cos\theta$

Now we can form the discretized approximate equation:

$$\sum_{i=1, i\neq j}^{N} \frac{\sigma_i(r_j \vec{-} r_i) \cdot \vec{n_j}}{2\pi |r_j - r_i|^2} R\delta\theta + \frac{\sigma_j}{2} = -V_o \cos\theta_j$$

Stability

When applying a numerical procedure to a problem in fluid mechanics, the result can diverge. In other words, the process is unstable. Such instabilities can be fundamentally fluid mechanical or they may come from inaccuracies in the numerical procedure.

For example, suppose a process is governed by the differential equation:

$$\frac{dy}{dt} = 3y \quad \text{with initial condition} \qquad y(0) = 1$$

We know that the solution to this equation is $y = e^{3t}$ which diverges as t increases. This is a fundamental instability in the process being modeled. A proper numerical solution will capture this instability.

Now we explore a numerical instability using numerical values with three decimal places. Consider the set of values, z_n defined for non-negative integers n by:

$$z_n = \int_0^1 \frac{x^n}{x+5} dx$$

A recursion relation for the z's can be made as follows:

$$z_n + 5z_{n-1} = \int_0^1 \frac{x^n + 5x^{n-1}}{x+5} dx = \int_0^1 \frac{x^{n-1}(x+5)}{x+5} dx = \frac{1}{n}$$

$$z_n = \frac{1}{n} - 5\, z_{n-1}$$

$$\text{For n} = 0, \quad z_o = \int_0^1 \frac{1}{x+5} dx = \ln(6) - \ln(5) = 0.182$$

$$\text{For n} > 0, \quad z_n = \frac{1}{n} - 5\, z_{n-1}$$

$$z_1 = 1.000 - 5 \times 0.182 = 0.090$$

$$z_2 = 0.500 - 5 \times 0.090 = 0.050$$

$$z_3 = 0.333 - 5 \times 0.050 = 0.083$$

$$z_4 = 0.250 - 5 \times 0.083 = -0.165$$

The above negative value for z_4 must be wrong since the integrand is positive. It comes from numerical instability associated with roundoff error.

An alternative recursion relation is:

$$z_{n-1} = \frac{1}{5n} - \frac{Z_n}{5} = 0.2\left(\frac{1}{n} - z_n\right)$$

This reduces the effect of the error by a factor of 5. We will start with an approximation to z_{10} and use the recursion relation for successively smaller values of n.

Approximate Equation: $z_{10} = \int_0^1 \frac{x^{10}}{5(1+0.2x)} \approx 0.2\int_0^1 x^{10}(1-0.2x)dx = 0.015$

$$z_{10} = 0.015$$

$$z_9 = 0.2\left(\frac{1}{10} - 0.015\right) = 0.017$$

$$z_8 = 0.2\left(\frac{1}{9} - 0.017\right) = 0.019$$

$$z_7 = 0.2\left(\frac{1}{8} - 0.019\right) = 0.021$$

$$z_6 = 0.2\left(\frac{1}{7} - 0.021\right) = 0.024$$

$$z_5 = 0.2\left(\frac{1}{6} - 0.024\right) = 0.029$$

$$z_4 = 0.2\left(\frac{1}{5} - 0.029\right) = 0.034$$

$$z_3 = 0.2\left(\frac{1}{4} - 0.034\right) = 0.043$$

$$z_2 = 0.2\left(\frac{1}{3} - 0.043\right) = 0.058$$

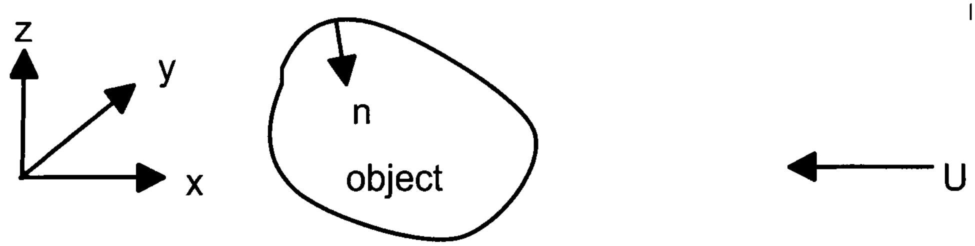
$$z_1 = 0.2\left(\frac{1}{2} - 0.058\right) = 0.088$$

$$z_0 = 0.2\left(\frac{1}{1} - 0.088\right) = 0.182$$

Note that the result for z_0 is cortrect even though an approximate value was used for Z_{10}. This iteration scheme is stable.

P1

PANEL METHODS



Sketch of an Object in a Uniform Stream

$$\Phi = -Ux + \phi$$

Boundary Condition on Perturbation Potential

$$\frac{\partial \Phi}{\partial n} = 0 \qquad \longrightarrow \qquad \frac{\partial \phi}{\partial n} = U\hat{i} \cdot \mathbf{n}$$

Three Dimensional Flows

$$\int\int_S \left[\phi \frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] dS = \begin{cases} 0 & (x,y,z) \text{ outside } S \\ -2\pi\phi(x,y,z) & (x,y,z) \text{ on } S \\ -4\pi\phi(x,y,z) & (x,y,z) \text{ inside } S \end{cases}$$

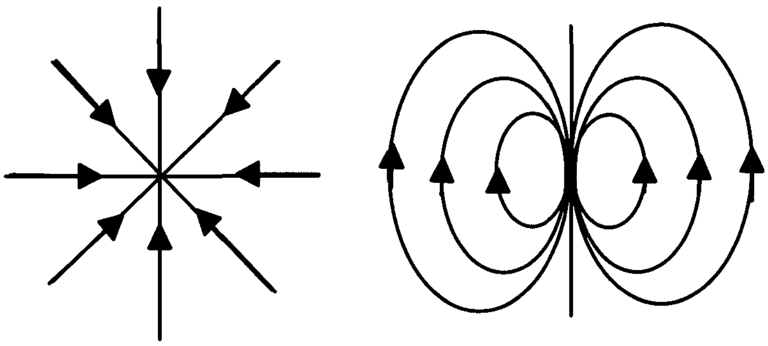
$$G(x,y,z,\xi,\eta,\zeta) = \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} = \frac{1}{r}$$

$$\text{where: } r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

G can be taken as: $G = \frac{1}{4\pi r} + H(x,y,z,\xi,\eta,\zeta)$ where H is an *analytic function* ($\nabla^2 H = 0$ in both (x,y,z) and in (ξ,η,ζ) coordinates. It is used sometimes when a particular H makes the integrand zero on flow boundaries external to an object, thereby removing the necessity of integrating over them.

Interpretation of Green's Theorem

Sketch of Streamlines from Sink and from Dipole



Sink Dipole

Sketch of Sink and Dipole Streamlines

The velocity potential inside the fluid domain and on object surface expressed as distributions of sources (or sinks for the Green function we have chosen) and dipoles on the surface of the object.

$G(x, y, z, \xi, \eta, \zeta)$ is the velocity potential at (x, y, z) due to a point sink of unit strength at (ξ, η, ζ)

A sheet of of sinks with strength σ per unit area causes the normal velocity to jump by $4\pi\sigma$ when crossing the surface from inside the object out into the fluid.

$\frac{\partial G}{\partial n}$ is the velocity potential at (x, y, z) due to a point dipole of unit dipole moment at (ξ, η, ζ) with the axis of the dipole normal to the object and pointing out of the fluid and into the interior of the object.

A sheet of dipoles with strength μ per unit area causes the velocity potential to jump by $-4\pi\mu$ when crossing the surface from inside the object out to the fluid. That's why the dipole moment per unit area needs to be ϕ to generate a velocity potential of $-4\pi\phi$ in the fluid just outside the surface and in the fluid.

Arrangement of the Integral Equation

- arrange the equation for ϕ in the form of an integral equation with unknowns on the left and known quantities on the right.
- The equation will be applied on the surface of the object where the boundary conditions specify part of the equation.
- Excluding an infinitesimal surface around the point $(\xi, \eta, \zeta) = (x, y, z)$ from the region of integration makes the constant on the right hand side of the equation equal to 2π instead of 4π.

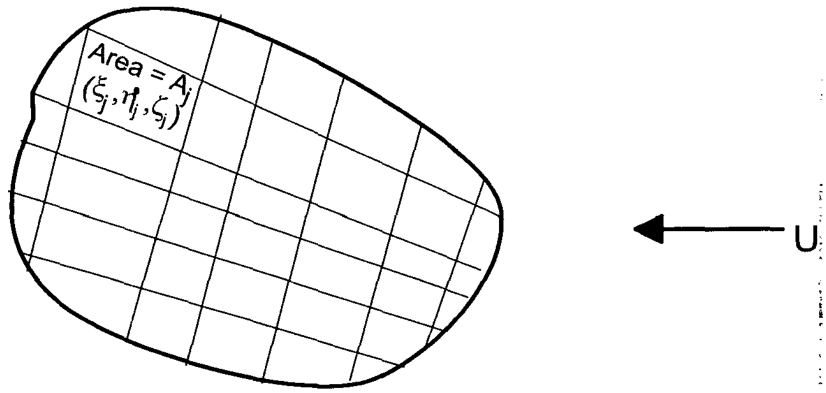
For problems where $\frac{\partial \phi}{\partial n}$ is known on the fluid side of the surface, we write the equation for the unknown ϕ with a known right hand side.

$$\int\int_S \phi \frac{\partial G}{\partial n} \, dS_{\xi,\eta,\zeta} + 2\pi\phi(x, y, z) = \int\int_S G \frac{\partial \phi}{\partial n} \, dS_{\xi,\eta,\zeta}$$

When we substitute $U\hat{\imath} \cdot \mathbf{n}$ for $\frac{\partial \phi}{\partial n}$ the integral equation for the unknown velocity potential ϕ is:

$$\int\int_S \phi \frac{\partial G}{\partial n} \, dS_{\xi,\eta,\zeta} + 2\pi\phi(x, y, z) = \int\int_S G U \hat{\imath} \cdot \mathbf{n}(\xi, \eta, \zeta) \, dS_{\xi,\eta,\zeta}$$

Numerical Form of the Integral Equation



Sketch of an Object Surface Divided into Quadrilateral Panels

- A quadrilateral panel with four corners on the surface will not necessarily be planar.
- The simplest approach is to use approximating planar panels with ϕ constant on each panel.
- For an approximating planar panel, j, place panel centroid, $c_j = (\xi_j, \eta_j, \zeta_j)$ on the actual object surface and orient the panel such that its normal is in the direction of the cross-product of the two diagonal vectors of the non-planar panel.
- This leaves gaps between the panels which are sources of error. However, the smaller the panels, the smaller the gaps. A less erroneous procedure uses non-planar panels, but the integrations for the discretized equation becomes more complicated.

Making the Numerical Equations

- Write a separate equation for values of (x, y, z) at the centroid of each panel.
- There are N centroids where ϕ will be calculated so there will be N equations.
- The calculation (field) points, which are the panel centroids will be labeled by the index i and the value of ϕ on the i^{th} panel is called ϕ_i These are the unknowns.
- For each equation (one for each of the N valies of i), the integrals on the right hand side are done over each panel (j) individually and the results are summed together.

$$\sum_{j=1}^{N} \int\int_{S_j} \phi(\xi,\eta,\zeta)\frac{\partial G_{ij}}{\partial n_j}\, dS_{\xi,\eta,\zeta} + 2\pi\phi_j\,\delta_{ij} = \sum_{j=1}^{N} \int\int_{S_j} G_{ij} U\,\hat{\imath}\cdot\mathbf{n_j}(\xi,\eta,\zeta)\, dS_{\xi,\eta,\zeta}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

The symbol G_{ij} means the Green function with the field point at the centroid of the i^{th} panel and with the source point varying over all locations in the j^{th} panel as the integration is carried out. Since, in the approximation being used, ϕ is a constant,

$$\sum_{j=1}^{N} \phi_j \int\int_{S_j} \frac{\partial G_{ij}}{\partial n_j}\, dS_{\xi,\eta,\zeta} + 2\pi\phi_j\,\delta_{ij} = \sum_{j=1}^{N} \int\int_{S_j} G_{ij} U\,\hat{\imath}\cdot\mathbf{n_j}(\xi,\eta,\zeta)\, dS_{\xi,\eta,\zeta}$$

Define:

$$\int\int_{S_j} \frac{\partial G_{ij}}{\partial n_j}\, dS_{\xi,\eta,\zeta} + 2\pi\,\delta_{ij} \equiv A_{ij}$$

and

$$\sum_{j=1}^{N} \int\int_{S_j} G_{ij} U\,\hat{\imath}\cdot\mathbf{n_j}(\xi,\eta,\zeta)\, dS_{\xi,\eta,\zeta} \equiv B_i$$

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For planar panels, $\mathbf{n_j}(\xi, \eta, \zeta)$ is a constant on each panel and we call it $\mathbf{n_j}$ and then $U\hat{\imath} \cdot \mathbf{n_j}(\xi, \eta, \zeta)$ is a constant on each panel so that:

$$B_i = \sum_{j=1}^{N} U\hat{\imath} \cdot \mathbf{n_j} \int\int_{S_j} G_{ij}\, dS_{\xi,\eta,\zeta}$$

The final set of equations for the N values of ϕ is:

$$\sum_{j=1}^{N} A_{ij}\phi_j = B_i \quad \text{equivalently} \quad \mathbf{A}\phi = \mathbf{B}$$

P<

Solution Steps

- Do the numerical integrals to generate the matrix **A** and the vector **B** which are integrals of G and $\frac{\partial G}{\partial n}$.
- For each field point i integrals must be done for all N values of j. .
- Once **A** and **B** are determined, solve the set of linear equations is solved for ϕ.
- When the number of panels is less than a few thousand, it is practical to solve the equations by Gaussian Elimination, or an **LU** decomposition.
- The burden of carrying out these procedures is removed from the MATLAB user by obtaining the entire solution for all the values of ϕ with the MATLAB statement: $\phi = \mathbf{A}\backslash\mathbf{B}$.
- After ϕ and $\frac{\partial \phi}{\partial n}$ have been determined on the surface of the object the numerical approximation to the left hand side of Green's Theorem can be used to compute ϕ at any point in the fluid.
- The usual goal of a panel method in fluid mechanics is to find the pressure distribution on an object from which the forces and moments can be computed. With inviscid fluid mechanics for which the panel method was developed, the local pressure, P is given by Bernoulli's equation:

$$P = \rho\left[-\frac{\partial \phi}{\partial t} - |\nabla\phi|^2 - gz\right]$$

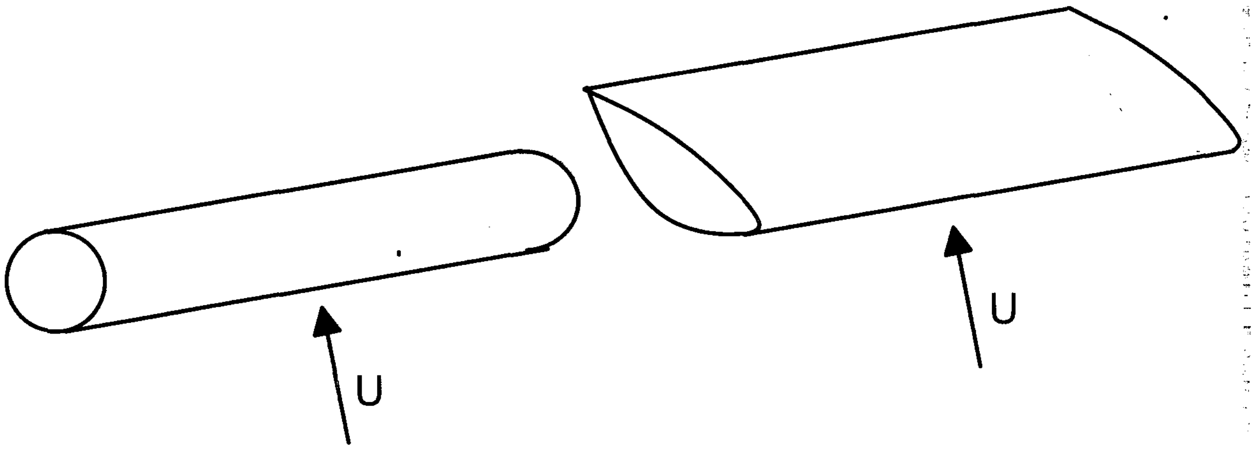
ρ is the fluid density and g is the acceleration due to gravity.

The first term on the right hand side applies to time-dependent motion. Although this has not been considered explicitely here, one can imagine an object moving sinusoidally so that in a reference frame attached to the object U is sinusoidal and the solution has time dependence. The third term is simply the hydrostatic pressure. The most difficult term on the right hand side to compute is generally the second. It is the square of the velocity at the object surface.

Two Dimensional Panel Methods

The development for two-dimensional flows is similar to the 3D case, except that two dimensional source functions are involved and the dimensionality of some integrals and associated constants are different.

Two dimensional flows are either mathematical abstractions with all flow directions in a two dimensional plane or physical approximations for long prismatic objects with an inflow that is perpendicular to the long axis.



Objects for which the Flow is Nearly Two-Dimensional

For the two-dimensional case, Green's Theorem, is:

$$\int_L \left[\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n}\right] d\ell = \begin{cases} 0 & (x,y) \text{ outside } S \\ -\pi\phi(x,y) & (x,y) \text{ on } S \\ -2\pi\phi(x,y) & (x,y) \text{ inside } S \end{cases}$$

and the Green function is:

$$G(x,y,\xi,\eta) = -\ln\sqrt{(x-\xi)^2 + (y-\eta)^2)} = -\ln r$$

$$\text{where: } r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$$

As in the 3D case, in 2D flows, sometimes the Green function is taken as $-\ln r + h(x,y)$ where h is an analytic function which means $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$. Green's Theorem, still holds with this Green function. It is used for problems with boundaries on which the integral in Green's Theorem vanishes to simplify the integrations that are necessary.

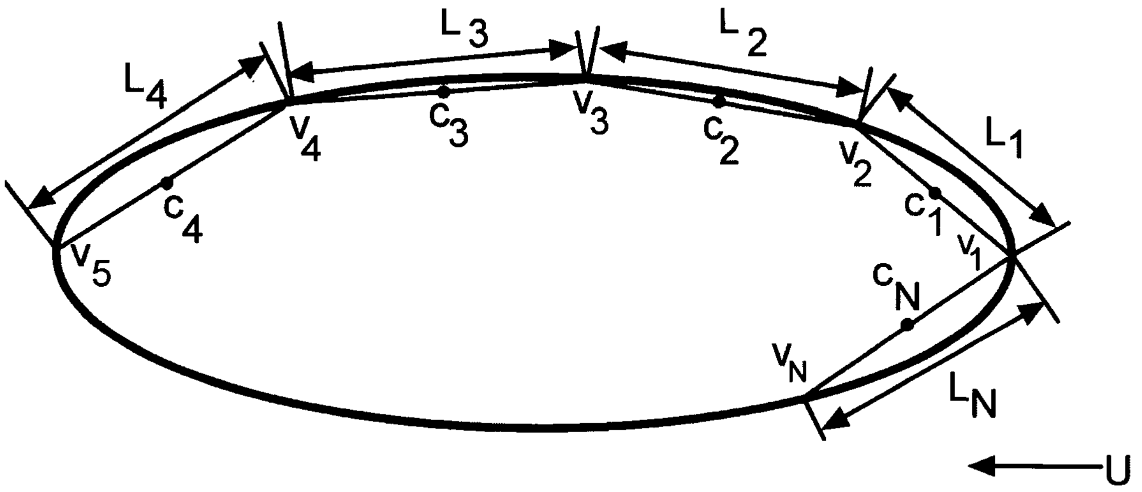
For an incomming stream in the $-x$ direction the total potential is:
$\Phi = -Ux + \phi$, the boundary condition for the perturbation potential, ϕ on the surface of the vehicle is $\frac{\partial \phi}{\partial n} = U\,\hat{\imath} \cdot \mathbf{n}$ on the object surface.

The integral equation for the unknown function, $\phi(x, y)$ on the object surface, having unknown quantities on the left and known quantities on the right hand is:

$$\int_L \phi(\xi, \eta) \frac{\partial G}{\partial n}\, d\ell_{\xi,\eta} + \pi\phi(x, y) = \int_L G\, U\,\hat{\imath} \cdot \mathbf{n}\, d\ell_{\xi,\eta}$$

Numerical Form of the Two Dimensional Integral equation

The 2D curve is divided into panels which become segments of a curve:



Panelization of a Two Dimensional Object

- Each panel(actually a line) is defined by two vertices on the object curve. Panel 1 extends from vertex 1 (labeled v_1 in the figure) to vertex 2, panel 2 extends from vertex 2 to vertex 3, etc.
- The centroids of each panel, are at the mid-points of the arc length of each panel. For flat panels, straight lines are drawn between adjacent vertices and the centroids, called c_i are moved to the midpoints of each straight line panel.
- The unit normal vector, $\mathbf{n}_i$ on panel i, is a constant along the length of the panel. The panels are defined by the (x, y) values of the two vertices at the panel ends.
- For the simplest implementation the potential, ϕ, is approximated as being a constant on each panel.

With these approximations and definitions, the numerical form of the integral equation is:

$$\sum_{i=1}^{N} \phi_j \int_{L_j} \frac{\partial G_{ij}}{\partial n_j} d\ell_j + \pi \phi_j \, \delta_{ij} = \sum_{j=1}^{N} U \, \hat{\imath} \cdot \mathbf{n_j} \int_{L_j} G_{ij} d\ell_j$$

Analogous to the 3D case, we define:

$$\int_{L_j} \frac{\partial G_{ij}}{\partial n_j} d\ell_j + \pi\,\delta_{ij} \equiv A_i j$$

and

$$\sum_{j=1}^{N} U\,\hat{\imath} \cdot \mathbf{n_j} \int_{L_j} G_{ij} d\ell_j \equiv B_i$$

Again, the final set of equations for the N unknown values of ϕ is:

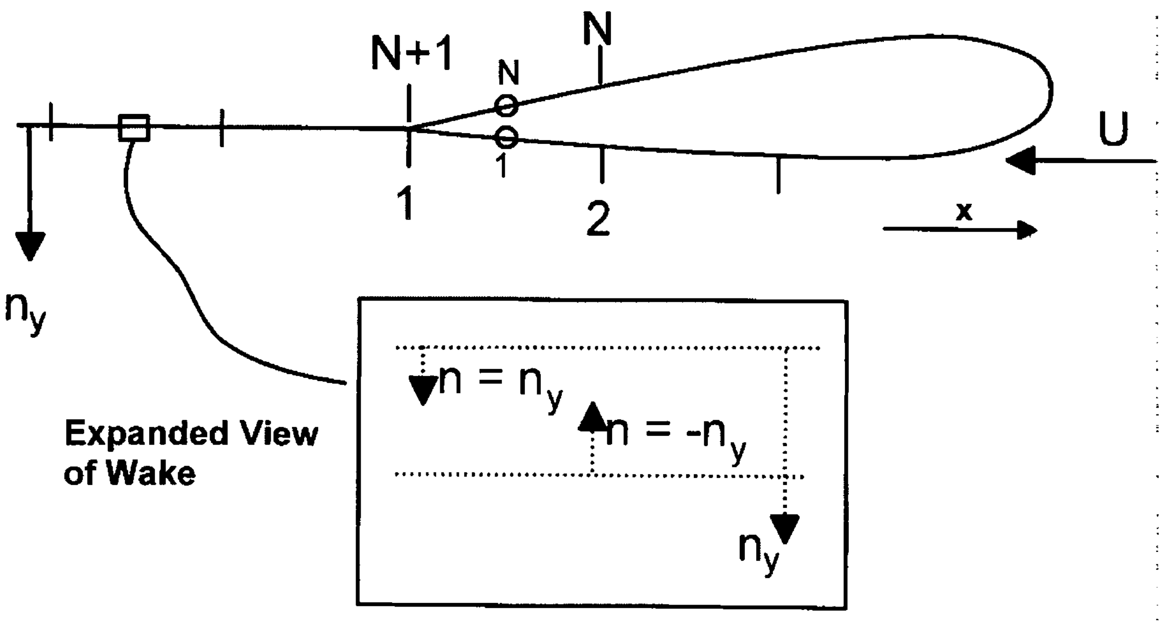
$$\sum_{j=1}^{N} A_{ij}\,\phi_j = B_i \quad \text{or} \quad \mathbf{A}\,\phi = \mathbf{B}$$

Calculation of A_{ij} and B_i

For a field point i located at the midpoint of the $i'th$ panel we need to determine for each panel j the value of A_{ij} by integrating $\frac{\partial G_{ij}}{\partial n_j}$ over the j'th panel. We also need to determine B_i by summing up the terms in the equation for B_i. General purpose MATLAB m-functions for doing these integrals exist and will be provided to students.

Situations With the Generation of Lift

- If the object in a streaminf flow has a sharp trailing edge, it will usually generate lift.
- The lift is related to circulation around the object so that ϕ jumps acorss the trailing edge. This results in a "wake" across which the potential jumps by the same amount as at the trailing edge.
- To obtain a flow domain in which the velocity potential is analytic (satisfies $\nabla^2\phi = 0$) the wake must be excluded from the domain. The line of the wake can be treated as part of the object upon which there is a uniform strength dipole sheet which makes the velocity potential jump when crossing it.
- Here we will use an approximation in which the wake is presumed to follow the free stream direction. Theoretically, the wake is infinitely long. For practical purposes, we can model the wake as being about two airfoil chord lengths long since dipoles further than this from the object (airfoil) will have negligible effect on the flow on the object.



A Two-Dimensional Lifting Airfoil With a Wake

- Green's theorem is applied to a curve which starts at the far end of the wake at the bottom, goes along the bottom of the wake, goes around the foil, and finally goes along the top of the wake. On the top and the bottom of the wake, on any wake panel, $\frac{\partial \phi}{\partial n}$ are equal and opposite For any control point on the foil where the potential is to be determined and a source panel on the wake, G is the same for the two elements of the integration path, one on the wake top and one exactly under it on the wake bottom. Therefore, for the control points on the foil, $\int_{wake} G \frac{\partial \phi}{\partial n} d\ell = 0$.

- At the start of the wake at its junction with the airfoil, there is a jump in potential in going from the bottom to the top.

$$\phi_{wake\ top} - \phi_{wake\ bottom} \equiv \Delta\phi = \phi_N - \phi_1$$

  This jump in potential is maintained all along the wake because as one moves aft along the wake the potential changes the same amount on the top and on the bottom by $\int_{tr\ edge}^{x} u(x)dx$.

- Thus, for control (field) points on the foil, with L being the path around the foil, W being the single line along the wake from its aft most considered point to the trailing edge of the foil, and **n** on the wake pointing downward for the configuration shown in the Figure , Green's theorem takes the form:

$$\int_L \phi(\xi, \eta) \frac{\partial G}{\partial n}\, d\ell_{\xi,\eta} + \int_W \Delta\phi \frac{\partial G}{\partial n} d\ell_{\xi,\eta} + \pi\phi(x, y) = \int_L G\, U\, \hat{\imath} \cdot \mathbf{n}\, d\ell_{\xi,\eta}$$

- If the wake panels are labeled $N+1$ to M with lengths ds_{j_w}, the discretized form of the equations appropriate for numerical solution for values of ϕ at control points, i, on the foil are:

$$\sum_{j=1}^{N} \phi_j \int_{L_j} \frac{\partial G_{ij}}{\partial n_j} d\ell_j + (\phi_N - \phi_1) \sum_{j_w=N+1}^{M} \int_{L_{j_w}} \frac{\partial G_{ij_w}}{\partial n_{j_w}} d\ell_{j_w} + \pi\phi_i = \sum_{j=1}^{N} U\, \hat{\imath} \cdot n_j \int_{L_j} G_{ij} d\ell_j$$

To put the preceding equation in the same format that was used for the non-lifting case, we define:

$$\int_{L_j} \frac{\partial G_{ij}}{\partial n_j} d\ell_j + \pi\,\delta_{ij} + \int_{Ljw} \frac{\partial G_{ij_w}}{\partial n_{j_w}} d\ell_{jw}\,(\delta_{jN} - \delta_{j1}) \equiv Q_{ij}$$

Then, the final set of equations to be solved is:

$$\sum_{j=1}^{N} Q_{ij}\,\phi_j = B_i$$

which has the matrix notation:

$$\mathbf{Q}\,\phi = \mathbf{B}$$

Computation of pressures and forces

For steady flow, Bernoulli's equation for dynamic pressure is: $P = -\rho|V|^2$ This presumes that in the very far field, the pressure is hydrostatic and there is no dynamic pressure there. In unsteady flows, there is an additional contribution to the dynamic pressure equal to $-\rho\frac{\partial\phi}{\partial t}$ which is straightforward to compute at each control point.

Here we concentrate on computation of the $-rho|V|^2$ term in the dynamic pressure. It is most convenient to know this pressure, P_i at each control point. Then the force is: $\mathbf{F} = \sum_{i=1}^{N} P_i \mathbf{n}_i d\ell_i$

The central problem is calculation of the velocity at the surface of the object. At each control point, i, the total flow is tangent to the object surface so we need only find tangential velocity. The tangential velocity is the tangential derivative of the total potential, Φ (perturbation potential plus any exterior potential such as $-Ux$ for the steady flow problems we have been considering). For two-dimensional flows the tangential velocity component has a single direction at each control point. We know how to determine the tangential derivative at a point using a modified central difference procedure for smooth objects. Even when the object has a sharp edge, this procedure can be used for all control points except for those adjacent to a trailing edge. In addition, the numerical derivative can be determined at a point that is midway between the first and second control points from the trailing edge. Then the tangential derivative can be approximated at a control point nearest the trailing edge by numerical extrapolation. An example follows.

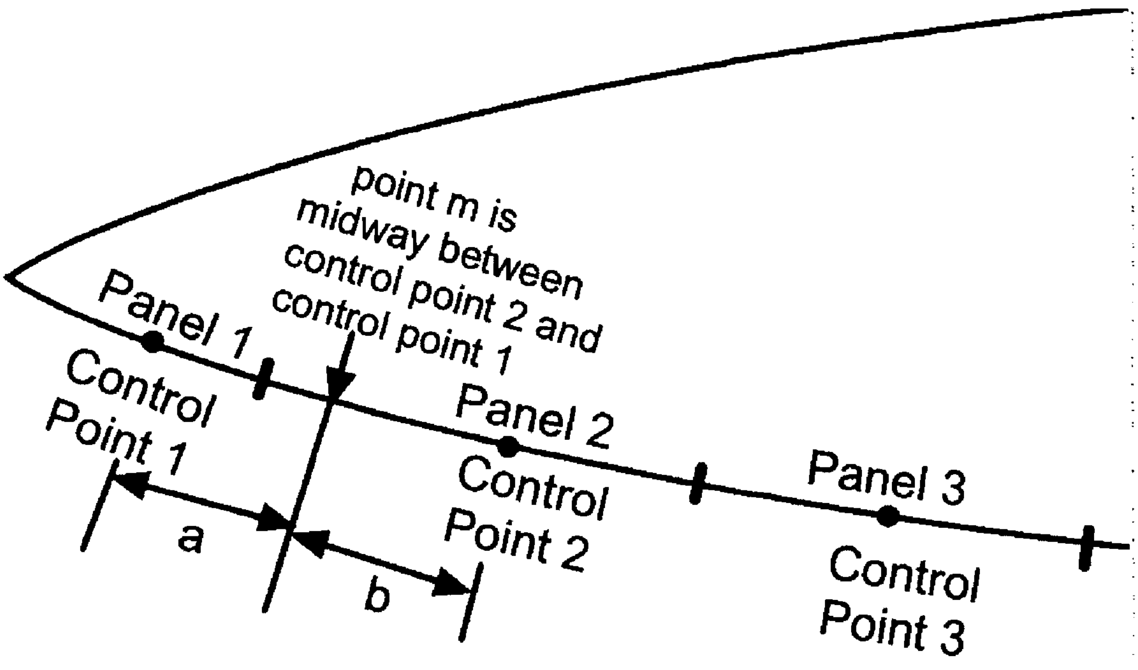
We know how to calculate the numerical tangential Derivative, called d_2 at-Control point 2. The lengths of the panels are called $L_1, L_2, ...$ The tangential derivative at point m, called d_m is given by:

$$d_m = \frac{\Phi_2 - \Phi_1}{0.5(L_1 + L_2)}$$

Then, by extrapolation, the derivative at control point 1, called d_1 is numerically approximated as:

$$d_1 = d_m - a\frac{d_2 - d_m}{b}$$

P17



Tangential Derivative at a Control Point Near a Sharp Edge

BLTITLE

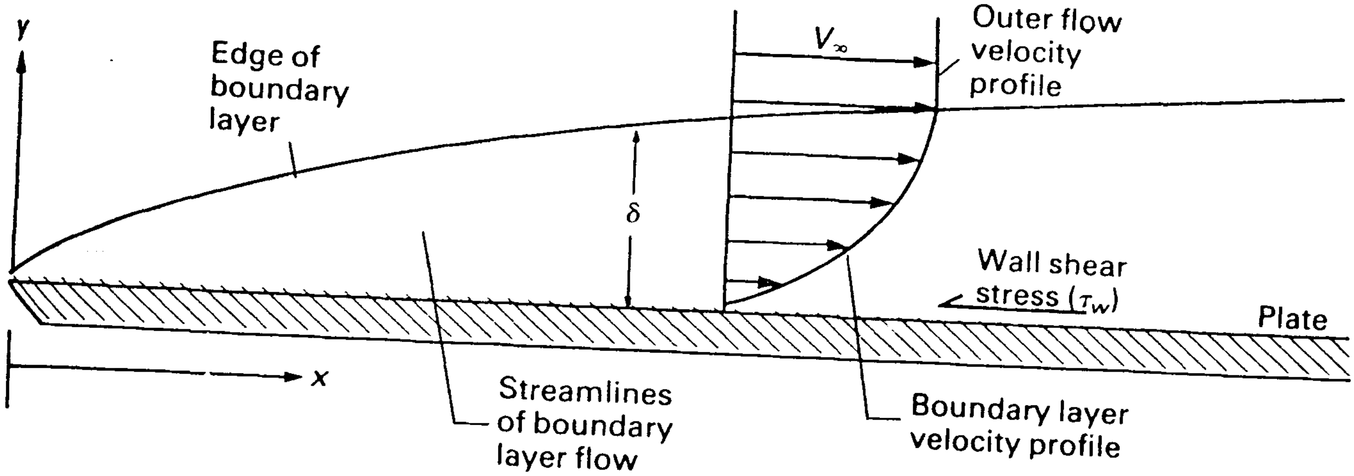
Boundary Layers

Two-Dimensional Steady Boundary Layer Equations

x is horizontal direction along direction of main flow velocity u. Velocity at outer edge of boundary layer is called U_∞ or V_∞ or U_e or V_e.

y is perpendicular to wall and velocity in this direction is v.

The boundary layer begins, say, at $x = 0$ and the boundary layer thickness is δ. $\delta \ll x$. Because the boundary layer is thin, to leading order the pressure is constant through the thickness of the boundary layer, $\frac{\partial P}{\partial y} = 0$. Also, $v \ll u$, and $\frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y}$.



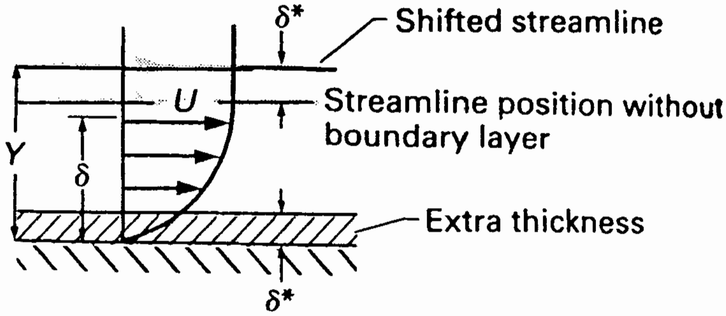
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

fmbbL2

Boundary Layer Parameters



Thickness of Boundary Layer defined as location where u is 99% of U_e.

$$\delta = y]_{u/U_e=0.99}$$

The wall shear stress τ_w is given by:

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{\text{wall}} = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

The skin friction coefficient, C_f, is:

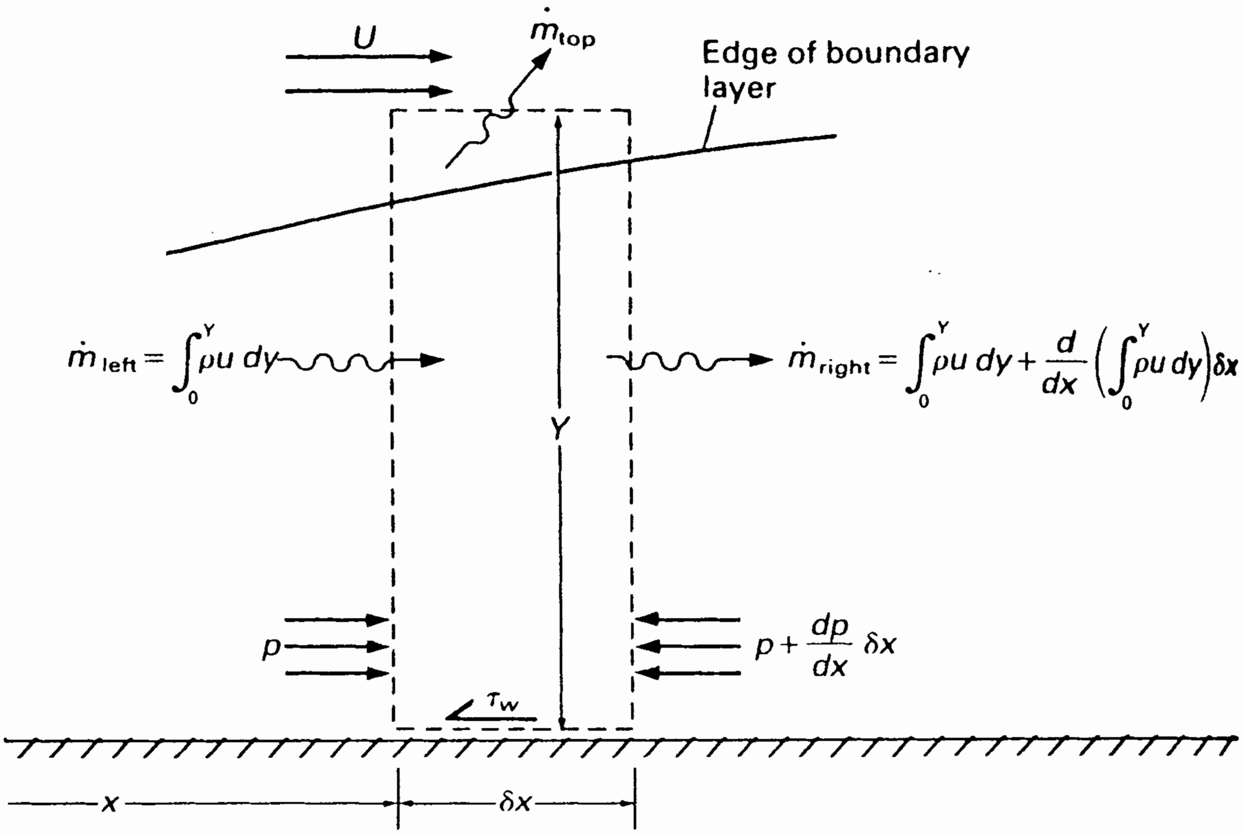
$$C_f = \tau_w / \left(\frac{1}{2}\rho U_e^2\right) = \frac{2\tau_w}{\rho U_e^2} = \frac{2\nu}{U_e^2}\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

The displacement thickness, δ^* is the thickness of a flow of speed U_e that carries a flow rate equal to the deficit in the boundary layer because its speed is less than U_e.

$$U_e\,\delta^* = \int_0^{\delta}(U_e - u)dy \qquad\qquad \delta^* = \int_0^{\delta}\left(1 - \frac{u}{U_e}\right)dy$$

FmbbL3

Mass Fluxes



$$\dot{m}_{\text{left}} = \dot{m}_{\text{right}} + \dot{m}_{\text{top}}$$

$$\dot{m}_{\text{top}} = -\frac{d}{dx}\left(\int_0^Y \rho u dy\right)\delta x$$

Momentum Equation in x direction

$$\dot{M}_{\text{right}} + \dot{M}_{\text{top}} + \dot{M}_{\text{left}} = F_{\text{pressure}} + F_{\text{stress}}$$

$$\dot{M}_{\text{left}} = -\int_0^Y \rho u^2 dy$$

$$\dot{M}_{\text{right}} = \int_0^Y \rho u^2 dy + \frac{d}{dx}\left(\int_0^Y \rho u^2 dy\right)\delta x$$

$$\dot{M}_{\text{top}} = \dot{m}_{\text{top}} U_e = -U_e \frac{d}{dx}\left(\int_0^Y \rho u dy\right)\delta x$$

$$F_{\text{pressure}} = -\frac{dp}{dx} Y\,\delta x = \rho U_e \frac{dU_e}{dx} Y\,\delta x \qquad\qquad F s = -\tau_w \delta x$$

One additional needed equation is:

$$Y = \int_0^Y dy$$

Then all the equations on the last two pages can be combined into:

$$\frac{d}{dx}\int_0^Y u(U_e - u)dy + \frac{dU_e}{dx}\int_0^Y (U_e - u)dy = \frac{\tau_w}{\rho}$$

For $y > \delta$ the integrands are zero so the upper limits can be changed to δ.

$$\frac{d}{dx}\int_0^\delta u(U_e - u)dy + \frac{dU_e}{dx}\int_0^\delta (U_e - u)dy = \frac{\tau_w}{\rho}$$

This is Von Karman's Momentum Integral Equation. It relates the integrals of the velocity profile in the boundary layer to the shear stress and U_e and U_e^2 whose x-derivative is proportional to the pressure gradient.

The momentum thickness Θ is defined as:

$$\Theta = \int_0^\delta \frac{u}{U_e}\left(1 - \frac{u}{U_e}\right) dy$$

With this definition, the momentum integral equation can be written in the following two forms:

$$\frac{d}{dx}\left[U_e^2 \Theta\right] + \delta^* U_e \frac{dU_e}{dx} = \tau_w/\rho$$

$$\frac{d\Theta}{dx} + (2 + H)\frac{\Theta}{U_e}\frac{dU_e}{dx} = \frac{C_f}{2} \qquad\qquad \text{where: } H \equiv \frac{\delta^*}{\Theta}$$

FMBBL4(2)

A second boundary layer equation comes from equating the kinetic energy change along x in the boundary layer to the energy input or output from the pressure distribution and the energy dissipation due to shear stresses in the boundary layer.

The kinetic energy thickness, θ^* is defined as:

$$\theta^* = \int_0^\delta \frac{u}{U_e}\left(1 - \frac{u^2}{U_e^*}\right) dy$$

The kinetic energy dissipation coefficient, C_D, is defines ad:

$$C_D = \frac{D}{\rho u_e^3}$$

where D is the dissipation per unit area (along and perpendicular to the surface).

Using these definitions, the kinetic energy equation is:

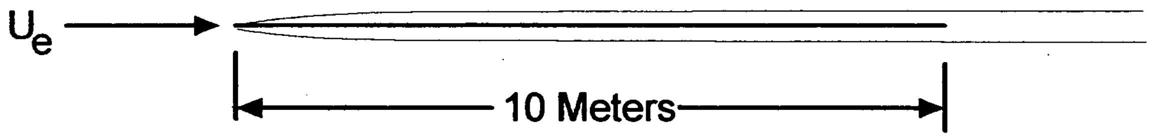
$$\frac{d\theta^*}{dx} + 3\frac{\theta^*}{u_e}\frac{du_e}{dx} = 2C_D$$

The energy thickness ratio, H^* is defined as: $H^* = \dfrac{\theta^*}{\theta}$

It is common to combine the kinetic energy equation and Von Karman's momentum equation to obtain:

$$\frac{\theta}{H^*}\frac{dH^*}{dx} = \frac{2C_D}{H^*} - \frac{C_f}{2} + (H-1)\frac{\theta}{u_e}\frac{du_e}{dx}$$

Example of Solution of Momentum Integral BL Equation



$U_e = 2m/s \qquad \delta(x) = 0.01*(1-e^{-0.1x}) \qquad \frac{u(y)}{U_e} = \left(1 - e^{-k(x)y}\right)^2 \qquad \rho = 1000\text{kg/m}^3$

Problem: Determine the shear stress, τ, at x = 5 meters.

Determination of $k\delta$ from BL thickness:

$$0.99 = \left(1 - e^{-k(x)\delta(x)}\right)^2 \longrightarrow k(x)\delta(x) = 5.3 \qquad k(x) = \frac{5.3}{\delta(x)}$$

At x = 5m, $k = 1347 \ m^{-1}$.

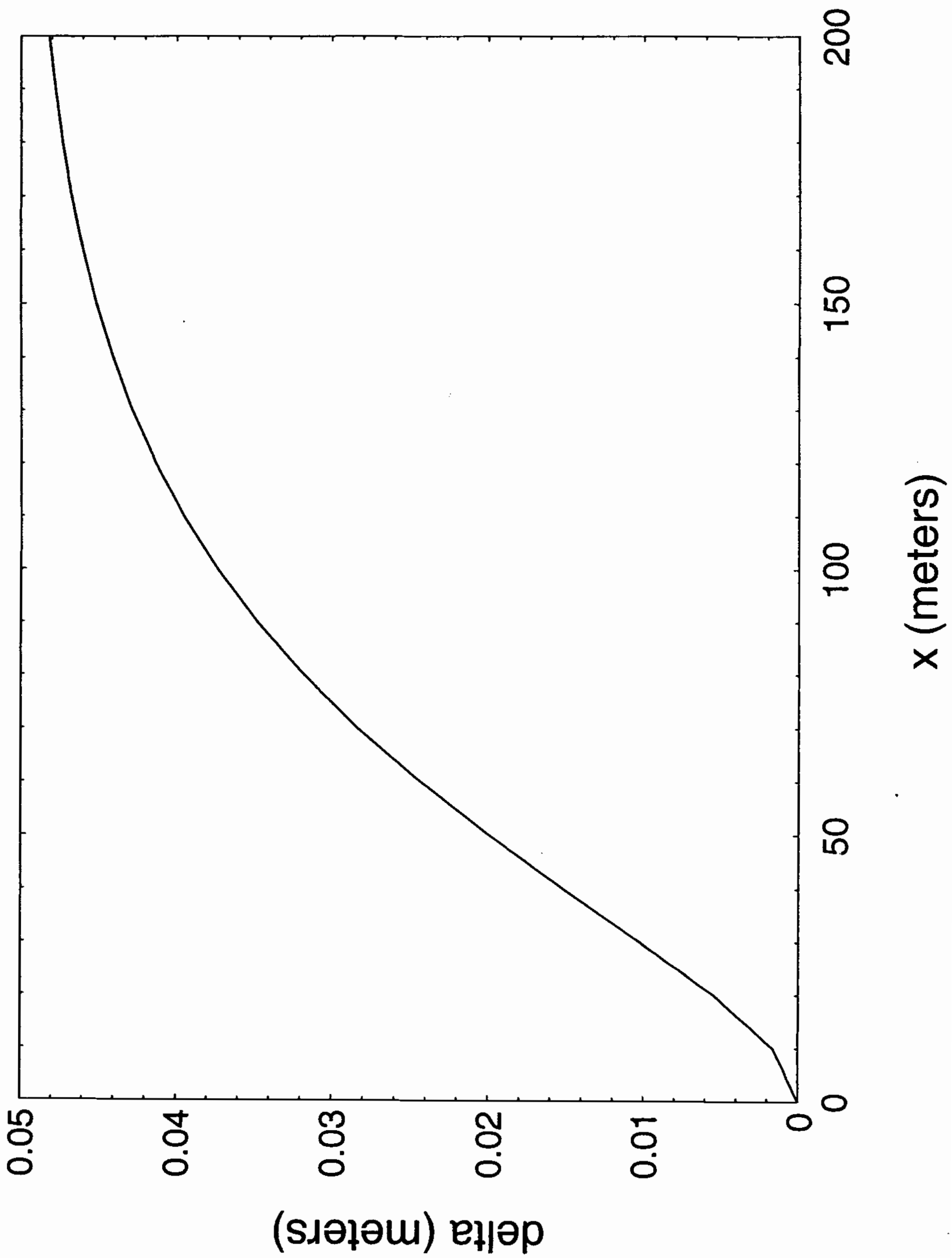
$$\frac{d}{dx}\int_0^{0.01[1-\exp(-0.1x)]} U_e(1 - e^{k(x)y})^2 \left[U_e - U_e(1 - e^{-ky})^2\right] dy + 0 = \frac{\tau}{\rho}$$

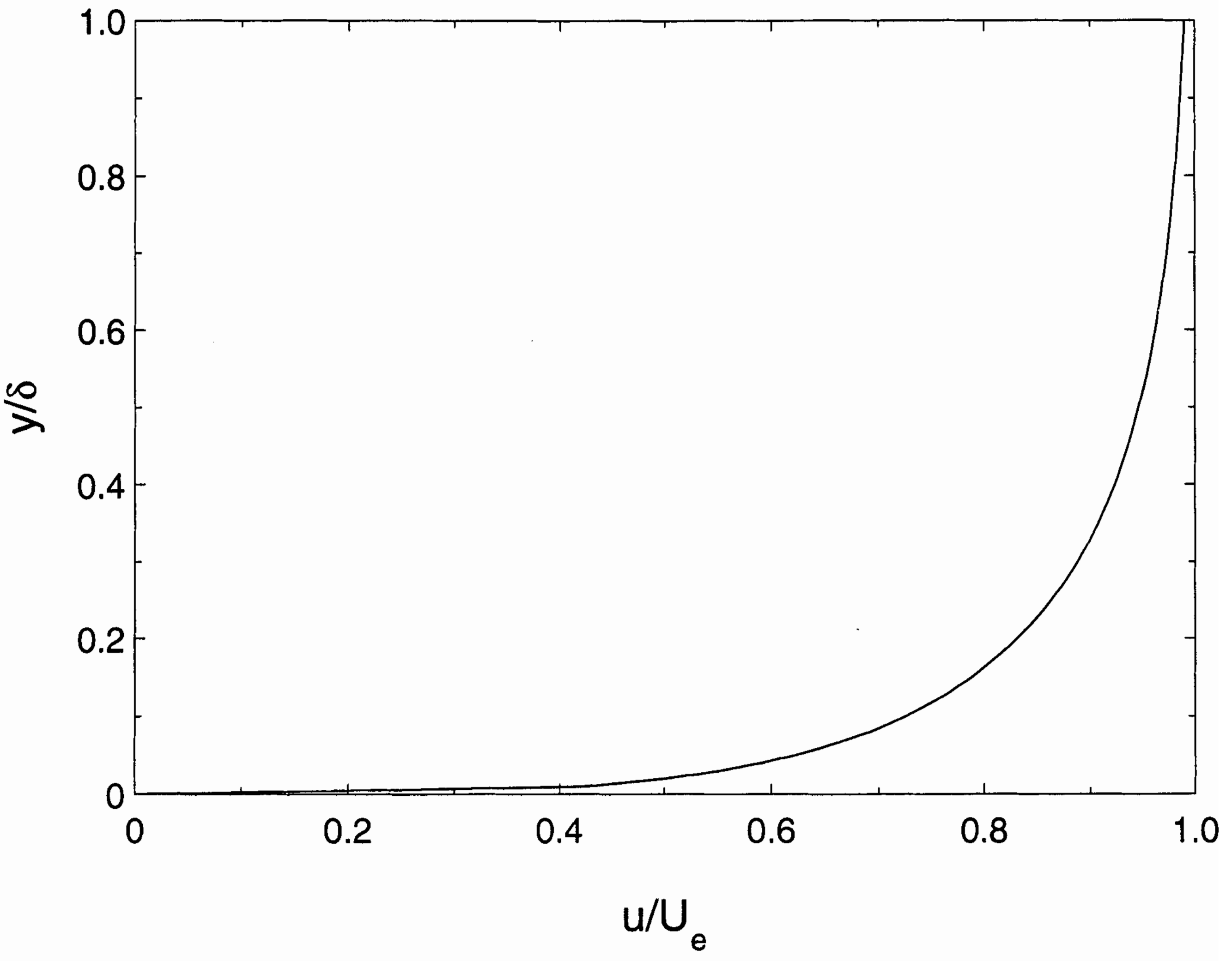
$$0.01(1 - e^{-0.1\times 5}) = 0.01(1 - e^{-0.5}) = 0.00393$$

$$\begin{aligned}\frac{\tau}{\rho} &= U_e^2(1 - e^{-5.3})^2 \left[1 - (1 - e^{-5.3})^2\right] \frac{d}{dx}\left[0.01(1 - e^{-0.1x})\right] \\ &\quad + \int_0^{0.00393} \frac{d}{dx}\left\{U_e(1 - e^{-k(x)y})^2 \left[U_e - U_e(1 - e^{-k(x)y})^2\right]\right\} dy \\ &= U_e^2(0.000060 + 0.000100) = 0.00016\, U_e^2\end{aligned}$$

$$\tau = 1000 \times 4 \times 0.00016 = 0.64 N/m^2$$

$$c_f = \frac{\tau}{\frac{1}{2}\rho U_e^2} = 0.00032$$


delta (meters)
0.05
0.04
0.03
0.02
0.01
0
0
50
100
150
200
x (meters)


1.0
0.8
0.6
0.4
0.2
0
y/δ
0
0.2
0.4
0.6
0.8
1.0
u/U_e

x	delta
0.000000	0.000000
10.000000	0.001643
20.000000	0.005434
30.000000	0.010179
40.000000	0.015162
50.000000	0.019979
60.000000	0.024416
70.000000	0.028381
80.000000	0.031848
90.000000	0.034836
100.000000	0.037382
110.000000	0.039534
120.000000	0.041340
130.000000	0.042848
140.000000	0.044104
150.000000	0.045145
160.000000	0.046007
170.000000	0.046718
180.000000	0.047305
190.000000	0.047788
200.000000	0.048185

udata

y/delta	u/Ue
0.000000	0.000000
0.010000	0.422358
0.020000	0.500261
0.030000	0.551427
0.040000	0.590201
0.050000	0.621605
0.060000	0.648047
0.070000	0.670886
0.080000	0.690975
0.090000	0.708884
0.100000	0.725018
0.110000	0.739674
0.120000	0.753079
0.130000	0.765409
0.140000	0.776804
0.150000	0.787378
0.160000	0.797225
0.170000	0.806423
0.180000	0.815038
0.190000	0.823127
0.200000	0.830737
0.210000	0.837911
0.220000	0.844685
0.230000	0.851092
0.240000	0.857160
0.250000	0.862915
0.260000	0.868379
0.270000	0.873573
0.280000	0.878515
0.290000	0.883221
0.300000	0.887707
0.310000	0.891986
0.320000	0.896072
0.330000	0.899974
0.340000	0.903705
0.350000	0.907274
0.360000	0.910689
0.370000	0.913960
0.380000	0.917093
0.390000	0.920096
0.400000	0.922976
0.410000	0.925739
0.420000	0.928390
0.430000	0.930936
0.440000	0.933381
0.450000	0.935730
0.460000	0.937988
0.470000	0.940158
0.480000	0.942245
0.490000	0.944252
0.500000	0.946184
0.510000	0.948042
0.520000	0.949832
0.530000	0.951554
0.540000	0.953213
0.550000	0.954812
0.560000	0.956351
0.570000	0.957835
0.580000	0.959264
0.590000	0.960643
0.600000	0.961971
0.610000	0.963253
0.620000	0.964488
0.630000	0.965680
0.640000	0.966830
0.650000	0.967939
0.660000	0.969009
0.670000	0.970042
0.680000	0.971039
0.690000	0.972001
0.700000	0.972930
0.710000	0.973827
0.720000	0.974693
0.730000	0.975529
0.740000	0.976336
0.750000	0.977116
0.760000	0.977870
0.770000	0.978597
0.780000	0.979300
0.790000	0.979979
0.800000	0.980636
0.810000	0.981270
0.820000	0.981882
0.830000	0.982475
0.840000	0.983047
0.850000	0.983600
0.860000	0.984135
0.870000	0.984651
0.880000	0.985151
0.890000	0.985634
0.900000	0.986101
0.910000	0.986552
0.920000	0.986989
0.930000	0.987411
0.940000	0.987819
0.950000	0.988214
0.960000	0.988595
0.970000	0.988964
0.980000	0.989321
0.990000	0.989666
1.000000	0.990000

FMBBL6A(1)

Calculation of Turbulent Boundary Layer when Pressure Distribution is Known

This result is approximate since the boundary layer thickness will alter the pressure distribution.

The principal unknowns (quantities to be determined) are: $\theta(x)$ and $\delta^*(x)$. An equivalent set of unknowns is $\theta(x)$ and $H(x)$.

There are two fundamental equations:

$$\frac{d\theta}{dx} = -(H+2)\frac{\theta}{U_e}\frac{dU_e}{dx} + \frac{C_f}{2} \tag{1}$$

$$\frac{\theta}{H^*}\frac{dH^*}{dx} = \frac{2C_D}{H^*} - \frac{C_f}{2} + (H-1)\frac{\theta}{u_e}\frac{du_e}{dx} \tag{2}$$

To be able to integrate the unknowns along the boundary layer, the derivatives of each of them are required: $d\theta/dx$ and dH/dx. Equation 1 is in the desired form. To put equation 2 in the desired form, use the chain rule:

$$\frac{dH^*}{dx} = \frac{dH}{dx}\frac{dH^*}{dH} \tag{3}$$

Empirical "closure relations" for $H^*(H)$ and dH^*/dH exist. Therefore we write the energy equation in the desired form as:

$$\frac{dH}{dx} = \frac{H^*}{\theta}\frac{1}{dH^*/dH}\left[\frac{2C_D}{H^*} - \frac{C_f}{2} + (H-1)\frac{\theta}{u_e}\frac{du_e}{dx}\right] \tag{4}$$

To do the integrals numerically, we need a means of determining C_f, C_D, H^* and dH^*/dH in terms of the principal quantities H and R_θ, where $R_\theta = U_e\theta/\nu$. These empirical "closure relations" have been determined by assembling a large amount of experimental data.

Laminar Closure Relations

$$H^* = \begin{cases} 0.76(H-4)^2/H + 1.515, & H < 4.0 \\ 0.015(H-4)^2/H + 1.515, & H \geq 4.0 \end{cases}$$

$$C_f = \begin{cases} \left[0.03954[(7.4-H)^2/(H-1.0)] - 0.134\right]/R_\theta, & H < 7.4 \\ \left[0.044[1.0 - 1.4/(H-6)]^2 - 0.134\right]/R_\theta, & H \geq 7.4 \end{cases}$$

$$\frac{2C_D}{H^*} = \begin{cases} \left[0.00205(4-H)^{5.5} + 0.207\right]/R_\theta, & H < 4.0 \\ \left[-0.003(H-4.0)^2/(1+0.02(H-4)^2) + 0.207\right]/R_\theta, & H \geq 4.0 \end{cases}$$

Turbulent Closure Relations

$$H_o = \begin{cases} 3 + 400/R_\theta, & R_\theta > 400 \\ 4, & R_\theta \leq 400 \end{cases}$$

$$R_{\theta z} = \begin{cases} R_\theta, & R_\theta > 200 \\ 200, & R_\theta \leq 200 \end{cases}$$

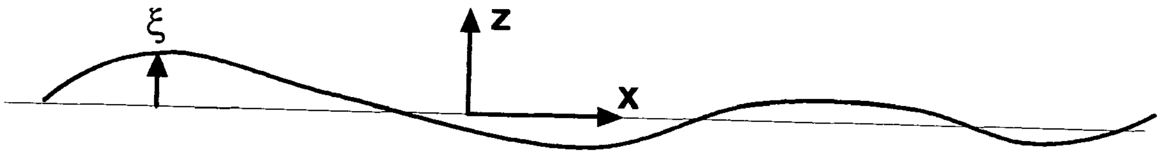
$$H^* = \begin{cases} 1.505 + 4/R_\theta + (0.165 - 1.6/\sqrt{R_\theta})\frac{(H_o-H)^{1.6}}{H}, & H < H_o \\ (H-H_o)^2[0.007\frac{\ln(R_{\theta z})}{[H-H_o+4/\ln(R_{\theta z})]^2} + 0.015/H] + 1.505 + 4.0/R_\theta, & H \geq H_o \end{cases}$$

$$C_f = 0.3e^{-1.33H}\left[\frac{\ln(R_\theta)}{2.3026}\right]^{-(1.74+0.31H)}$$

$$\frac{2C_D}{H^*} = 0.5C_f\frac{4.0/H - 1}{3} + 0.03\left(1 - \frac{1}{H}\right)^3$$

FM813

Sea Waves



Dominated by inviscid irrotational solution ($\nabla^2\phi = 0$)

Boundary Conditions

$$\frac{\partial\phi}{\partial z} = \frac{\partial\zeta}{\partial t} + \left[\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y}\right]_{z=\zeta} \qquad \text{(kinematic)}$$

$$\left\{\frac{\partial\phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]\right\}_{z=\zeta} + g\zeta = \text{constant } (0) \qquad \text{(dynamic)}$$

Linearized Boundary Conditions

Case of onset flow velocity of $-\hat{\imath}U$.
Now ϕ is the perturbation potential and the total potential is $-Ux + \phi$.

$$\left[\frac{\partial\phi}{\partial z}\right]_{z=0} = \frac{\partial\zeta}{\partial t} - U\frac{\partial\zeta}{\partial x} \qquad \left[\frac{\partial\phi}{\partial t} - U\frac{\partial\phi}{\partial x}\right]_{z=0} + g\zeta = 0$$

For steady flow with onset flow:

$$\frac{\partial\phi}{\partial z} = -U\frac{\partial\zeta}{\partial x} \qquad U\frac{\partial\phi}{\partial x} = g\zeta \qquad \frac{\partial\phi}{\partial z} = -\frac{U^2}{g}\frac{\partial^2\phi}{\partial x^2}$$

Case of 2D waves and zero onset flow so ϕ is the total potential.

$$\left[\frac{\partial\phi}{\partial z}\right]_{z=0} = \frac{\partial\zeta}{\partial t} \qquad \left[\frac{\partial\phi}{\partial t}\right]_{z=0} + g\zeta = 0$$

Dispersion Relations for waves of ciruclar frequency $\omega = 2\pi f$ and wavenumber $= k = 2\pi/\lambda$ and zero onset flow.

$$\omega^2 = gk \qquad \text{deep water}$$

$$\omega^2 = gk \tanh kh \qquad \text{water of depth } h$$

FMB13(B)

$$\left.\frac{\partial^2\phi}{\partial t^2}\right]_{z=0} = -g\frac{\partial\zeta}{\partial t} \qquad \left.\frac{\partial\phi}{\partial z}\right]_{z=0} = -\frac{1}{g}\left.\frac{\partial^2\phi}{\partial t^2}\right]_{z=0}$$

$$\zeta = Ae^{i(kx-\omega t)} \quad \text{and} \quad \zeta = -\frac{1}{g}\frac{\partial\phi}{\partial t}$$

Deep Water

$\phi = Be^{kz}e^{i(kx-\omega t)}$ Traveling wave that satisfies Laplace's Equation

$$Bke^{kz}e^{i(kx-\omega t)} = \frac{1}{g}\omega^2 Be^{kz}e^{i(kx-\omega t)} \qquad k = \frac{\omega^2}{g} \qquad \omega^2 = kg$$

$$\zeta = -\frac{1}{g}(-i\omega)B\, e^{i(kx-\omega t)}$$

$$A = \frac{i\omega}{g}B \qquad B = -\frac{ig}{\omega}A = -i\frac{\omega}{k}A$$

Finite Depth

$$\phi = B\cosh\, k(z+h)\, e^{i(kx-\omega t)}$$

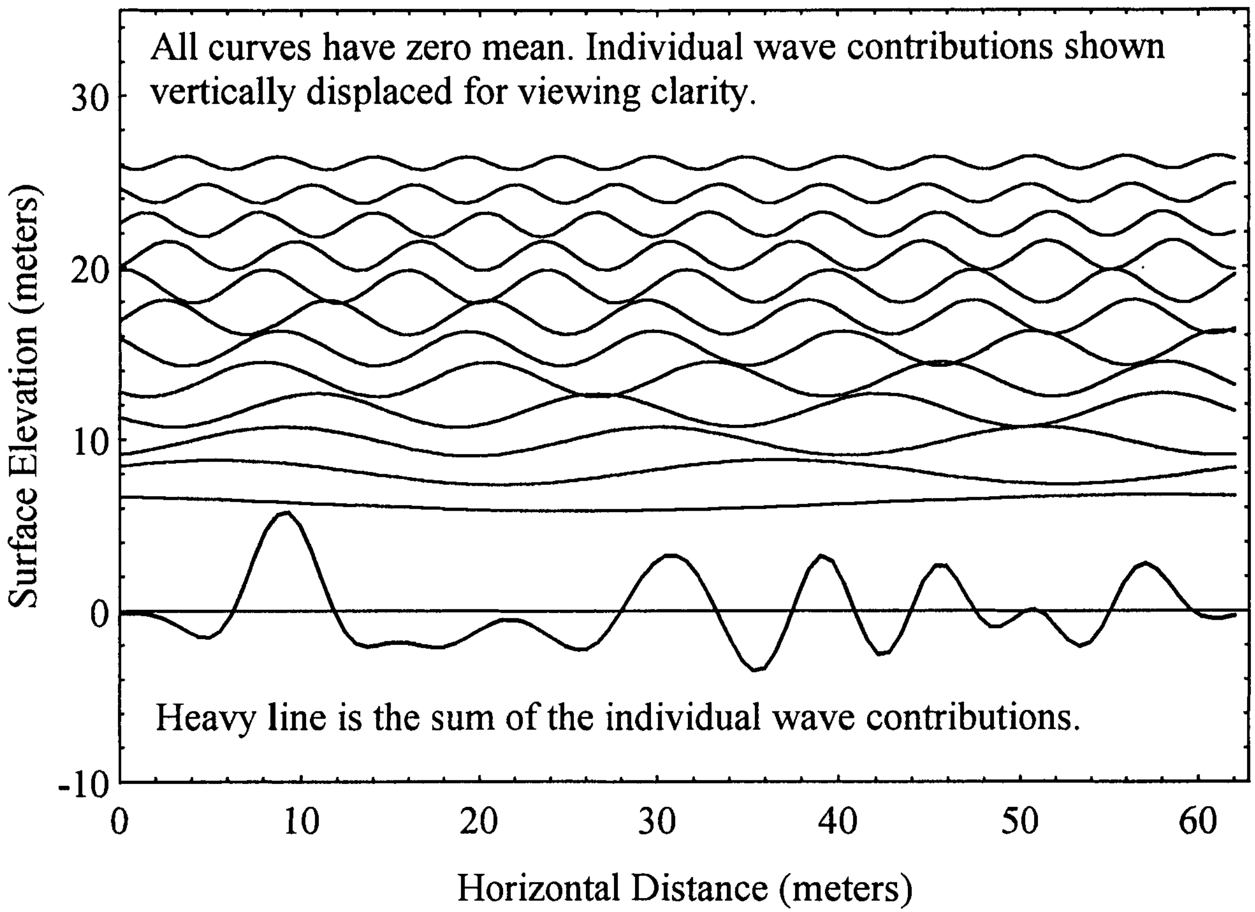
$$Bk\sinh\, kh\, e^{i(kx-\omega t)} = B\omega^2\frac{1}{g}\cosh\, kh\, e^{i(kx-\omega t)}$$

$$k\tanh\, kh = \frac{\omega^2}{g} \qquad \omega^2 = gk\tanh\, kh$$

$$\zeta = -\frac{1}{g}(-i\omega)B\, e^{i(kx-\omega t)} = \frac{i\omega}{g}B\cosh(kh)\, e^{i(kx-\omega t)} \qquad A = \frac{i\omega}{g}\cosh(kh)\, B$$

wavedem.ep

Generation of Random Wave Form From Sinusoidal Components



AU m er S

Example of Simulation

Suppose a two dimensional (long crested) wave is generated with a wave-maker in a wave tank with an elevation at a specified location given by $z(t)$, where:

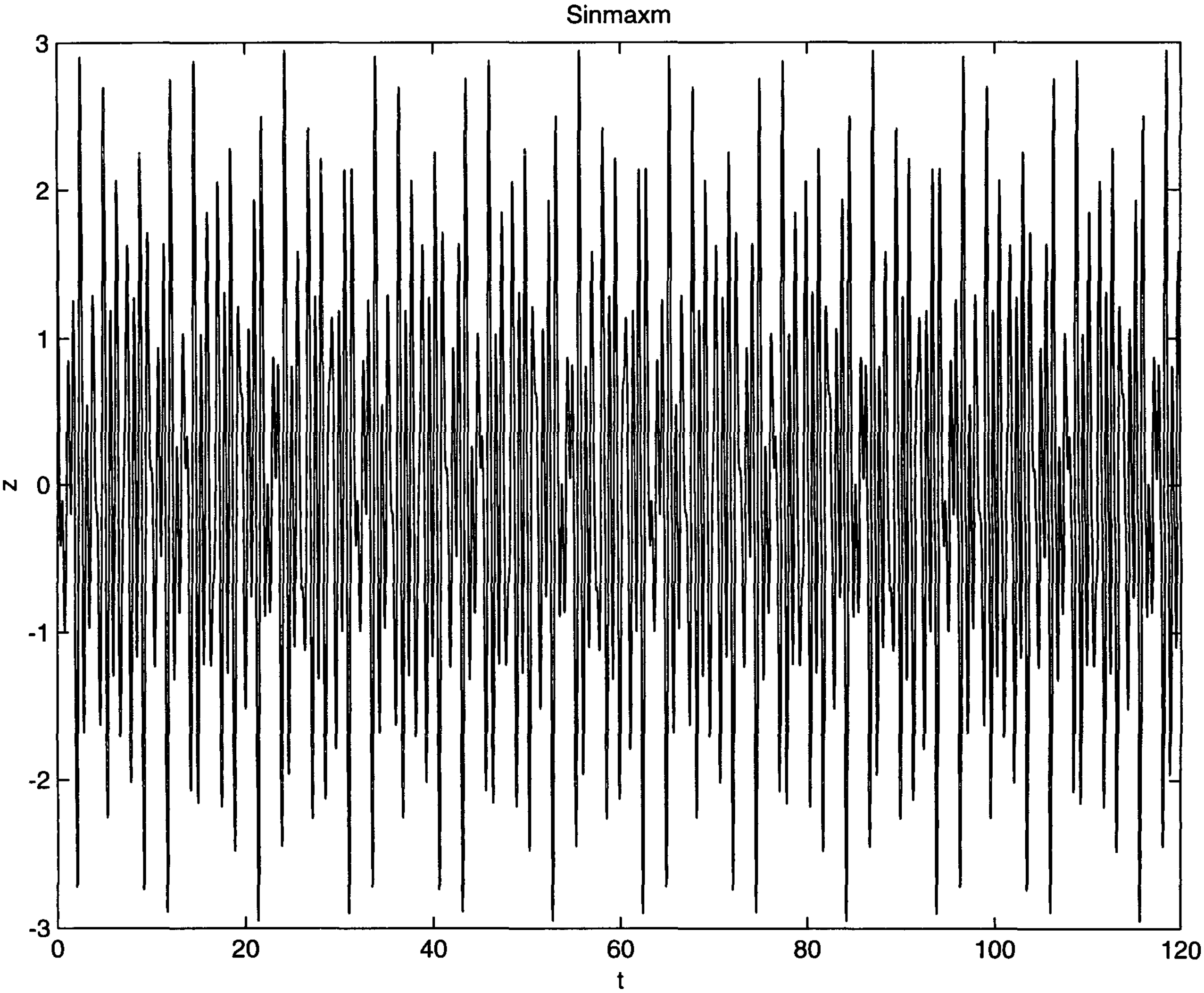
$$z(t) = 0.97\sin(5.2t + 0.82) + 0.99\sin(7.8t + 1.24) + 1.08\sin(9.8t + 2.72)$$

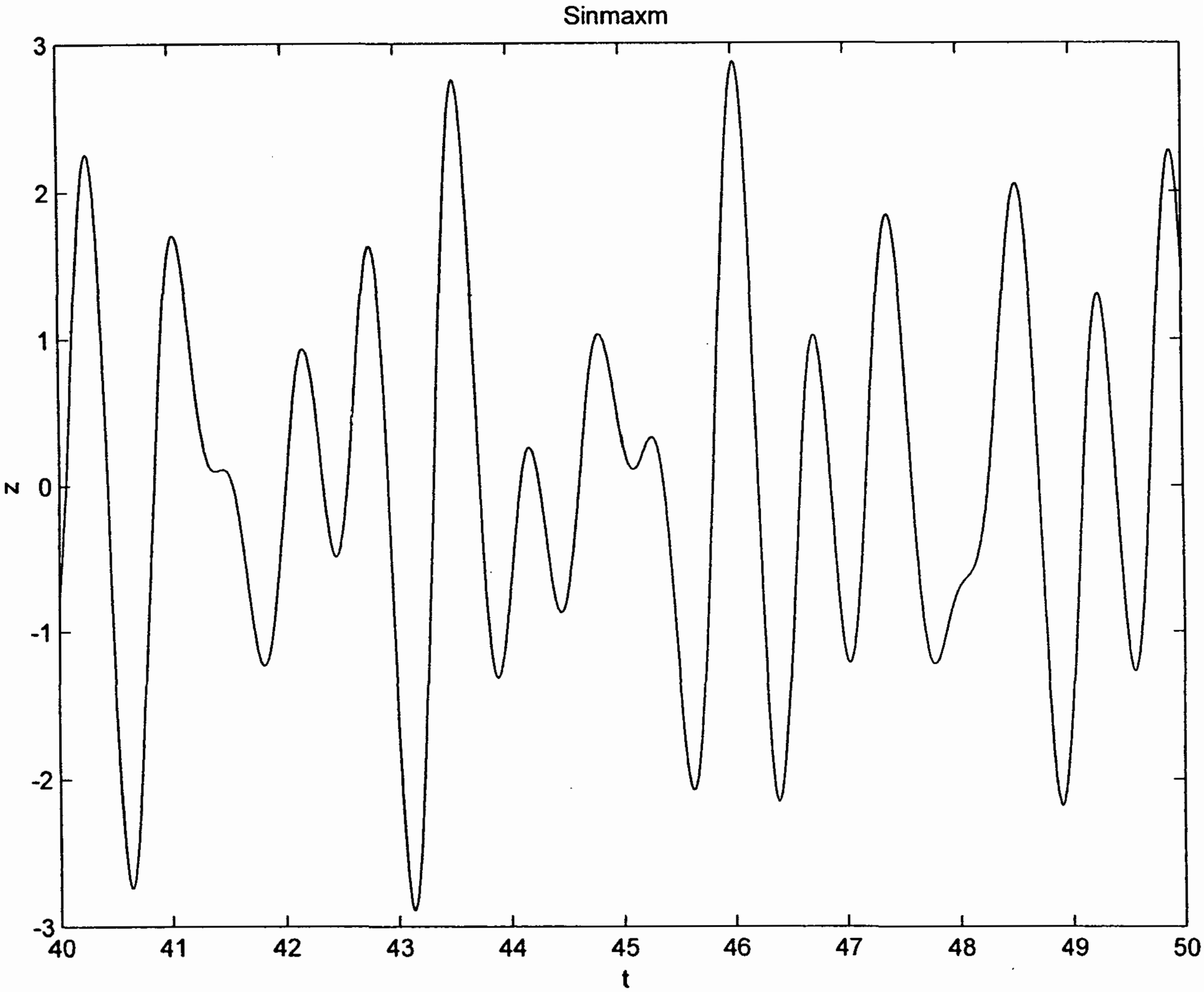
What is the maximum elevation that occurs in the time interval of 0 to 120 seconds (2 minutes). The usual way of finding maxima of analytic functions by setting the derivative to zero is not practical here because there are a great many maxima and the largest of these must be determined. However, because of the great computational speed of common computers, this can be done numerically without much effort.

```
% MATLAB Version of program sinmax
t = 0:0.01:120;
z = 0.97*sin(5.2*t + 0.82) ...
    + 0.99*sin(7.8*t +1.24) + 1.08*sin(9.8*t + 2.72);
zmax = max(z);
mmax= find(z == zmax);
tmax =(t(mmax));
fprintf (1,'tmax = %7.3f     zmax = %8.4f\n',tmax,zmax);
q = [t;z];
fid = fopen('zmaxm.dat','w');
fprintf(fid,'%f7.2f  %8.4f\n',q);
plot(t,z);
xlabel('t');
ylabel('z');
title('Sinmaxm');
```

```
>> sinmaxm
tmax = 118.490     zmax =    2.9447
>>
```

NUmet 8




Sinmaxm
z
t
3
2
1
0
-1
-2
-3
40
41
42
43
44
45
46
47
48
49
50

Sea Spectra

We consider wave fields whose statistics are both stationary and homogeneous in the horizontal plane.

A sea spectrum function $S_T(k, \omega, \theta)$ is a partial description of the statistics of the wave field defined such that $S_T(k, \omega, \theta)\delta k\, \delta\omega\, \delta\theta$ is the contribution to the *average wave energy per unit surface area*, E, in the wavenumber, wave circular frequency and propagation angle bands; $\delta k\, \delta\omega\, \delta\theta$.

For surface elevation $\zeta(\mathbf{x}, t)$ the average wave energy is defined as:

$$E =< \zeta^2 >$$

where $< >$ signifies the statistical, temporal or spatial average.

$$\text{Thus:} \quad < \zeta^2 >= \int_0^{2\pi} \int_0^{\infty} \int_0^{\infty} S_T(k, \omega, \theta)\delta k\, \delta\omega\, \delta\theta$$

Similar definitions apply when frequency, f, is used instead of circular frequency, ω, and/or when spatial frequency, $\frac{1}{\lambda}$, is used instead of wavenumber, k.

For the frequently encountered case of linear, deep water gravity waves the circular frequency and the wavenumber are related to each other through the *dispersion relation*

$$\omega^2 = gk$$

so that ω and k are not independent of each other. Then the spectrum is a function of only one or the other of these variables and can be written as: $S_t(\omega, \theta)$ or $S_x(k, \theta)$. These functions are related by:

$$S_x(k, \theta) = \frac{g}{2\omega} S_t(\omega, \theta)$$

$$\text{Hence:} \quad < \zeta^2 >= \int_0^{2\pi} \int_0^{\infty} S_x(k, \theta) dk d\theta = \int_0^{2\pi} \int_0^{\infty} S_x(\omega, \theta) d\omega d\theta$$

For unidirectional (long crested) seas, all the waves are in a single direction and the spectra are described by $S_t(\omega)$ or $S_x(k)$.

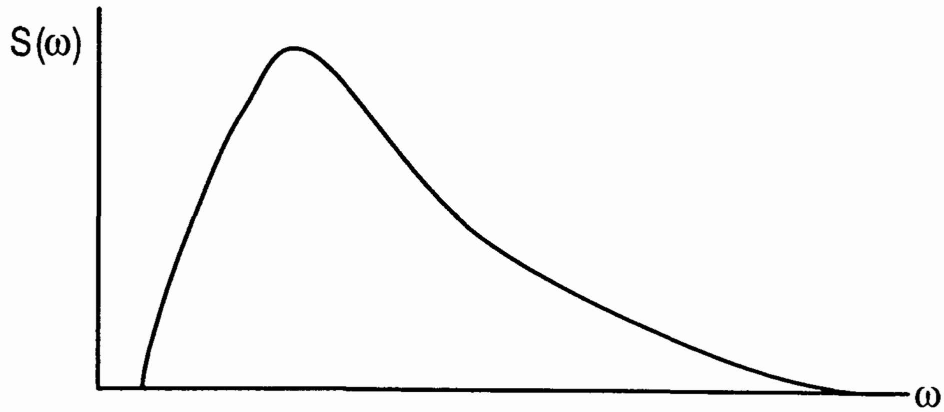
$$< \zeta^2 >= \int_0^{\infty} S_t(\omega) d\omega = \int_0^{\infty} S_x(k) dk$$

The fundamental linearized plane progressive wave is:

$$\zeta = Ae^{i(kx-\omega t)}$$

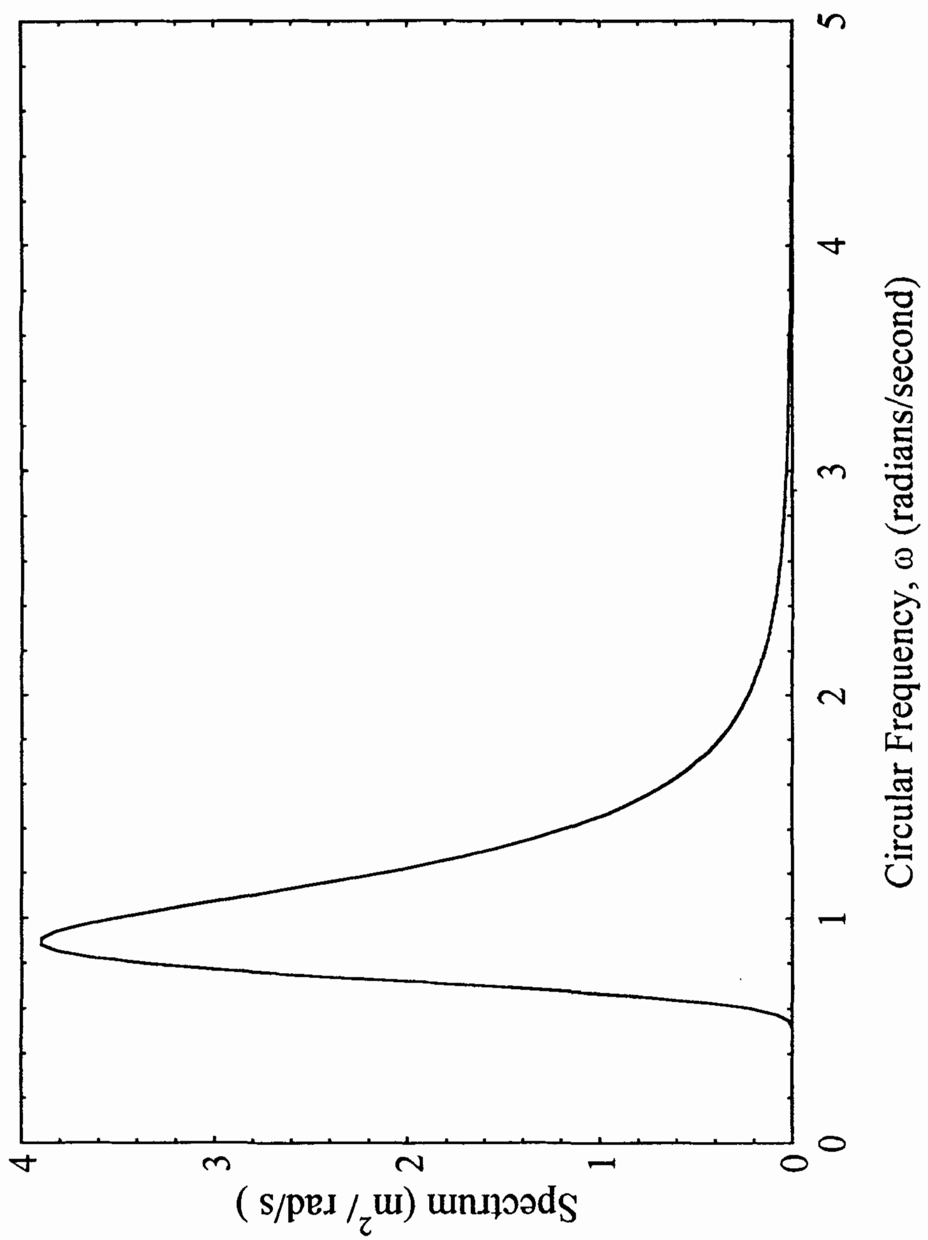
$$\phi = -\frac{i\omega A}{k} e^{kz} e^{i(kx-\omega t)}$$

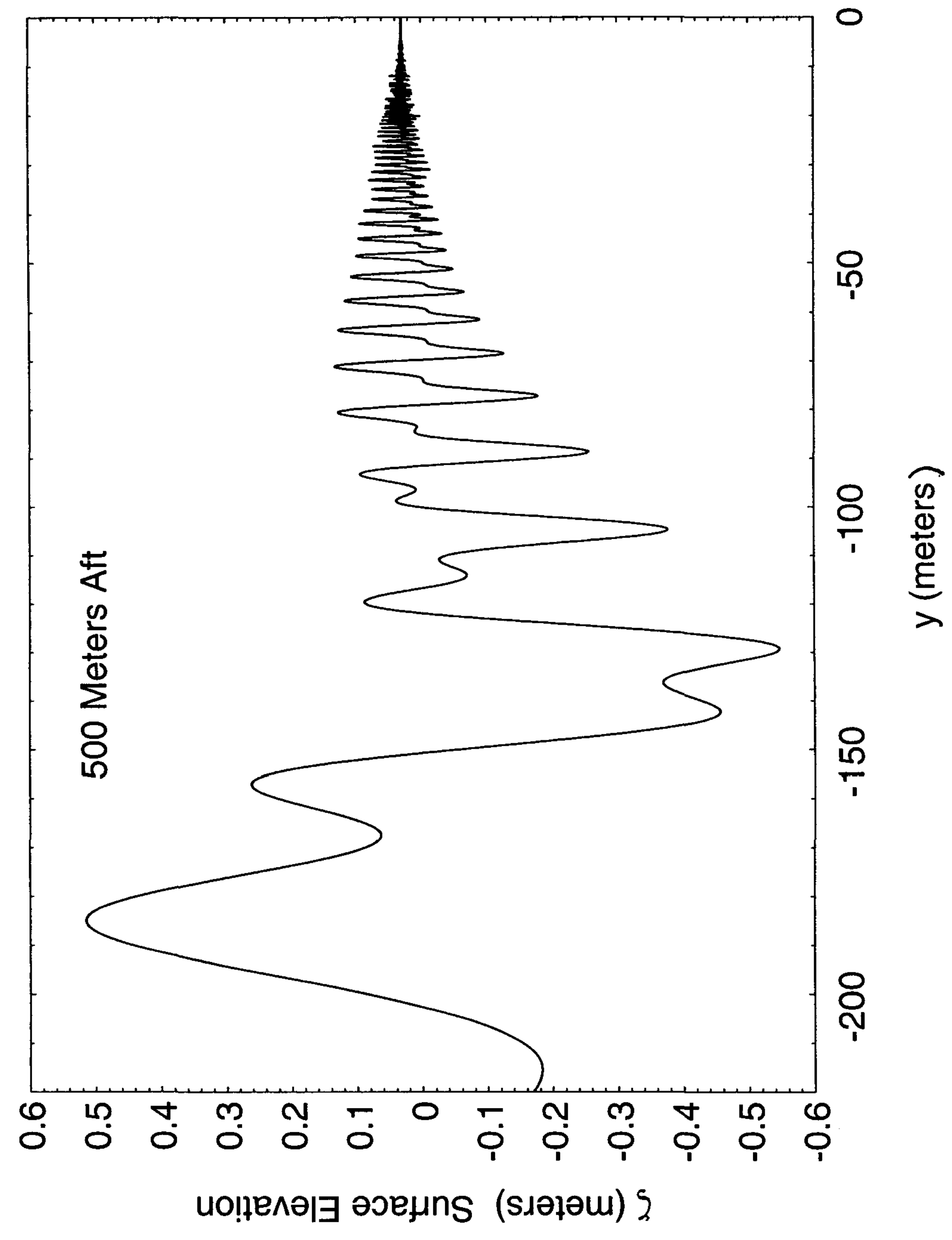
Random sea waves have spectrum $S(\omega, \theta)$.
For the 2D case the spectrum is $S(\omega)$.

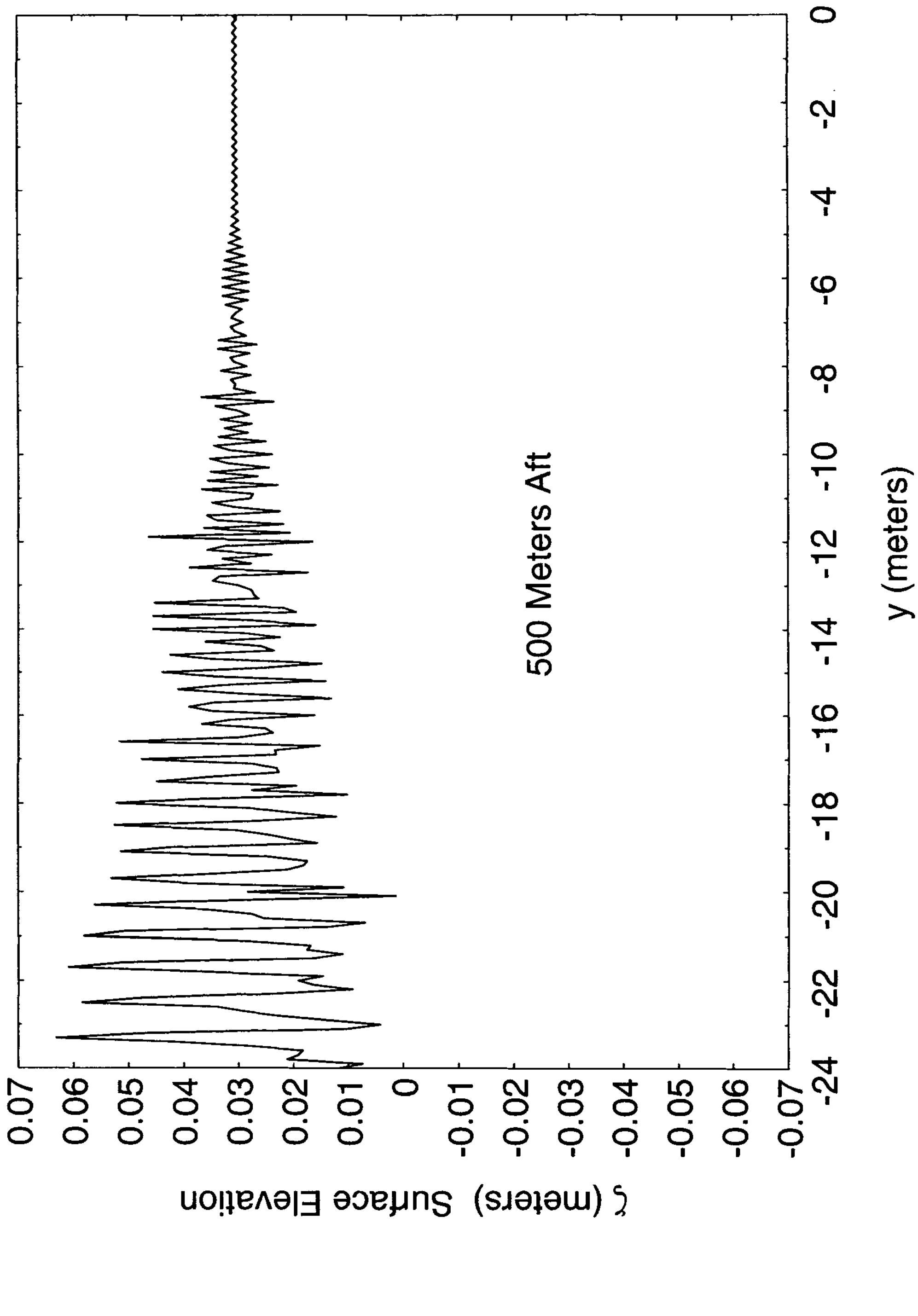


$\int_{\omega_1}^{\omega_2} S(\omega)\, d\omega$ is the contribution to $\overline{\zeta^2}$ of waves with circular frequencies between ω_1 and ω_2.

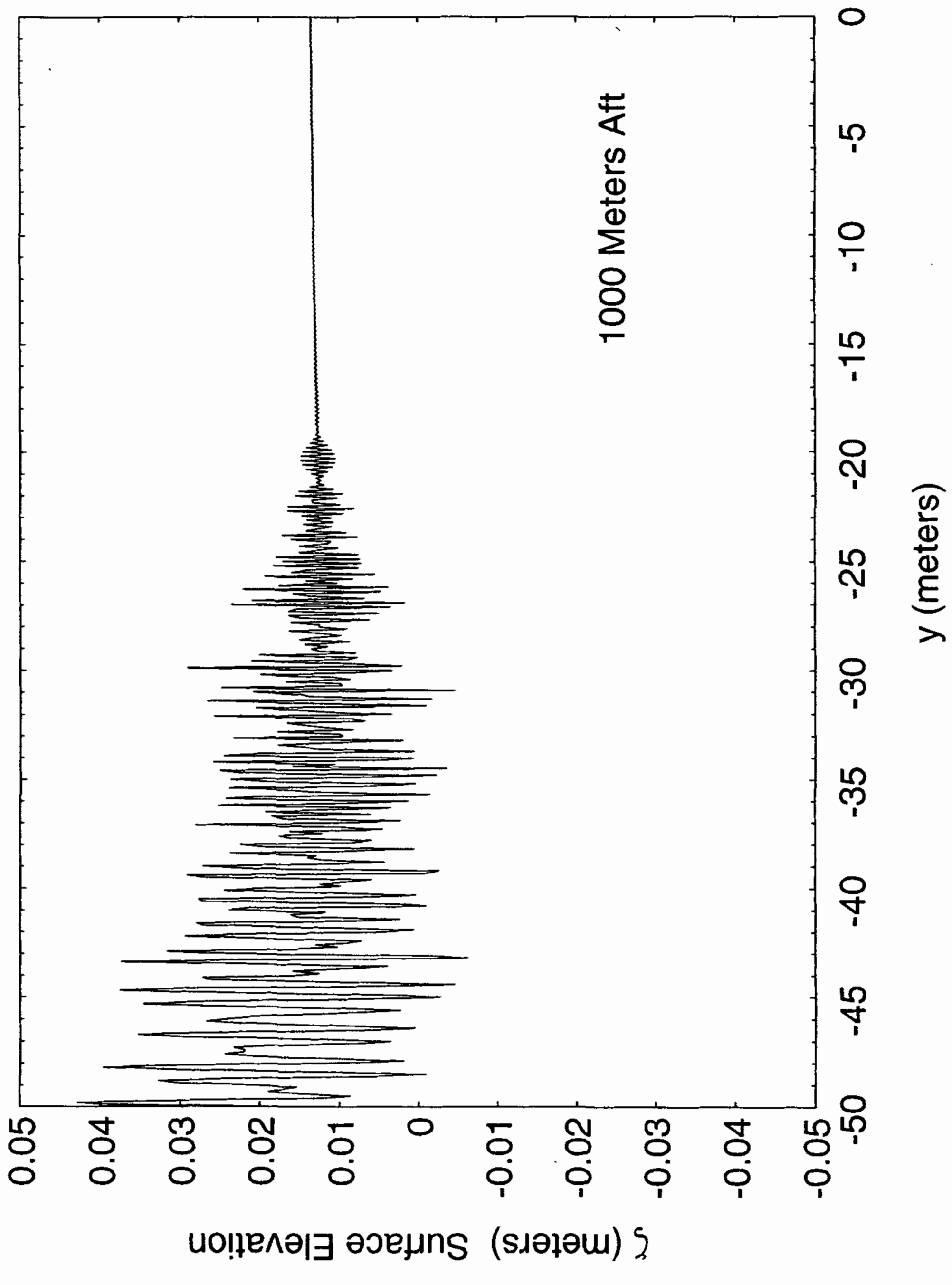
spec.ef

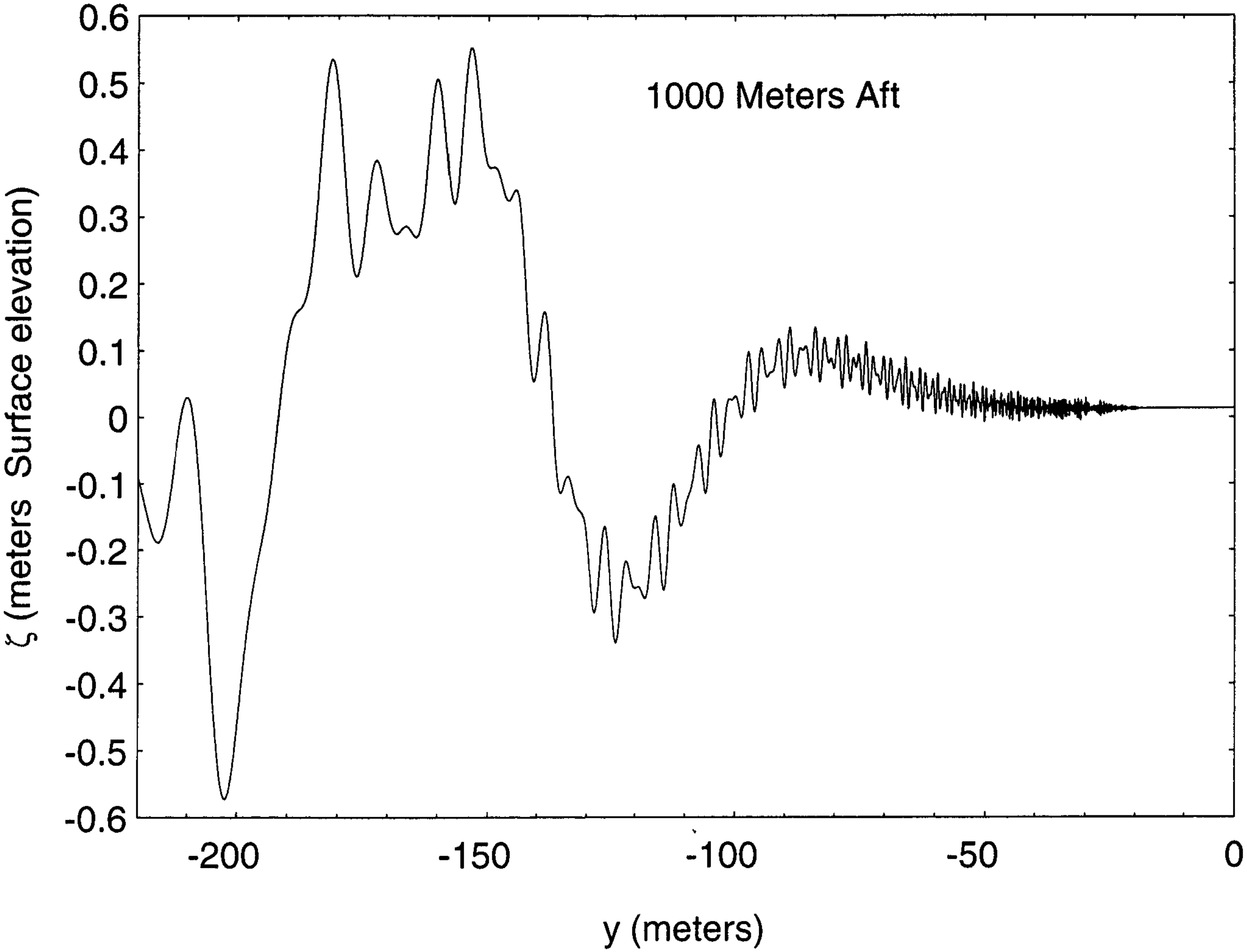

Spectrum (m2/ rad/s)
Circular Frequency, ω (radians/second)
0
1
2
3
4
0
1
2
3
4
5


500 Meters Aft
y (meters)
ζ (meters) Surface Elevation
0
-50
-100
-150
-200
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6



500 Meters Aft
y (meters)
ζ (meters) Surface Elevation
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
-22
-24
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
-0.07


1000 Meters Aft
y (meters)
ζ (meters) Surface Elevation


1000 Meters Aft
ζ (meters Surface elevation)
y (meters)
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-200
-150
-100
-50
0

FT1

Fourier Transforms

Fourier Transforms are valuable tools in numerical hydrodynamics because a number of problems can be described in the form of Fourier transforms and they can be computed very quickly by the Fast Fourier Transform (FFT) method. Two of these problems are solving a certain class of differential equations, and in simulating sea waves.

$$X(f) = \mathcal{F}x(t) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt$$

$$x(t) = \mathcal{F}^{-1}X(f) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df$$

As an example, consider a differential equation with constant coefficients of the form:

$$A_n\frac{d^n y}{dx^n} + A_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + A_0 y(x) = g(x)$$

Consider Fourier Transforms from x to f where:

$$\mathcal{F}[y(x)] \equiv Y(f) \qquad \text{and} \qquad \mathcal{F}[g(x)] \equiv G(f)$$

Take the Fourier transform of the differential equation to get:

$$(i2\pi f)^n A_n Y(f) + (i2\pi f)^{n-1}A_{n-1}Y(f) + ... + A_0 Y(f) = G(f)$$

This is an algebraic equation which can be numerically solved for $Y(f)$:

$$Y(f) = \frac{G(f)}{(i2\pi f)^n A_n + (i2\pi f)^{n-1}A_{n-1} + A_0}$$

$y(x)$ can be determined by inverse Fourier transformation. Not only is this less computationally intensive than solving the differential equation by direct numerical methods, but the error in the integration rule is avoided.

Fourier Transforms (continued)

$$X(f) = \mathcal{F}x(t) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt$$

$$x(t) = \mathcal{F}^{-1}X(f) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df$$

Suppose $x(t) \approx 0$ for $t < 0$ and $t > T$. Then:

$$X(f) \approx \int_0^T x(t)e^{-i2\pi ft}dt$$

Also, suppose x(t) is *band limited* such that: $X(f) = 0$ for $|f| \geq F_{max}$. Then:

$$x(t) = \mathcal{F}^{-1}X(f) = \int_{-Fmax}^{Fmax} X(f)e^{i2\pi ft}df$$

Now, consider a periodic function having period t that is identical to $x(t)$ for $0 \leq t \leq T$. This function has a Fourier series given by:

$$x(t) = \sum_{n=-\infty}^{\infty} A_n e^{i2\pi nt/T}, \qquad A_n = \frac{1}{T}\int_0^T x(t)e^{-i2\pi nt/T}dt$$

The expression for A_n is identical to $\frac{1}{T}$ times the Fourier Transform evaluated at $f = \frac{n}{T}$. These Fourier coefficients, $A_n = \frac{1}{T}X\left(\frac{n}{T}\right)$ can be numerically evaluated very quickly by an algorithm called the *Fast Fourier Transform* (FFT).

From the A_n's, the function $x(t)$ can be constructed over the t-range $0 < t < T$. Outside this range the reconstruction is periodic whereas the real value of $x(t) \approx 0$.

Evaluate A_n by the following rectangular rule integration:

$$\delta t = \frac{1}{2F_{max}} \qquad t = j\delta t \qquad j_{max} \equiv N \qquad T = N\delta t \qquad x_j \equiv x(j\delta t)$$

$$A_n = \frac{1}{N\delta t}\sum_{j=0}^{N-1} x(j\delta t)\exp\left[-\frac{i2\pi nj\delta t}{N\delta t}\right]\delta t = \frac{1}{N}\sum_{j=0}^{N-1} x_j e^{-i2\pi nj/N}$$

$$f_t = \frac{n_t}{T} \qquad n_{max} = M$$

$$F_{max} = \frac{M}{T} \qquad M = T F_{max}$$

$$\delta t = \frac{1}{2F_{max}} \; ; \qquad \text{Sampling Theorem}$$

$$F_{max} = \frac{1}{2\delta t} \qquad M = \frac{T}{2\delta t} \qquad \boxed{\delta t = \frac{1}{2F_{max}}}$$

$$F_{max} = M\,\delta f \qquad \frac{1}{2\delta t} = \frac{T}{2\delta t}\,\delta f$$

$$\boxed{\delta f = \frac{1}{T}}$$

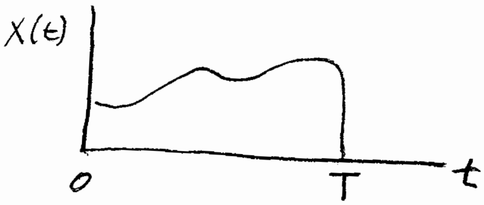
$$t = j\,\delta t \qquad T = j_{max}\,\delta t$$

$$M = \frac{j_{max}\,\delta t}{2\delta t} \qquad j_{max} = 2M$$

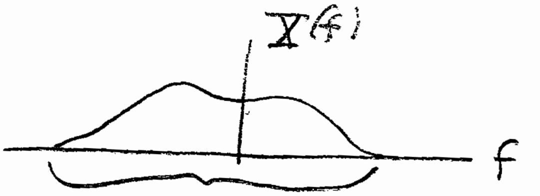
Let $2M = N$ $\qquad j_{max} = N$

$$\boxed{\delta t = \frac{T}{2M} = \frac{T}{N}}$$

$$\delta f = \frac{1}{T} = \frac{1}{M/F_{MAX}} = \frac{F_{max}}{M} = \frac{2F_{max}}{N}$$



$T = N\delta_t$



$2F_{max} = N\delta f$

$$-\frac{i\,2\pi n t}{T} = -\frac{i\,2\pi\, n\, j\,\delta t}{N\,\delta t} = -\frac{i\,2\pi\, n\, j}{N}$$

$$F_{MAX} = M\,\delta f = M\,\frac{1}{T} = M\,\frac{1}{N\delta t} = \frac{1}{2\delta t}$$

$$f = n\,\delta f = \frac{n}{T}$$

$$n_{max} = T\,F_{max} = T\,\frac{1}{2\delta t} = \frac{N}{2}$$

$$\delta t = \frac{T}{N} \qquad df = \frac{1}{T} \qquad dt\,df = \frac{1}{N}$$

Fourier Transforms (continued)

$$X(f) = 0 \text{ for } f \geq F_{max} = \frac{1}{2\delta t} \text{ and } f = \frac{n}{T}, \text{ so } \delta f = \frac{1}{T} \text{ and } n_{max} = T f_{max} = \frac{N}{2}$$

$$\delta t = \frac{1}{2F_{max}}, \qquad \delta f \delta t = \frac{1}{2F_{max}T} = \frac{1}{N}$$

$$x(j\delta t) = x_j = \sum_{n=-N/2}^{N/2} T A_n \exp\left[\frac{i2\pi n j \delta t}{N\delta t}\right] \frac{1}{T} = \sum_{n=-N/2}^{N/2} A_n e^{i2\pi nj/N}$$

Fast Computing

Computing speed is minimized by minimizing the number of complex exponentials that must be computed.

Let: $q_1 = e^{-i2\pi/N}$ $\qquad\qquad q_2 = e^{i2\pi/N}$

$$e^{-i2\pi nj/N} = e^{-i2\pi(n-1)j/N}\, q_1^j = e^{-i2\pi n(j-1)/N}\, q_1^n$$

$$e^{i2\pi nj/N} = e^{i2\pi(n-1)j/N}\, q_2^j = e^{i2\pi n(j-1)/N}\, q_2^n$$

Even the powers of q can be avoided:

$$e^{-i2\pi(0)(0)/N} = 1$$

$$e^{-i2\pi(1)(1)/N} = e^{-i2\pi(0)(0)/N}\, q_1$$
$$e^{-i2\pi(1)(2)/N} = e^{-i2\pi(1)(1)/N}\, q_1$$
$$e^{-i2\pi(2)(1)/N} = e^{-i2\pi(1)(1)/N}\, q_1$$
$$e^{-i2\pi(1)(3)/N} = e^{-i2\pi(1)(2)/N}\, q_1$$
$$e^{-i2\pi(2)(2)/N} = e^{-i2\pi(1)(3)/N}\, q_1$$

etc.

Fourier Transforms (continued)

Periodicity

The actual integral transforms are of limited extent.

$$x(t) = 0 \text{ except for } 0 \leq t \leq T$$

$$X(f) = 0 \text{ except for } -F_{max} \leq f \leq F_{max}$$

However, the mathematical constructions, while consistent with the integral transforms for : $0 \leq t \leq T$, and $-Fmax \leq f \leq Fmax$, are periodic outside these ranges.

$$A_{n+N} = \frac{1}{N}\sum_{j=0}^{N-1} x_j e^{-i2\pi(n+N)j/N} = \frac{1}{N}\sum_{j=0}^{N-1} x_j e^{-i2\pi nj/N} e^{-i2\pi j} = \frac{1}{N}\sum_{j=0}^{N-1} x_j e^{-i2\pi nj/N} = A_n$$

$$x_{j+N} = \sum_{n=-N/2}^{N/2} A_n e^{i2\pi n(j+N)/N} = \sum_{n=-N/2}^{N/2} A_n e^{i2\pi nj/N} e^{i2\pi n} = \sum_{n=-N/2}^{N/2} A_n e^{i2\pi nj/N} = x_j$$

$$\text{Therefore: } x_j = \sum_{n=0}^{N-1} A_n e^{i2\pi nj/N}$$

Computational FFT and IFFT of REAL Numbers

$$A_n = \frac{1}{N}\sum_{j=0}^{N-1} x_j e^{-i2\pi nj/N} \quad ; \quad x_j = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} A_n e^{i2\pi nj/N}$$

$$A_{\frac{N}{2}} = \frac{1}{N}\sum_{j=0}^{N-1} x_j e^{i2\pi j\frac{N}{2}\frac{1}{N}} = \frac{1}{N}\sum_{j=0} x_j e^{i\pi j}$$

If x_j's are real $\quad A_{\frac{N}{2}}$ is real

$A_{-1} = A_{N-1}$

$A_{-2} = A_{N-2} \Rightarrow A_{-K} = A_{N-K}$

$A_{-\frac{N}{2}} = A_{\frac{N}{2}}$

also, if the x_j are real,

$$A_{-n} = \frac{1}{N}\sum_{j=0}^{N-1} x_j e^{i2\pi(-n)j/N} = A_n^*$$

Since $e^{i2\pi(-n)j/N} = e^{i2\pi(N-n)j/N}$

$$A_{-n} e^{i2\pi(-n)j/N} = A_n^* e^{i2\pi(N-j)/N}$$

$$A_{-\frac{N}{2}} e^{i2\pi(+\frac{N}{2})j/N} = A_{\frac{N}{2}} e^{i2\pi(N-\frac{N}{2})j/N}$$

$$= A_{N/2} e^{i2\pi N/2\, j/N}$$

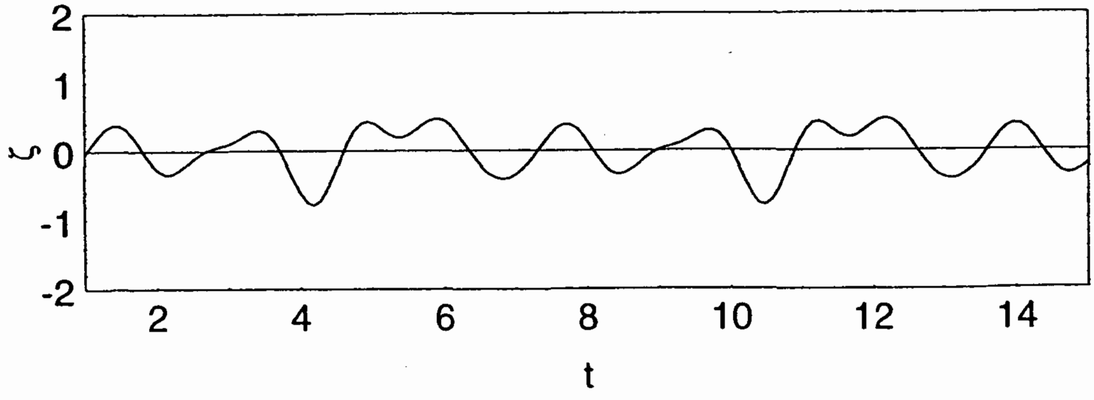
Therefore, $x_j = \sum_{n=0}^{N-1} A_n' e^{i2\pi nj/N}$

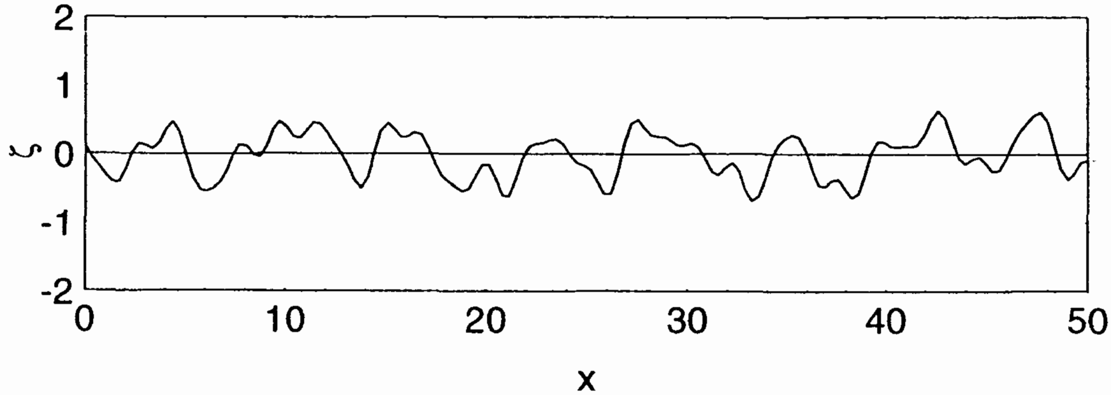
where $A_n' = \begin{cases} A_n, & 0 \le n' \le N/2 - 1 \\ 2A_n, & n' = N/2 \\ A_{N-n'}^*, & \frac{N}{2} < n' < N \end{cases}$

ws1

Simulation of Random Waves

Here we consider two-dimensional (long crested) waves. The waves are approximated as hydrodynamically linear in the sense that wave breaking and other nonlinear effects are neglected.





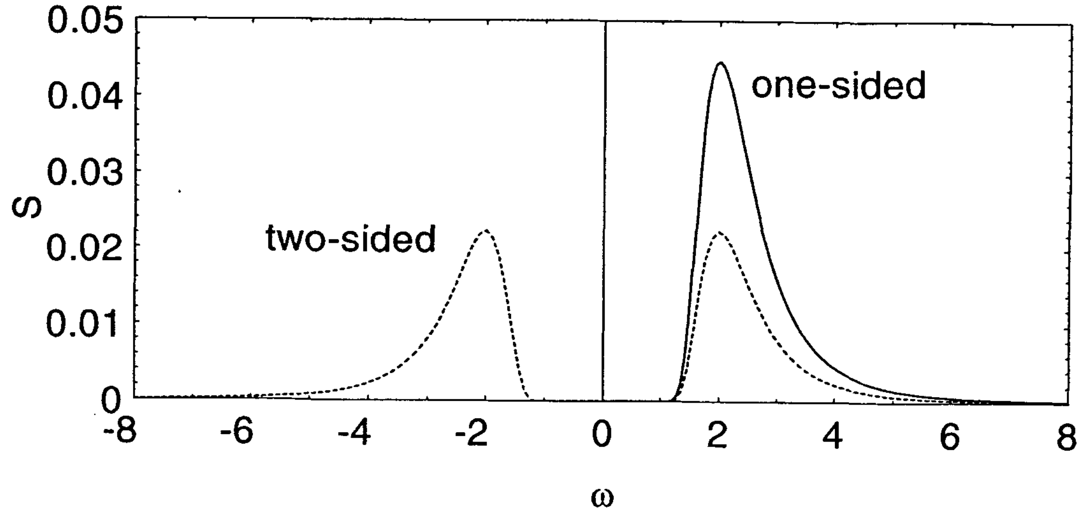
$$\zeta(x,t) = \sum_{n=0}^{\infty} Z_n \cos\left(-\frac{\omega_n^2}{g}x + \omega_n t + \alpha_n\right)$$

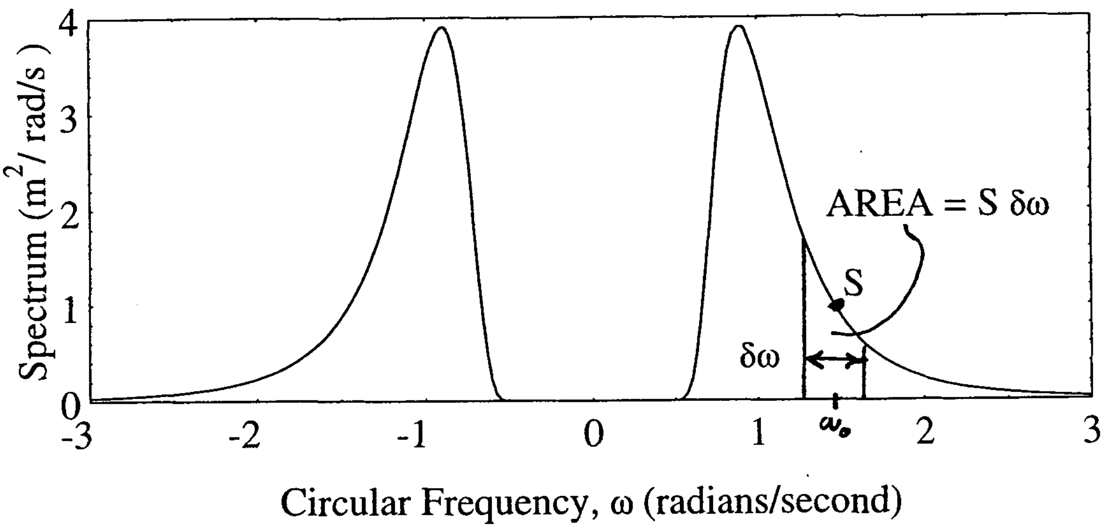
where the Z_n's are chosen to provide the desired wave spectrum and the α_n's are random numbers uniformly distributed on $0 \leq \alpha < 2\pi$.

An alternate expression is: $\zeta(x,t) = \sum_{n=-\infty}^{\infty} Z_n \exp\left[i\left(-\frac{\omega_n^2}{g}x + \omega_n t + \alpha_n\right)\right]$

Combining $e^{i\alpha_n}$ into Z_n, the surface elevation vs time at $x = 0$ is:

$$\zeta(t) = \sum_{n=-\infty}^{\infty} Z_n e^{i\omega_n t}$$





The region in the "almost trapezoid" is represented by a sinusoidal wave having frequency ω_o and the same energy, E, of this region of the spectrum. The sinusoidal wave $Ae^{i\omega_o t}$ has energy $|A^2|$. Thus,

$$|A^2| = S(\omega_o)\delta\omega$$

The waves are random processes and can be represented in two different ways. One way is to have stochastic waves and a stochastic spectrum whose expectation is equal to the spectrum being simulated (Type 1). The other way has stochastic waves and a deterministic spectrum equal to the spectrum being simulated (type 2).

Similarly, at $t = 0$ the surface elevation vs x is:

$$\zeta(x) = \sum_{n=-\infty}^{\infty} Z'_n e^{-ik_n x} = \sum_{n=-\infty}^{\infty} Z_n e^{ik_n x} \quad \text{where } k_n = \frac{\omega_n^2}{g}$$

With $\omega_n = 2\pi n \delta f$, $k_n = 2\pi n \delta b$, $(b = 1/\lambda)$, $t = j\delta t$, $x = j\delta x$, and n limited to $-\frac{N}{2} \leq n \leq \frac{N}{2}$ with $\delta f\, \delta t$ or $\delta b\, \delta x$ equal to $1/N$, the expressions for ζ have the form of an inverse discrete Fourier transform. Hence, by first choosing the Z_n's so they are consistent with the wave spectrum, the surface elevation for all values of t or for all values of x can be computed very rapidly by using an FFT program.

Either set $Z_{-n} = Z_n^*$ or use non-negative n and take the real (or imaginary) part.

We will use the method in which $Z_{-n} = Z_n^*$.

This corresponds to a two-sided spectrum whose levels are half the levels of the corresponding 1-sided spectrum.

WS3

Type 1

At a fixed value of x, the sea elevation is $\zeta(t)$ which is a sample function of a random process having a 2-sided power density function, $S_w(\omega)$. The associated 1-sided spectrum is $S_W(\omega) = 2S_w(\omega)$ for $\omega = 0$. The fourier transform of $\zeta(t)$ is $Z(\omega)$. The spectrum and the Fourier transform of $\zeta(t)$ are truncated at $|\omega| = \omega_c = 2\pi f_c$.

$\zeta(t)$ is discretized with the time interval $\delta t = \pi/\omega_c$ to satisfy the sampling theorem. Thus, $\zeta(t)$ is specified at the discrete times $\zeta_j = \zeta(j\delta t),\ j = 0, 1, 2, ..., N$.

The Fourier coefficient Z_j corresponds to the circular frequency $\omega_j = j\delta\omega$, $j = -\frac{N}{2} + 1, ..., 0, ...\frac{N}{2}$, where $\delta\omega = \frac{2\pi}{N\delta t}$. N is usually chosen as a power of 2 for computational efficiency.

For the Type 1 approach, each Fourier coefficient is separated into its real and imaginary parts and each of these is an uncorrelated Gaussian variate.

$$Z_j = Z_{r_j} + iZ_{i_j}$$

Z_{r_j} and Z_{i_j} are identically distributed with the probability density function:

$$p(Z_{r_j}) = \frac{1}{\sigma_j\sqrt{2\pi}} \exp\left(-\frac{Z_{r_j}^2}{2\sigma_j^2}\right)$$

From the physics of the modeling, where here E means "Expectation":

$$E\left[|Z_j^2|\right] = S_w(\omega_j)\delta\omega$$

$$E\left[Z_{r_j}^2\right] = E\left[Z_{i_j}^2\right] = \frac{1}{2}S_w(\omega_j)\delta\omega$$

From the mathematics of the Gaussian pdf: $\sigma_j^2 = E\left[Z_{r_j}^2\right]$

$$\sigma_j = \sqrt{\frac{1}{2}S_w(\omega_j)\delta\omega} = \sqrt{\frac{1}{4}S_W(|\omega_j|)\delta\omega}$$

There are computer programs which give Gaussian distributed random numbers for which the user specifies σ_j.

Type 2 $\qquad Z_j = e^{i\alpha_j}\sqrt{S_w(\omega_j)\delta\omega} = e^{i\alpha_j}\sqrt{\frac{1}{2}S_W(\omega_j)\delta\omega}\,, \qquad \omega_j \geq 0$

α_j is uniformly distributed on $0 \leq \alpha_j < 2\pi$ and can be obtained from a random number computer program.

We truncate the spectrum at frequencies $\pm N/2\delta\omega$.

Thus the expression for a simulated two-dimensional (long-crested) random wave elevation at a point on the ocean surface is:

$$\zeta(t) = \sum_{-N/2}^{N/2} e^{i\alpha_n} \sqrt{\frac{1}{2} S_W(|n\delta\omega|)\delta\omega} \; e^{i(n\delta\omega)t}$$

where α^n is a random number, $\leq \alpha_n < 2\pi$, and $\alpha_n = -\alpha_{-n}$.

This can be extended to a long-crested wave *field*, dependent on x and t as:

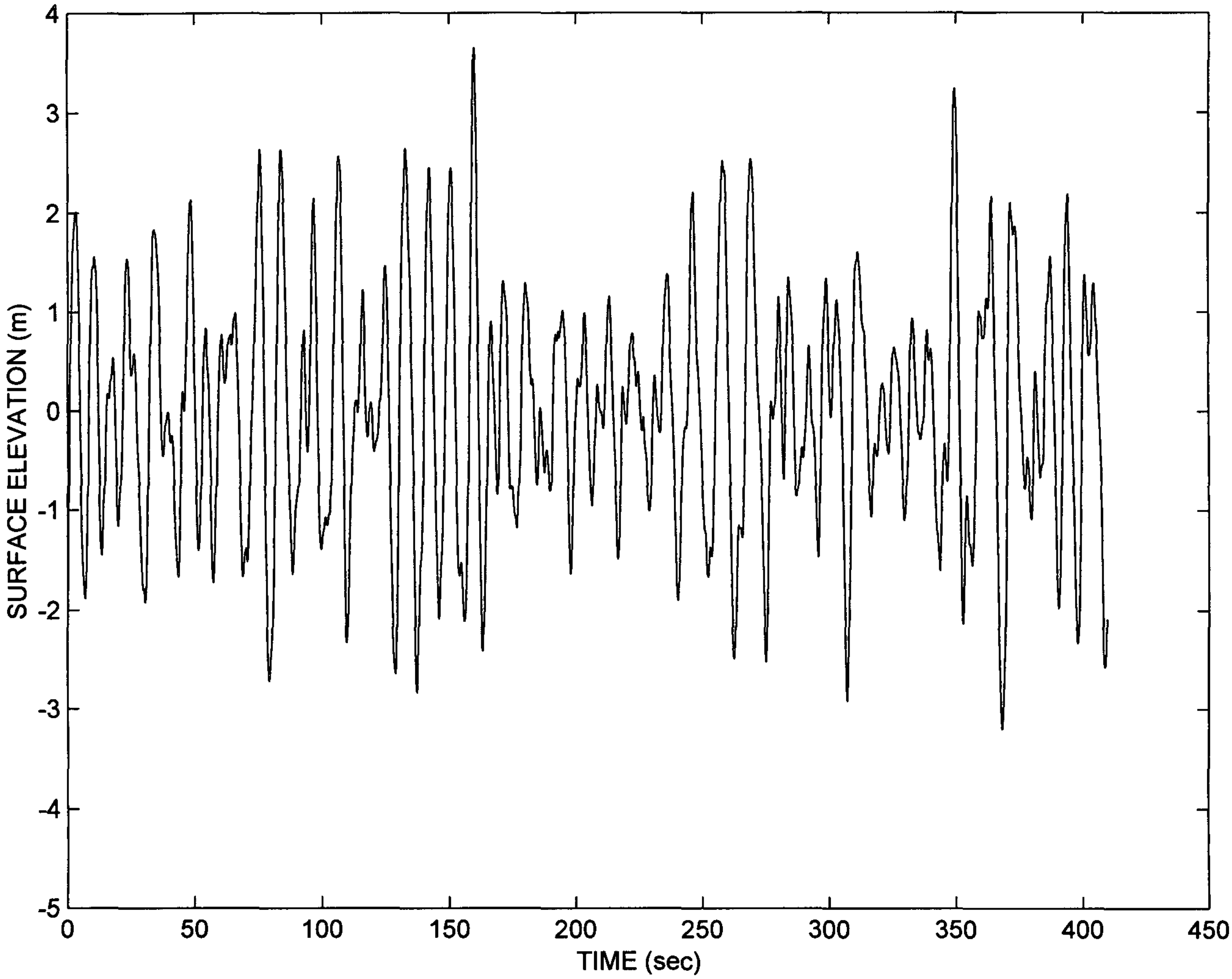
$$\zeta(x,t) = \sum_{-N/2}^{N/2} e^{i\alpha_n} \sqrt{\frac{1}{2} S_W(|n\delta\omega|)\delta\omega} \; e^{i[(n\delta\omega)t - (n\delta\omega)|n\delta\omega|x/g]}$$

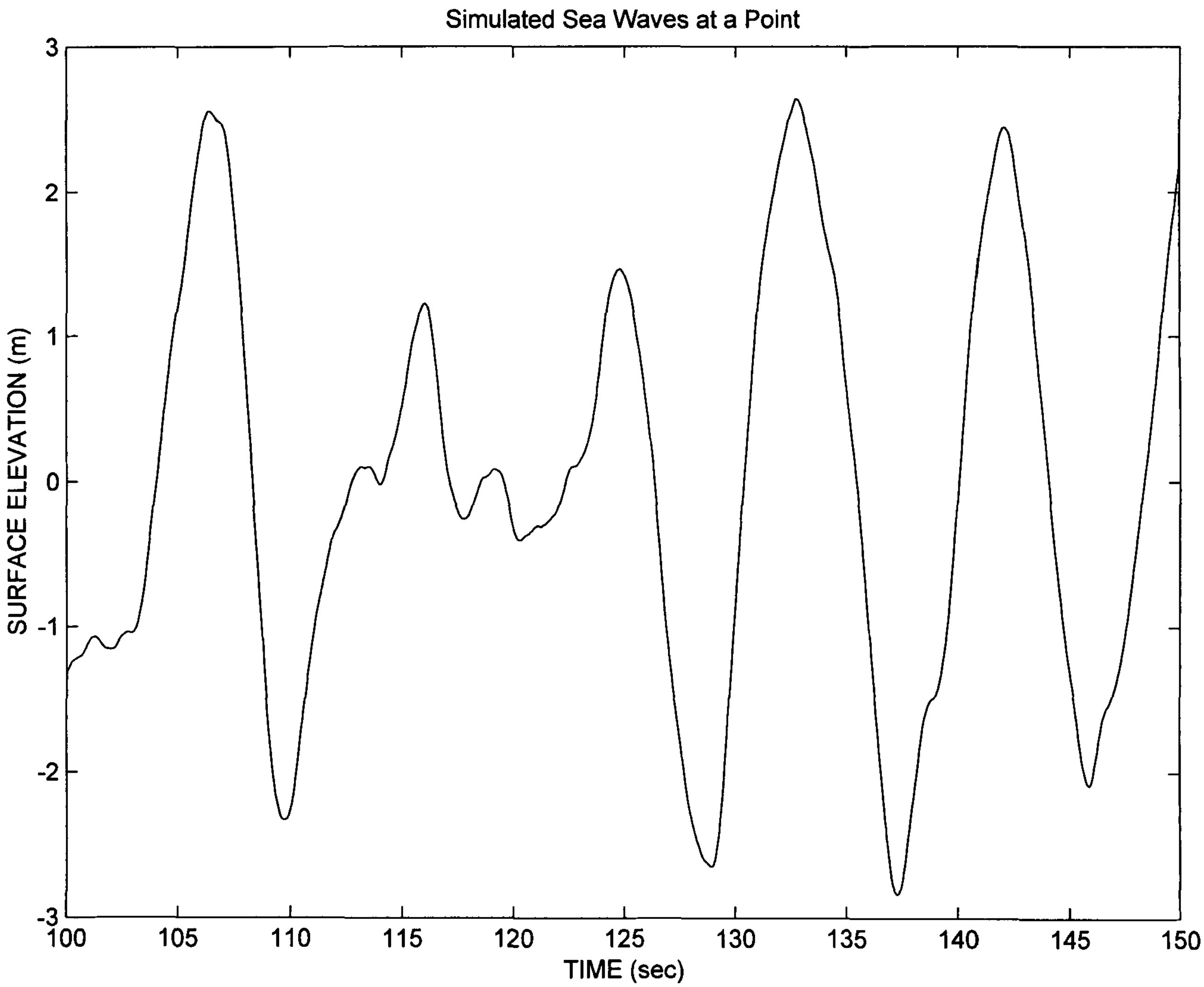
This is because $|k| = \omega^2/g$.

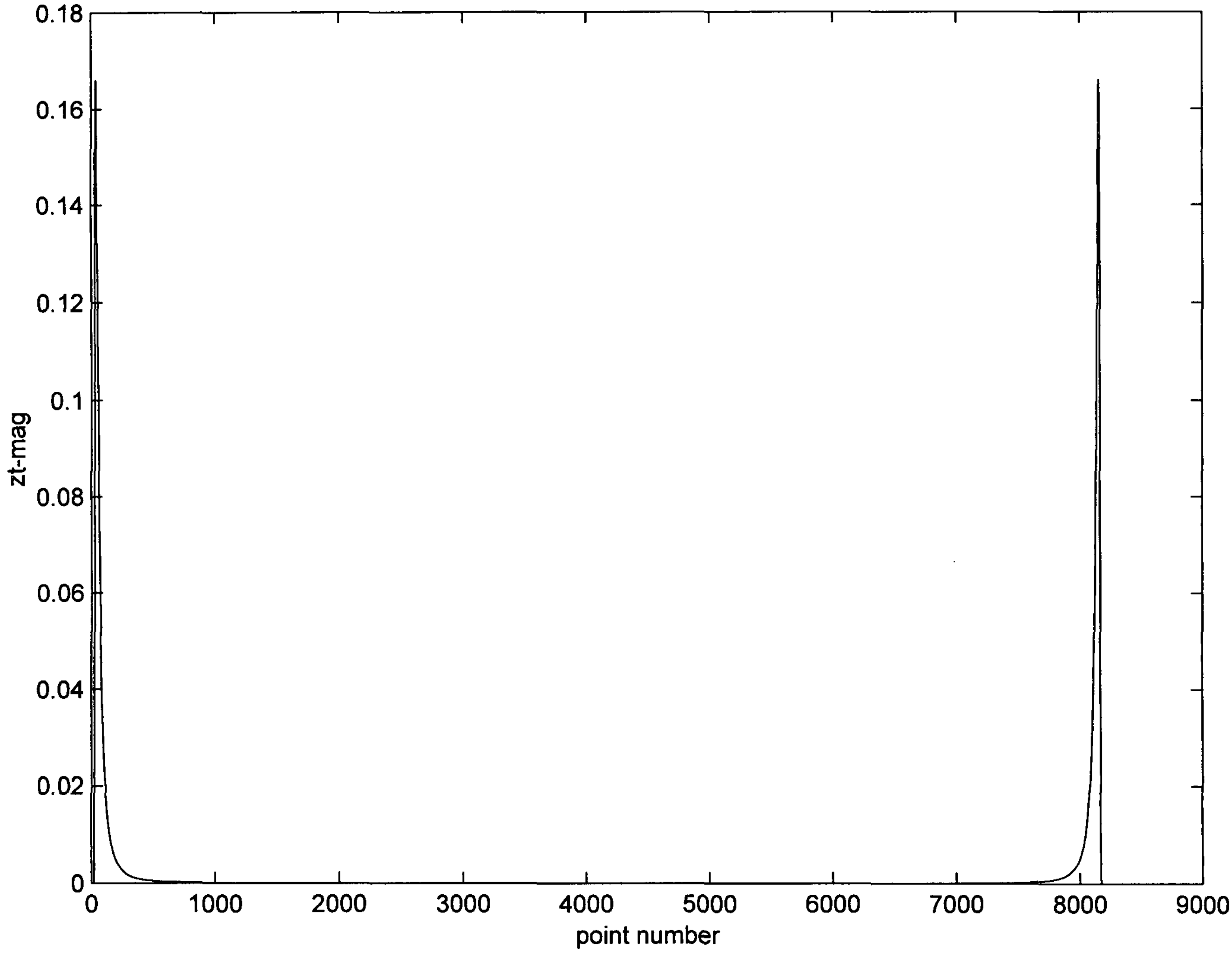
wavesims.m

```
% Wavesims
dt = 0.05;
npts = 8192;
nptso2 = npts/2.0;
tr = dt* npts; %dt * 8192 for 8192 total points
t = 0:dt:(tr-dt);
tp4 =  (2.0*pi) .^ 4;
g = 9.81;
v = 15.0;
df = 1.0/tr;
ffold = df * nptso2; %df * 4096 for 8192 total points
f = 0:df:ffold;
f = f+eps;
fac1 = 0.0081 *g*g/tp4;
fac2 = 0.74 *(g/v)^4/tp4;
s = 0.5*fac1 ./ f.^5 .* exp( -fac2 ./f .^ 4);
rand ('state',sum(100*clock));
p = 2.0 * pi * rand(1,nptso2);
p(nptso2+1) = 0.0;     %4097 for 8192 total points
z = exp(i*p) .* sqrt(s*df);
zt = [ z  conj(fliplr(z(2:4096)))];
zeta = real(fft(zt));
%The above gives same result as zeta = npts*real(ifft(zt))
plot (t,zeta);
xlabel('TIME (sec)')
ylabel('SURFACE ELEVATION (m)');
title('Simulated Sea Waves at a Point');
```

Simulated Sea Waves at a Point




Simulated Sea Waves at a Point
SURFACE ELEVATION (m)
TIME (sec)
3
2
1
0
-1
-2
-3
100
105
110
115
120
125
130
135
140
145
150



0.18
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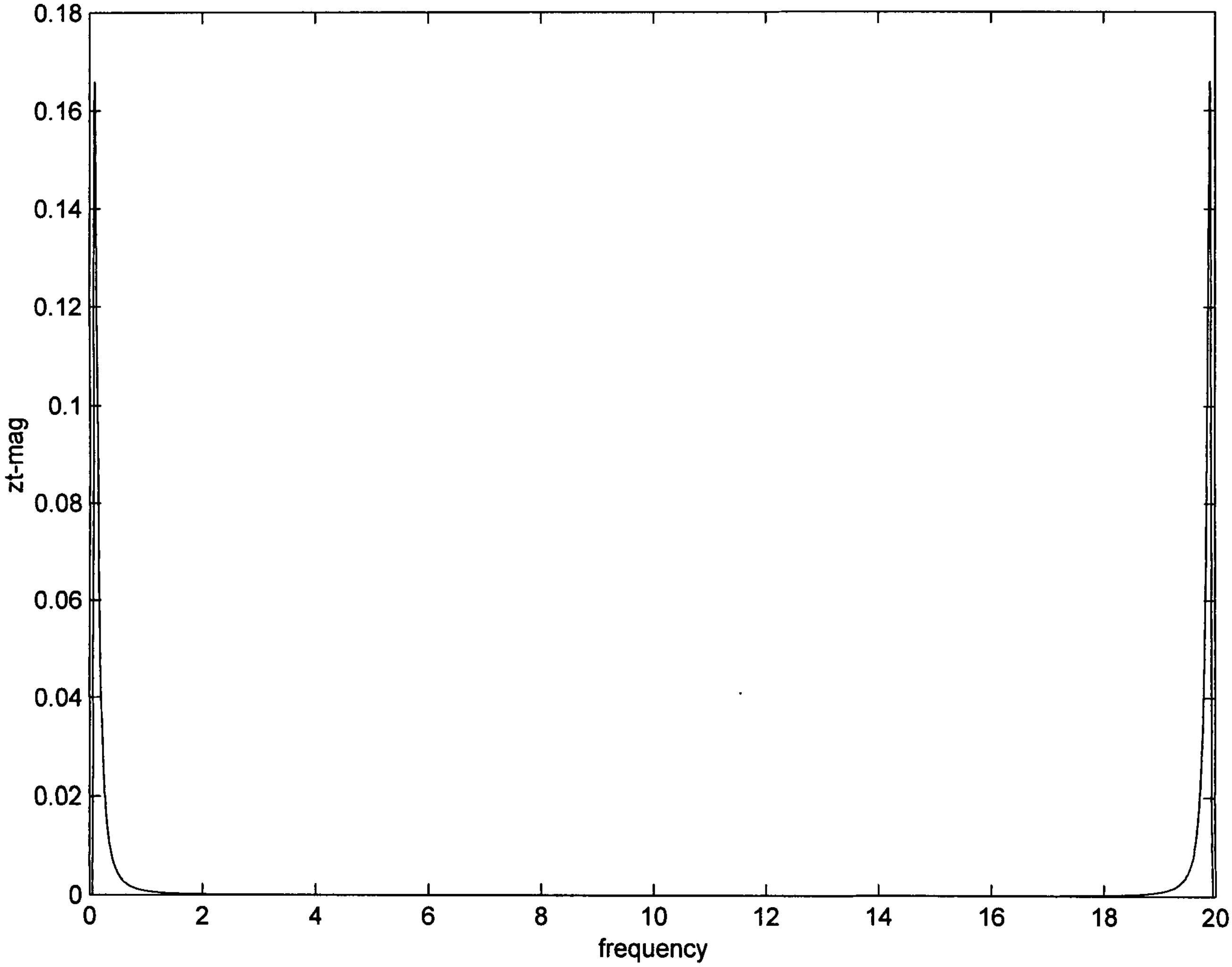
8000
9000
point number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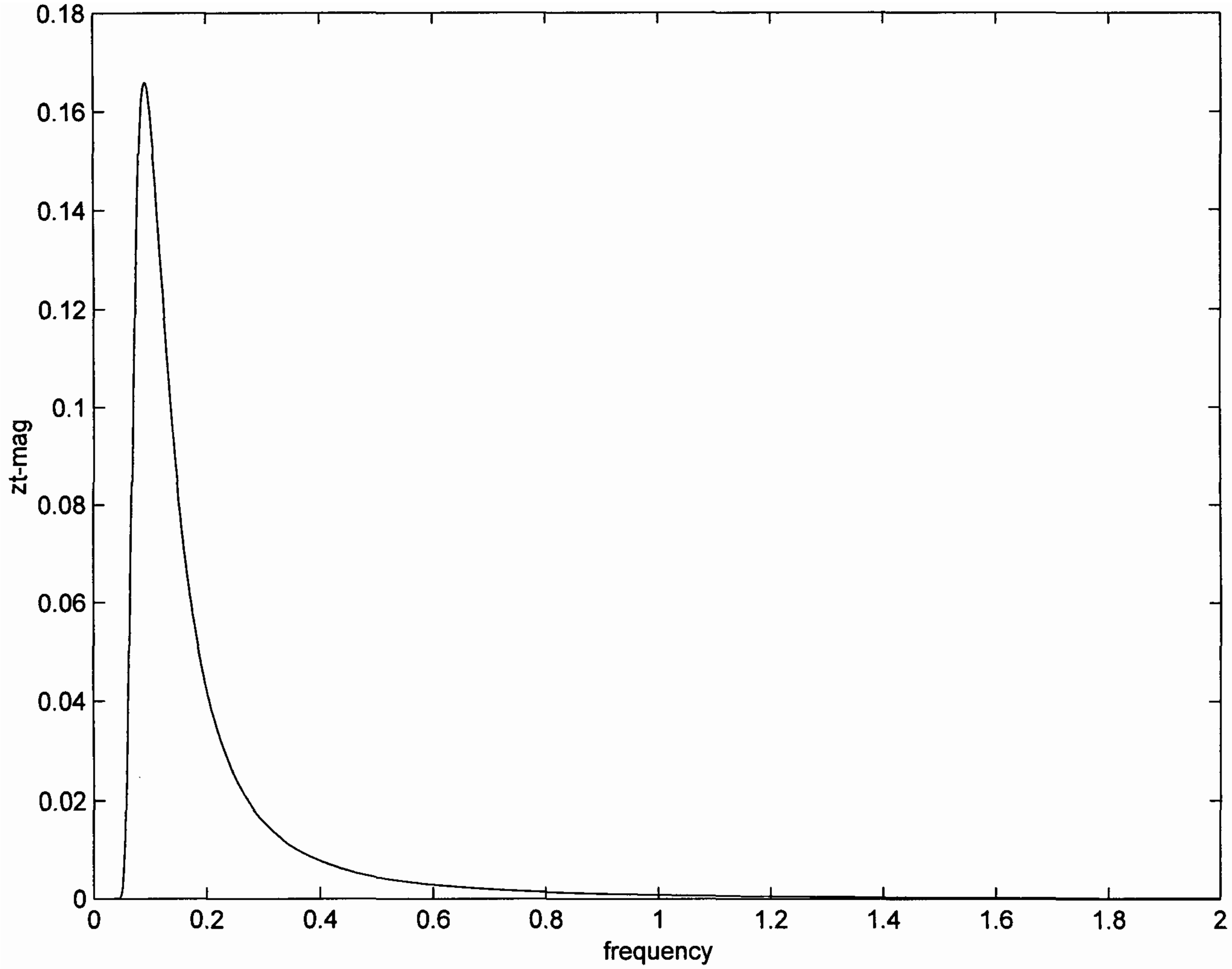
16
18
20
frequency


0.18
0.16
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0.12
0.1
0.08
0.06
0.04
0.02
0
zt-mag
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
frequency

Review of Fourier Transforms, Inverse Fourier Transforms, FFT'S, IFFT'S and Wave Simulation

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt, \quad x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f t} df$$

Consider functions of the form:

$x(t) = 0$ except for $0 \le t \le T$

$X(f) = 0$ except for $-F_{max} \le f \le F_{max}$

Then:

$$X(f) = \int_0^T x(t) e^{-i2\pi f t} dt, \quad x(t) = \int_{-F_{max}}^{F_{max}} X(f) e^{i2\pi f t} df$$

Construct a periodic function, $x_p(t)$ of period T

$x_p(t) = x(t)$, for $0 \le t < T$

$x_p(t)$ Has a Fourier Series Representation

$$x_p(t) = \sum_{n=-\infty}^{\infty} A_n e^{i2\pi f t}$$

where: $A_n = \frac{1}{T}\int_0^T x(t) e^{-i2\pi \frac{n}{T} t} dt$

Note that $X(\frac{n}{T}) = T A_n$

If T is very large, values of $\frac{n}{T}$ $n = 0, 1, 2, \ldots$ are dense so $X(f)$ can be determined from the Fourier Coefficients at closely spaced frequencies.

Note that $A_n = \frac{1}{T} X(\frac{n}{T})$

Since $f = \frac{n}{T}$, $A_n = 0$ for $|n| > T f_{max}$, also $\boxed{\delta f = \frac{1}{T}}$

In the range $0 \leq t \leq T$, $x(t) = \sum_{n=-T f_{max}}^{T f_{max}} A_n e^{i 2\pi f t}$

Let $T f_{max} = M$ $\qquad x(t) = \sum_{n=-M}^{M} A_n e^{i 2\pi f t}$

$2M = N$

Set $\boxed{dt = \frac{1}{2 f_{max}}}$, $t = j\delta t$, $x(j\delta t) \equiv x_j$, $\boxed{dt = \frac{T}{N}}$

We need to Evaluate $i 2\pi f t = i 2\pi n \delta f \, j \frac{T}{N}$

$= i 2\pi n \frac{1}{T} j \frac{T}{N} = i 2\pi \frac{nj}{N}$

Discretized approximation of $A_n = \frac{1}{T} X(\frac{n}{T})$

$A_n \cong \frac{1}{N \delta T} \sum_{n=0}^{N-1} x_j e^{-i 2\pi nj/N}$

$x(j\delta t) = x_j = \sum_{n=-N/2}^{N/2} A_n e^{i 2\pi nj/N} = \sum_{n=0}^{N-1} A'_n e^{i 2\pi nj/N}$

If $x(t)$ is real,

$$A'_n = \begin{cases} A_n, & 0 \leq n \leq \frac{N}{2} - 1 \\ 2A_n, & n = N/2 \\ A^*_{N-n}, & \frac{N}{2} < n < N \end{cases}$$

To Simulate waves having a one-sided frequency Spectrum $S_F(f)$ whose equivalent two-sided spectrum is $S_f(f) = \frac{1}{2} S_F(|f|)$;

the elevation $\zeta(t)$ at a point is

$$\zeta(j\delta t) = \sum_{n=0}^{N-1} A_n' e^{i 2\pi n j/N} \quad , \quad \delta t = \frac{T}{N}$$

where:

$$A_n' = e^{i\alpha_n} \sqrt{S_f(n\delta f)\,\delta f}$$

$$= e^{i\alpha_n} \sqrt{\tfrac{1}{2} S_F(n\delta f)\,\delta f}$$

where: α_n is a random number in the range $0 \le \alpha_n < 2\pi$
and the rules for A_n' are as given on the previous page for real ζ.

This is precisely the form of an inverse Fast Fourier transform

Generating Gaussian Random Numbers

This note is about the topic of generating **Gaussian** pseudo-random numbers given a source of **uniform** pseudo-random numbers. This topic comes up more frequently than I would have expected, so I decided to write this up on *one* of the best ways to do this. At the end of this note there is a list of references in the literature that are relevant to this topic. You can see some code examples that implement the technique, and a step-by-step example for generating **Weibull** distributed random numbers.

There are many ways of solving this problem (see for example **Rubinstein, 1981**, for an extensive discussion of this topic) but we will only go into one important method here. If we have an equation that describes our desired distribution function, then it is possible to use some mathematical trickery based upon the *fundamental transformation law of probabilities* to obtain a transformation function for the distributions. This transformation takes random variables from one distribution as inputs and outputs random variables in a new distribution function. Probably the most important of these transformation functions is known as the **Box-Muller** (1958) transformation. It allows us to transform uniformly distributed random variables, to a new set of random variables with a Gaussian (or Normal) distribution.

The most basic form of the transformation looks like:

```
y1 = sqrt( - 2 ln(x1) ) cos( 2 pi x2 )
y2 = sqrt( - 2 ln(x1) ) sin( 2 pi x2 )
```

We start with *two* independent random numbers, x1 and x2, which come from a uniform distribution (in the range from 0 to 1). Then apply the above transformations to get two new independent random numbers which have a Gaussian distribution with zero mean and a standard deviation of one.

This particular form of the transformation has two problems with it,

1. It is slow because of many calls to the math library.
2. It can have numerical stability problems when x1 is very close to zero.

These are serious problems if you are doing stochastic modelling and generating millions of numbers.

The **polar form** of the Box-Muller transformation is both faster and more robust numerically. The algorithmic description of it is:

```
float x1, x2, w, y1, y2;

do {
        x1 = 2.0 * ranf() - 1.0;
        x2 = 2.0 * ranf() - 1.0;
        w = x1 * x1 + x2 * x2;
} while ( w >= 1.0 );

w = sqrt( (-2.0 * log( w ) ) / w );
y1 = x1 * w;
y2 = x2 * w;
```

where **ranf()** is the routine to obtain a random number uniformly distributed in [0,1]. The polar form is faster because it does the equivalent of the sine and cosine geometrically without a call to the trigonometric function library. But because of the possiblity of many calls to **ranf()**, the uniform

WSS

random number generator should be fast (I generally recommend R250 for most applications).

Probability transformations for Non Gaussian distributions

Finding transformations like the Box-Muller is a tedious process, and in the case of empirical distributions it is not possible. When this happens, other (often approximate) methods must be resorted to. See the reference list below (in particular **Rubinstein, 1981**) for more information.

There are other very useful distributions for which these probability transforms *have* been worked out. Transformations for such distributions as the **Erlang**, **exponential**, **hyperexponential**, and the Weibull distribution can be found in the literature (see for example,**MacDougall, 1987**).

Useful References

Box, G.E.P, M.E. Muller 1958; **A note on the generation of random normal deviates**, Annals Math. Stat, V. 29, pp. 610-611

Carter, E.F, 1994; The Generation and Application of Random Numbers , Forth Dimensions Vol XVI Nos 1 & 2, Forth Interest Group, Oakland California

Knuth, D.E., 1981; **The Art of Computer Programming, Volume 2 Seminumerical Algorithms**, Addison-Wesley, Reading Mass., 688 pages, ISBN 0-201-03822-6

MacDougall,M.H., 1987; **Simulating Computer Systems**, M.I.T. Press, Cambridge, Ma., 292 pages, ISBN 0-262-13229-X

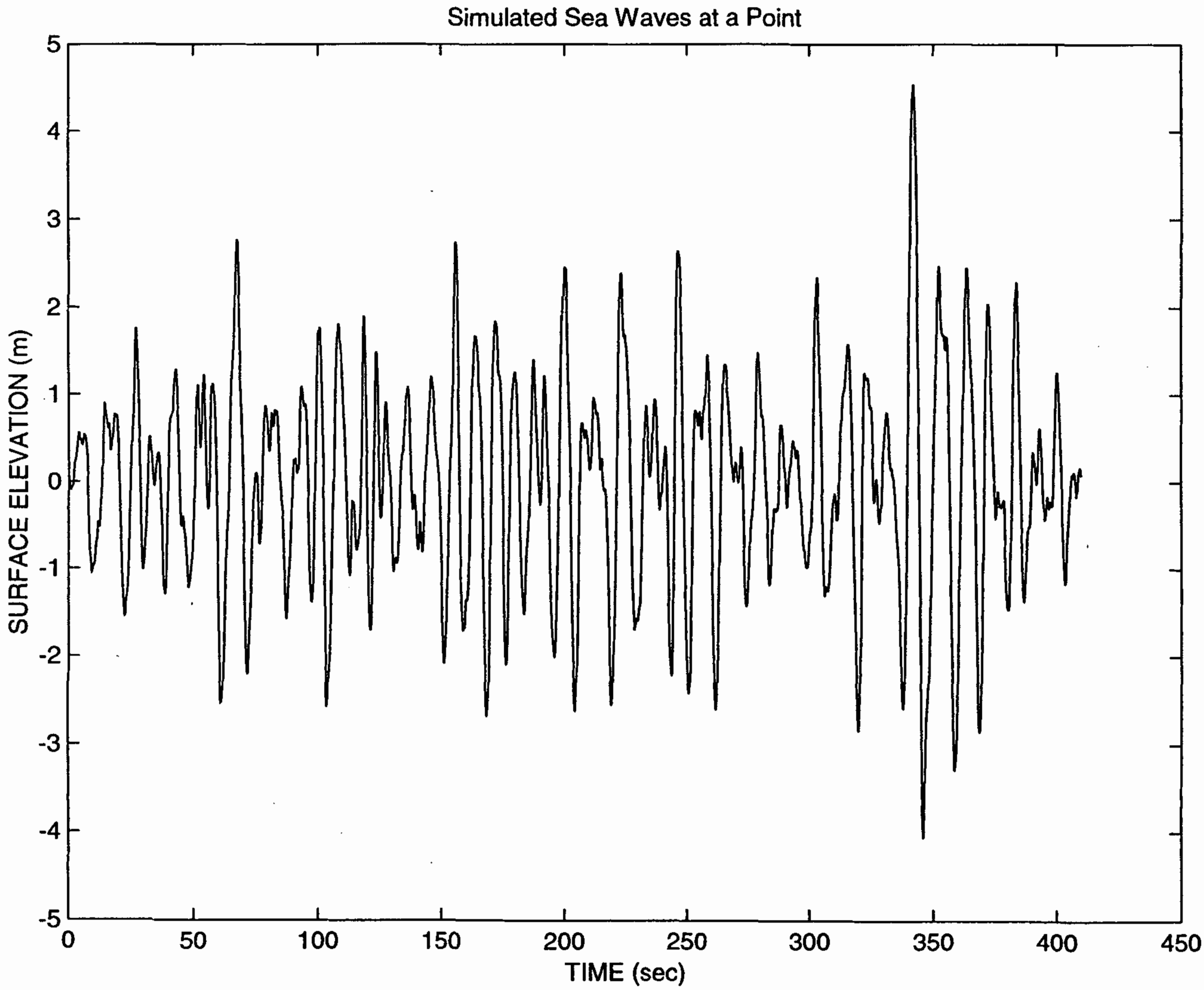
Press, W.H., B.P. Flannery, S.A. Teukolsky, W.T. Vetterling, 1986; **Numerical Recipes, The Art of Scientific Computing**, Cambridge University Press, Cambridge, 818 pages, ISBN 0-512-30811-9

Rubinstein, R.Y., 1981; **Simulation and the Monte Carlo method**, John Wiley & Sons, ISBN 0-471-08917-6

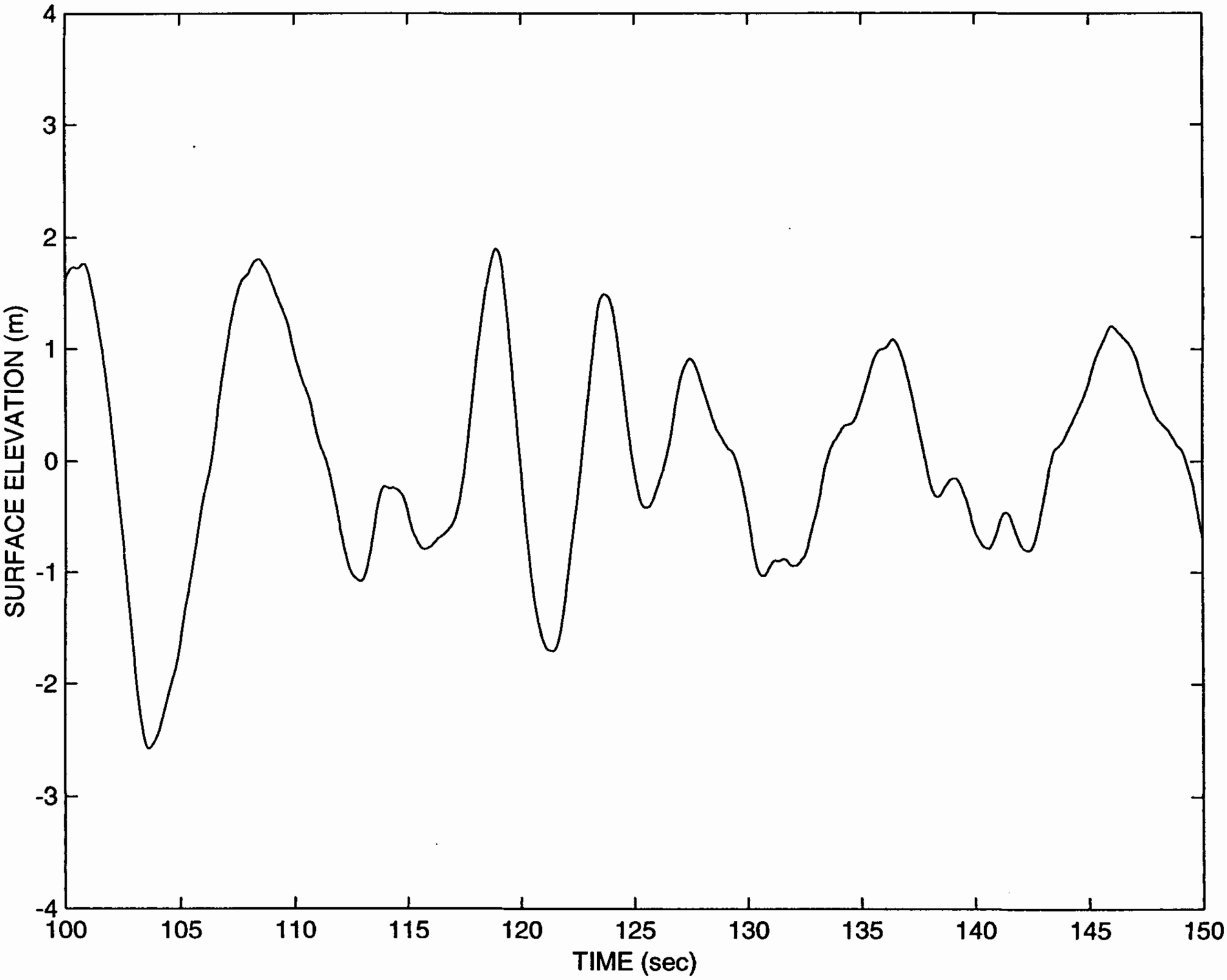
See Also: A Reference list of papers on Random Number Generation.

```
Everett (Skip) Carter
Taygeta Scientific Inc.
                         UUCP:     ...!uunet!taygeta!skip
                         WWW: http://www.taygeta.com/
```

```
/* boxmuller.c           Implements the Polar form of the Box-Muller
                         Transformation

                      (c) Copyright 1994, Everett F. Carter Jr.
                          Permission is granted by the author to use
                    this software for any application provided this
                    copyright notice is preserved.

*/

#include <math.h>

extern float ranf();         /* ranf() is uniform in 0..1 */


float box_muller(float m, float s)  /* normal random variate generator */
{                                   /* mean m, standard deviation s */
	float x1, x2, w, y1;
	static float y2;
	static int use_last = 0;

	if (use_last)                   /* use value from previous call */
	{
		y1 = y2;
		use_last = 0;
	}
	else
	{
		do {
			x1 = 2.0 * ranf() - 1.0;
			x2 = 2.0 * ranf() - 1.0;
			w = x1 * x1 + x2 * x2;
		} while ( w >= 1.0 );

		w = sqrt( (-2.0 * log( w ) ) / w );
		y1 = x1 * w;
		y2 = x2 * w;
		use_last = 1;
	}

	return( m + y1 * s );
}
```


Simulated Sea Waves at a Point
SURFACE ELEVATION (m)
TIME (sec)
5
4
3
2
1
0
-1
-2
-3
-4
-5
0
50
100
150
200
250
300
350
400
450

Simulated Sea Waves at a Point



WS 7

Wave Statistics

One way to calculate wave statistics is directly from long-term simulations.

Example What is the expected value of the largest wave elevation in a day?

Solution by simulation from a known wave spectrum.

1. Simulate waves for many days.
2. List the largest elevation in each day.
3. Calculatethe average of the values in the list.

Another Example What is the probability that the largest wave elevation in one day is less than the value V. Solution by simulation.

1. Simulate waves for many days.
2. Determine the fraction of days that the elevation does not exceed V.
3. This fraction is an estimate of the desired probability.

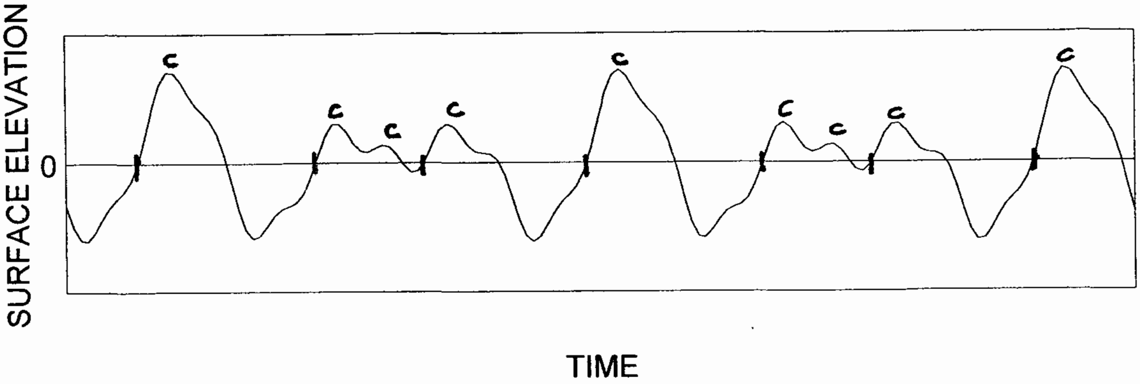
The above direct approach is cumbersome and computationally intensive. Many wave statistics have been theoretically determined in terms of the wave specturm. The associated formulae can be determined using numerical integration.

Results from Theory

The spectral moments, m_n, are defined in terms of the one-sided spectrum, $S_W(\omega)$, as:

$$m_n = \int_0^\infty S_W(\omega) d\omega$$

The following results apply when the surface elevation is a gaussian random process.



Number of Waves per Unit Time

The average number of times the wave elevation, ζ, crosses the mean sea level ($\zeta = 0$) per unit time while increasing is called f_o and given by:

$$f_o = \frac{1}{2\pi}\sqrt{\frac{m_2}{m_0}}$$

The average number of wave crests per unit time is called f_c and is given by:

$$f_c = \frac{1}{2\pi}\sqrt{\frac{m_4}{m_2}}$$

The bandwidth, ϵ, is given by:

$$\epsilon = \sqrt{1 - f_0^2/f_c^2}$$

Definition of a gaussian random process For any number of variables, the joint probability density (pdf) of all the variables is a joint gaussian random variable at each time for a gaussian random process. This probability density function is given by:

$$p(x_1, x_2, ..., x_n) = \frac{1}{\sqrt{(2\pi)^n|\Delta|}} \exp\left\{-\frac{1}{2}[X]^T[\Delta^{-1}][X]\right\}$$

$[X]$ is the column vector of the variables. Δ is the n-by-n covariance matrix whose elements are given by:

$$\Delta_{ij} = E[x_i x_j]$$

For most wave statistics of interest, the doubly joint pdf between surface elevation, ζ and vertical surface velocity, $\dot{\zeta}$, and the triply joint pdf where the surface acceleration, $\ddot{\zeta}$, is included are all that are needed.

$$p(\zeta, \dot{\zeta}) = \frac{1}{2\pi\sqrt{m_0 m_2}} \exp\left[-\frac{m_2\zeta^2 + m_0\dot{\zeta}^2}{2m_0 m_2}\right]$$

$$p(\zeta, \dot{\zeta}, \ddot{\zeta}) = \frac{1}{(2\pi)^{3/2}\sqrt{m_2(m_0 m_4 - m_2^2)}} \exp\left[-\frac{m_2 m_4\zeta^2 + (m_0 m_4 - m_2^2)\dot{\zeta}^2 + m_0 m_2\ddot{\zeta}^2 + 2m_2^2\zeta\ddot{\zeta}}{2m_2(m_0 m_4 - m_2^2)}\right]$$

The normalized Gaussian probability distribution function (pdf), $\Psi(x)$, is:

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz$$

Call the crest height ξ.

The normalized crest height, η, is defined by: $\eta = \dfrac{\xi}{\sqrt{m_0}}$

The probability distribution function for η is:

$$P(\eta) = \Psi\left(\frac{\eta}{\epsilon}\right) - \sqrt{1-\epsilon^2}\, e^{-\eta^2/2}\, \Psi\left(\frac{\sqrt{1-\epsilon^2}}{\epsilon}\eta\right)$$

and the pdf for η is:

$$p(\eta) = \frac{\epsilon}{2\pi} \exp\left[-\frac{\eta^2}{2\epsilon^2}\right] + \sqrt{1-\epsilon^2}\, \eta\, e^{-\eta^2/2}\, \Psi\left(\frac{\sqrt{1-\epsilon^2}}{\epsilon}\eta\right)$$

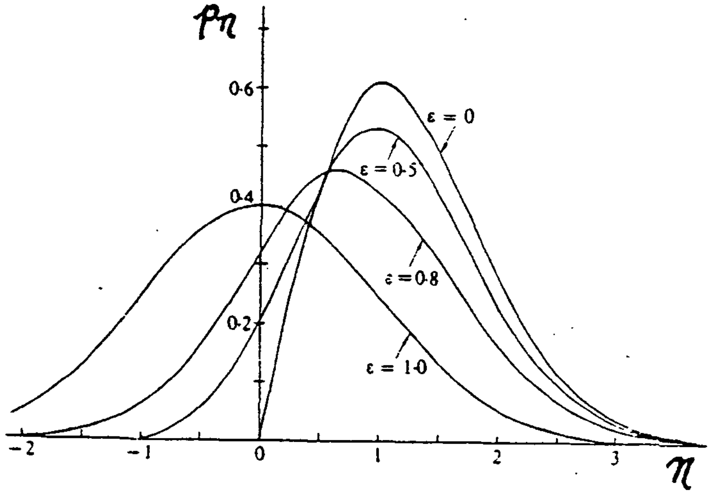


Figure 7.5 Probability density function of η for various values of the band width ϵ.

Typically, $\epsilon \approx 0.6$.

For engineering purposes we are interested in large seas ($\eta >> 1$). This corresponds to the tail of the pdf for η. In this region:

$$p(\eta) = \sqrt{1-\epsilon^2}\, \eta\, e^{-\eta^2/2} \qquad P(\eta) = 1 - \sqrt{1-\epsilon^2}\, e^{-\eta^2/2}$$

Average Amplitude of the 1/n'th Highest waves

Call the smallest normalized wave amplitude in the 1/n'th highest Waves $\eta_{1/n}$.

$$\frac{1}{n} = 1 - P(\eta_{1/n})$$

Example: n = 10.
1 - (probability that a wave amplitude is less than the smallest of the 10% largest waves) is 1/10.
This is because the probability that a (random) wave is smaller than 10% is 90%.

For $n >> 1$, use the approximate P.

$$\frac{1}{n} = \sqrt{1-\epsilon^2}\exp\left[-\frac{1}{2}\eta_{1/n}^2\right]$$

$$\eta_{1/n} = \sqrt{2\ln(n\sqrt{1-\epsilon^2})}$$

Amongst the 1/n'th highest waves, the conditional pdf is:

$$p_{\eta>\eta_{1/n}}(\eta) = np(\eta) = n\sqrt{1-\epsilon^2}\,\eta\,\exp(-\eta^2/2), \quad \eta_{1/n} < \eta < \infty$$

The expectation of these amplitudes is the average of the 1/n'th highest waves.

$$\overline{\eta_{1/n}} = n\sqrt{1-\epsilon^2}\int_{\eta_{1/n}}^{\infty} \eta^2 e^{-\eta^2/2} d\eta$$

Let $n' = \sqrt{1-\epsilon^2}\,n$. Then, n' is the number of zero up-crossings in a record with n crests. The result of the integration is:

$$\overline{\eta_{1/n}} = n'\left\{\frac{\sqrt{2\ln n'}}{n'} + \sqrt{2\pi}\left[1 - \Psi(\sqrt{2\ln n'})\right]\right\}$$

Extreme Waves

Consider n non-dimensional random wave Amplitudes. Each has same pdf.

What are the probabilities of the largest waves in the set?

Aproach
Order the waves from smallest to largest.
ϕ_1 is the smallest and ϕ_n is the largest wave amplitude. Now, each of the ϕ's has a different pdf.
We want to find the pdf for ϕ_n.

Probability that ϕ_n is less than a particular value ϕ_{n_o} is equal to the probability that all the waves are smaller than ϕ_{n_o}.

$$P_{\phi_n}(\phi_{n_o}) = [P_\eta(\phi_{n_o})]^n$$

The amplitude that has a probability, α, of being exceeded by ϕ_n is called ${}_\alpha\phi_n$.

$$P_{\phi_n}({}_\alpha\phi_n) = [P_\eta({}_\alpha\phi_n)]^n = 1 - \alpha$$

Meaning of the Nomenclature
Suppose $\alpha = 0.01$. Then the amplitude whose probability of being exceeded by ϕ_n is 0.01 is named ${}_{0.01}\phi_n$.
The probability that ϕ_n is less than ${}_{0.01}\phi_n$ is 0.99.

$$P_\eta({}_\alpha\phi_n) = (1 - \alpha)^{1/n}$$

$$\Psi\left(\frac{{}_\alpha\phi_n}{\epsilon}\right) - \sqrt{1-\epsilon^2}\,\exp\left[-\frac{1}{2}\,{}_\alpha\phi_n^2\right]\Psi\left(\frac{\sqrt{1-\epsilon^2}}{\epsilon}\,{}_\alpha\phi_n\right) = (1-\alpha)^{1/n}$$

Since we are interested large waves, we can use the expressions for the tails of the probability functions:

$$P(\eta) = 1 - \sqrt{1-\epsilon^2}\, e^{-n^2/2}$$

$$\text{Then,} \quad 1 - \sqrt{1-\epsilon^2}\exp\left[-\frac{1}{2}\,{}_{\alpha}\phi_n^2\right] = (1-\alpha)^{1/n}$$

Solve for ${}_{\alpha}\phi_n$:

$${}_{\alpha}\phi_n = \sqrt{2\ln\left(\frac{\sqrt{1-\epsilon^2}}{1-(1-\alpha)^{1/n}}\right)}$$

Note: The value of n for a given period of time T can be obtained from:

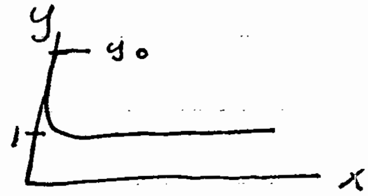
$$f_c = \frac{1}{2\pi}\sqrt{\frac{m_4}{m_2}}$$

$$n = f_c T = \frac{T}{2\pi}\sqrt{\frac{m_4}{m_2}}$$

STiff Equations

$$\frac{dy}{dx} = -100y + 100 \quad ; \text{ initial Condition, } y(0) = y_0$$

Exact Solution $y(x) = (y_0 - 1)e^{-100x} + 1$



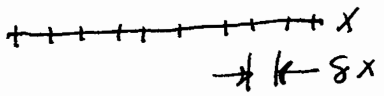
This is stable in sence that small change in initial condition causes small change in Solution.

Example, if $y(0) = y_0 + \epsilon$,

$$y(x) = (y_0 + \epsilon - 1)e^{-100x} + 1$$

Change in Solution, δ, is ϵe^{-100x}

Solution by the forward Euler method



$$y_{n+1} = y_n + (-100y_n + 100)\delta x = (1 - 100\,\delta x)\,y_n + 100\,\delta x$$

This difference equation has an exact Solution

$$y_n = (y_0 - 1)(1 - 100\,\delta x)^n + 1$$

For example, if $y_0 = 2$, $\quad y(x) = e^{-100x} + 1$

$$y_n = (1 - 100\,\delta x)^n + 1$$

Note; if $\delta x > 0.02$, the Solution (numerical) diverges

$(1 - 100\,\delta x)^n$ is an approximation to e^{-100x}. It is a poor approximation unless δx is very small even though e^{-100x} hardly contributes to the Solution for $x > .01$

This problem is often overcome by implicit methods. One is the backward Euler method

$$y_{n+1} = y_n + f(x_{n+1}, y_{n+1})\,\delta x$$

For our example $f(x_{n+1}, y_{n+1}) = (-100\,y_{n+1} + 100)$

$$y_{n+1} = y_n + (-100\,y_{n+1} + 100)\,\delta x$$

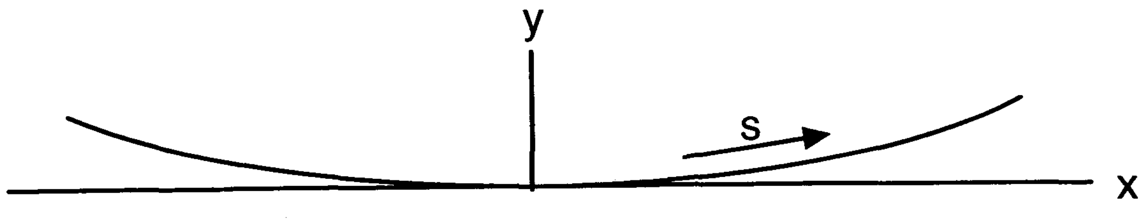
$$y_{n+1}(1 + 100\,\delta x) = y_n + 100\,\delta x \;; \quad y_{n+1} = \frac{y_n + 100\,\delta x}{1 + 100\,\delta x}$$

The exact Solution to this is:

$$y_n = \frac{1}{(1 + 100\,\delta x)^n} + 1$$

This is not unstable for any δx

Dynamics of Horizontal Shallow Sag Cables in Water



Static Solution

H is the horizontal component of the Tension.
w is the weight in water/unit length.
T is the tension.
L is the static length.

$$y = \frac{H}{w}\cosh\frac{w}{H}x - \frac{H}{w} \qquad\qquad T = H\cosh\frac{w}{H}x$$

For $T \gg wL$

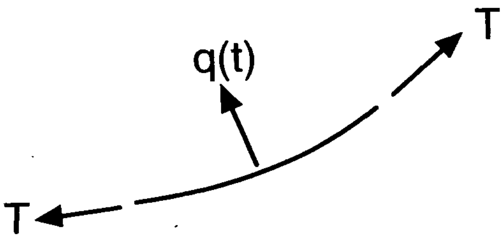
$$y = \frac{T_o}{w}\left[1 + \frac{w}{2T_o}x^2 + ...\right] - \frac{T_o}{w} \qquad\qquad T \equiv T_o \cong H$$

$$\frac{dy}{dx} = \frac{wx}{T_o}$$

$$\frac{d^2y}{dx^2} = \frac{w}{T_o} \equiv \alpha \qquad\qquad \text{static curvature } = \alpha$$

cab1(2)

Dynamics



$$\text{vertical mechanical force/unit length} \;=\left(T_o+\widetilde{T}\right)\left(\alpha+\frac{\partial^2 q}{\partial t^2}\right)$$

q is the displacement normal to the cable towards the inside of the static curvature.

$$\text{dynamic vertical mechanical force/unit length} \;=\left(T_o+\widetilde{T}\right)\left(\alpha+\frac{\partial^2 q}{\partial s^2}\right)-T_o\alpha$$

$$\text{hydrodynamic vertical force/unit length} \;=-b\frac{dq}{dt}\left|\frac{dq}{dt}\right|$$

where: $b=\frac{1}{2}\rho C_d D$, ρ is the density of water, C_d is the drag coefficient and D is the diameter of the cable.

Equation of Motion

$$m\frac{\partial^2 q}{\partial t^2}=(T_o+\widetilde{T})\left(\alpha+\frac{\partial^2 q}{\partial s^2}\right)-b\frac{dq}{dt}\left|\frac{dq}{dt}\right|-T_o\alpha$$

Strain Compatibility

$$\text{Tension increase due to } q \;=\text{increased length }\times\frac{EA}{L}$$

E is the elastic modulus and A is the cross sectional Area of the cable.

$$\widetilde{T}=\frac{EA}{L}\left[p_o-\alpha\int_0^L q\,ds+\frac{1}{2}\int_0^L\left(\frac{\partial q}{\partial s}\right)^2 ds\right]$$

where: p_o is the sum of the tangential extensions of the ends of the cable.

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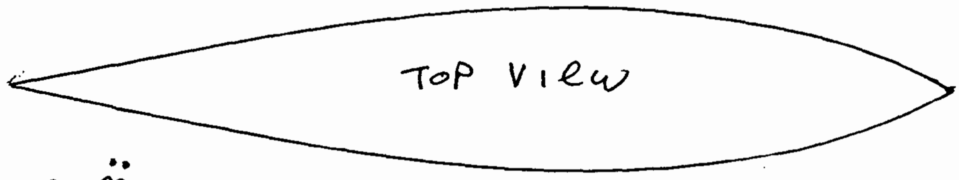
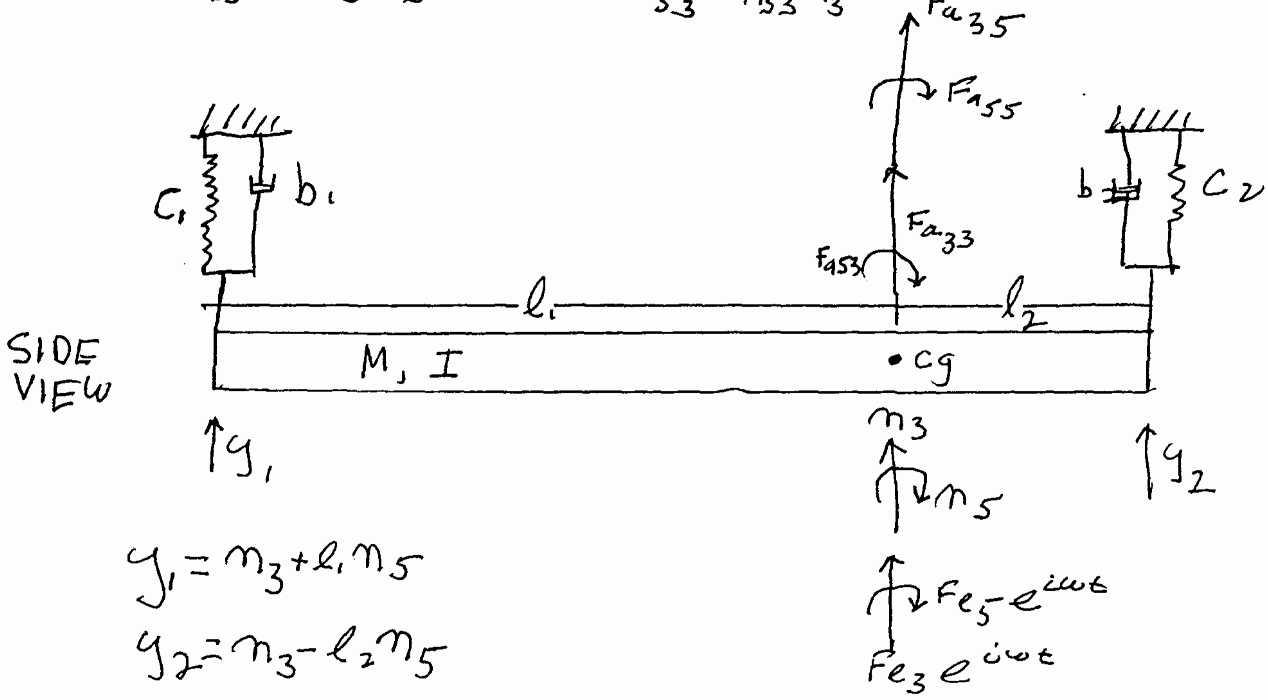
Oscillating Rigid Objects



$$F_{a_{33}} = -A_{33}\ddot{\eta}_3 \qquad F_{a_{55}} = -A_{55}\ddot{\eta}_5$$

$$F_{a_{35}} = -A_{35}\ddot{\eta}_5 \qquad F_{a_{53}} = -A_{53}\ddot{\eta}_3$$



$$y_1 = \eta_3 + \ell_1\eta_5$$
$$y_2 = \eta_3 - \ell_2\eta_5$$

$$M\ddot{\eta}_3 = -A_{33}\ddot{\eta}_3 - A_{35}\ddot{\eta}_5 - C_1\eta_3 - C_1\ell_1\eta_5 - b_1\dot{\eta}_3 - b_1\ell_1\dot{\eta}_5$$
$$- C_2\eta_3 + C_2\ell_2\eta_5 - b_2\dot{\eta}_3 + b_2\ell_2\dot{\eta}_5$$
$$+ F_{e_3} e^{i\omega t}$$

$$(M+A_{33})\ddot{\eta}_3 + \overbrace{(b_1+b_2)}^{B_{33}}\dot{\eta}_3 + \overbrace{(C_1+C_2)}^{C_{33}}\eta_3 + A_{35}\ddot{\eta}_5$$
$$+ \underbrace{(b_1\ell_1 - b_2\ell_2)}_{B_{35}}\dot{\eta}_5 + \underbrace{(C_1\ell_1 - C_2\ell_2)}_{C_{35}}\eta_5 = F_{e_3} e^{i\omega t}$$

$$I\ddot{\eta}_5 = -A_{55}\ddot{\eta}_5 - A_{53}\ddot{\eta}_3 - c_1\ell_1\eta_3 - c_1\ell_1^2\eta_5 - b_1\ell_1\dot{\eta}_3 - b_1\ell_1^2\dot{\eta}_5$$

$$+ c_2\ell_2\eta_3 - c_2\ell_2^2\eta_5 + b_2\ell_2\dot{\eta}_3 - b_2\ell_2^2\dot{\eta}_5 + F_{e_5}e^{i\omega t}$$

$$A_{53}\ddot{\eta}_3 + \overbrace{(b_1\ell_1 - b_2\ell_2)}^{B_{53}}\dot{\eta}_3 + \overbrace{(c_1\ell_1 + c_2\ell_2)}^{C_{53}}\eta_3 + (I_5 + A_{55})\ddot{\eta}_5$$

$$+ \underbrace{(b_1\ell_1^2 + b_2\ell_2^2)}_{B_{55}}\dot{\eta}_5 + \underbrace{(c_1\ell_1^2 + c_2\ell_2^2)}_{C_{55}}\eta_5 = F_{e_5}e^{i\omega t}$$

STPoTbc1

Potentials and Boundary Conditions

$$\phi_T = \phi_I + \phi_D + \sum_1^6 \zeta_j \phi_j$$

On the Free Surface:

$$\left[-\omega^2 - Ui\omega\frac{\partial}{\partial x} + u^2\frac{\partial^2}{\partial x^2} + g\frac{\partial}{\partial z}\right]\phi_D = 0$$

and: $$\left[-\omega^2 - Ui\omega\frac{\partial}{\partial x} + u^2\frac{\partial^2}{\partial x^2} + g\frac{\partial}{\partial z}\right]\phi_j = 0$$

On the Hull:

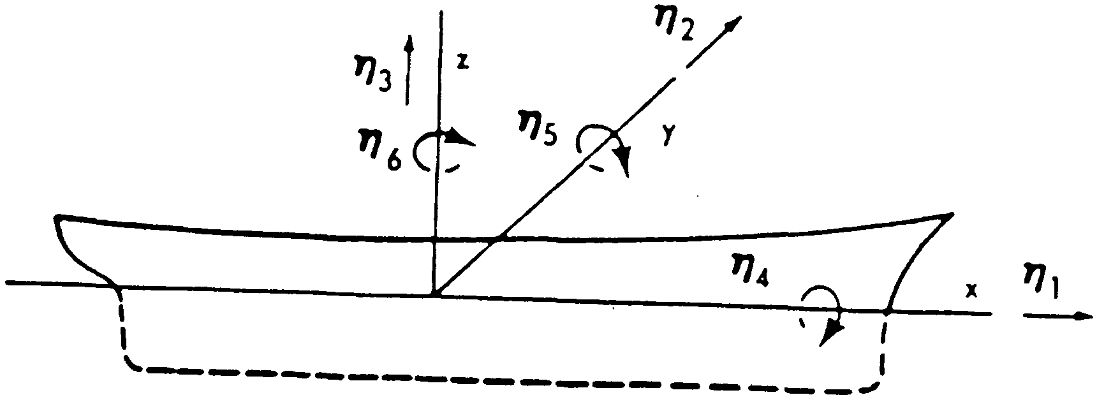
$$\frac{\partial \phi_I}{\partial n} + \frac{\partial \phi_D}{\partial n} = 0$$

Pressure on the hull:

$$p = -\rho\left(i\omega - U\frac{\partial}{\partial x}\right)\phi_T e^{i\omega t} - \rho g(\zeta_3 + \zeta_4 y - \zeta_5 x)e_{-\omega t}$$

ST1.TeX
→ ST1(a), ST1(b) ST1 (a)

Strip Theory



η_1 = **surge**	η_3 = **heave**	η_5 = **pitch**
η_2 = **sway**	η_4 = **roll**	η_6 = **yaw**

Sign convention for translatory and angular displacements

For inviscid, irrotational theory,

$$\phi_T = \phi_I + \phi_D + \sum_{k=1}^{6} \eta_j \phi_j + \text{interaction terms}$$

In linear theory the interaction terms are neglected. Here we focus on the ϕ_j's and the forces and moments associated with them.

In general, a force in the j'th direction will lead to motions in the six degrees of freedom, η_k, $k = 1, 2, 3, 4, 5, 6$.

Sinusoidal forces which generate sinusoidal motions are considered. The equations of motion for sinusoidal excitation are:

$$\sum_{k=1}^{6} \{(M_{jk} + A_{jk})\ddot{\eta}_k + B_{jk}\dot{\eta}_k + C_{jk}\eta_k\} = F_j e^{i\omega t}, \qquad j = 1, 2, 3, 4, 5, 6$$

$$\eta_k(t) = \zeta_k e^{i\omega t}$$

$$\sum_{k=1}^{6} \left\{-\omega^2 (M_{jk} + A_{jk})\zeta_k + i\omega B_{jk}\zeta_k + C_{jk}\zeta_k\right\} = F_j, \qquad j = 1, 2, 3, 4, 5, 6$$

$$\sum_{k=1}^{6} \left\{\left[-\omega^2 (M_{jk} + A_{jk}) + i\omega B_{jk} + C_{jk}\right]\zeta_k\right\} = F_j, \qquad j = 1, 2, 3, 4, 5, 6$$

STF(6)

$-M_{jk}\ddot{\eta}_k$ is an inertial force in the j'th direction due to motion in the k'th direction.

$-A_{jk}\ddot{\eta}_k$, $-B_{jk}\dot{\eta}_k$, and $-C_{jk}\eta_k$ are hydrodynamic and hydrostatic forces in the j'th direction due to motion in the k'th direction.

577

For a ship having port/starboard symmetry:

$$M_{jk} = \begin{bmatrix} M & 0 & 0 & 0 & Mz_c & 0 \\ 0 & M & 0 & -Mz_c & 0 & 0 \\ 0 & 0 & M & 0 & 0 & 0 \\ 0 & -Mz_c & 0 & I_4 & 0 & -I_{46} \\ Mz_c & 0 & 0 & 0 & I_5 & 0 \\ 0 & 0 & 0 & -I_{64} & 0 & I_6 \end{bmatrix}$$

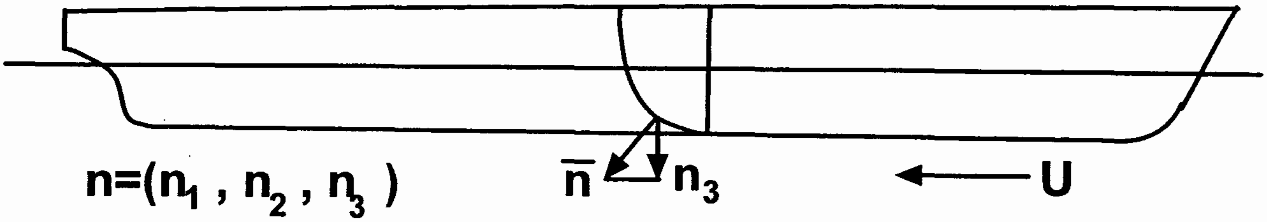
$$A_{jk} = \begin{bmatrix} A_{11} & 0 & a_{13} & 0 & A_{15} & 0 \\ 0 & A_{22} & 0 & A_{24} & 0 & A_{26} \\ A_{31} & 0 & A_{33} & 0 & A_{35} & 0 \\ 0 & A_{42} & 0 & A_{44} & 0 & A_{46} \\ A_{51} & 0 & A_{53} & 0 & A_{55} & 0 \\ 0 & A_{62} & 0 & A_{64} & 0 & A_{66} \end{bmatrix}$$

$$B_{jk} = \begin{bmatrix} B_{11} & 0 & B_{13} & 0 & B_{15} & 0 \\ 0 & B_{22} & 0 & B_{24} & 0 & B_{26} \\ B_{31} & 0 & B_{33} & 0 & B_{35} & 0 \\ 0 & A_{B2} & 0 & B_{44} & 0 & B_{46} \\ B_{51} & 0 & B_{53} & 0 & B_{55} & 0 \\ 0 & B_{62} & 0 & B_{64} & 0 & B_{66} \end{bmatrix}$$

$$C_{jk} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & C_{35} & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & C_{53} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

STIP

Boundary Conditions on Hull

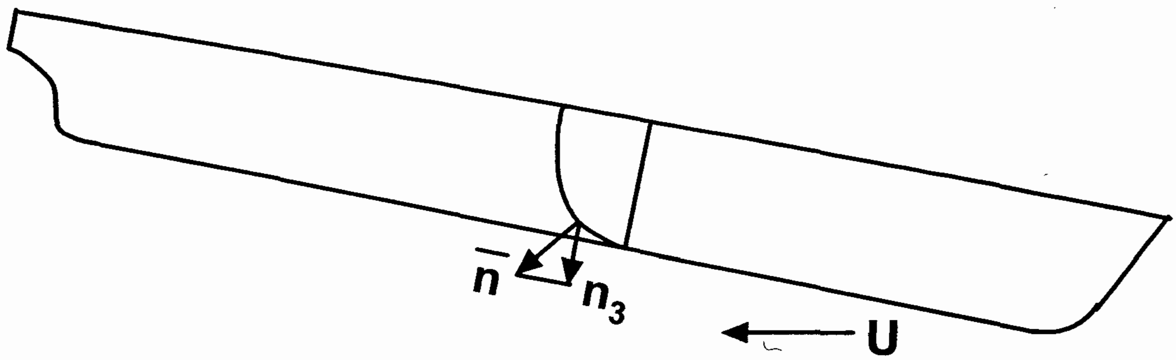


Definitions: $(n_1, n_2, n_3) = \vec{n}$ $(n_4, n_5, n_6) = \vec{r} \times \vec{n}$

Except for j = 5 and j = 6:

$$\frac{\partial}{\partial n}\zeta_j\phi_j e^{i\omega t} = \frac{\partial}{\partial t}\zeta_j e^{i\omega t} n_j \qquad \zeta_j\frac{\partial\phi_j}{\partial n}e^{i\omega t} = \zeta_j i\omega n_j e^{i\omega t} \qquad \frac{\partial\phi_j}{\partial n} = i\omega n_j$$

For j = 5 (pitch) and for j = 6 (yaw) there is a change in the normal velocity associated with U:



Here, for positive pitch, there is an upward (positive z) component of velocity equal to $-U\zeta_5 e^{i\omega t}$. This has a component normal to the hull surface of $-U\zeta_5 e^{i\omega t} n_3$.

For n = 5: $$\frac{\partial\phi_5}{\partial n} = i\omega n_5 + U n_3$$

For n = 6: $$\frac{\partial\phi_6}{\partial n} = i\omega n_6 - U n_2$$

ST3e

For slender ships surge forces F_1 are much smaller than the other forces. Furthermore, surge motions have little effect except for the special case of towing. We neglect surge here.

Under these conditions, the pitch and heave equations are decoupled from the sway, roll and yaw equations.

PITCH AND HEAVE EQUATIONS

$$\left[-\omega^2(M + A_{33}) + i\omega B_{33} + C_{33}\right]\zeta_3 + \left[-\omega^2 A_{35} + i\omega B_{35} + C_{35}\right]\zeta_5 = F_3$$

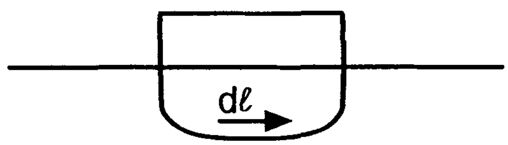
$$\left[-\omega^2 A_{53} + i\omega B_{53} + C_{53}\right]\zeta_3 + \left[-\omega^2(I_5 + A_{55}) + i\omega B_{55} + C_{55}\right]\zeta_5 = F_5$$

Hydrodynamic and Hydrostatic Coefficients

The hydrodynamic forces on the ship due to oscillatory motions of the ship are called radiation forces. For a motion of the form $\eta_k(t) = \zeta_k e^{i\omega t}$, the linearized force in the j direction is called $T_{jk}e^{i\omega t}$. T_{jk} will be a complex number having a real part and an imaginary part. It is conventionally written in the following form:

$$T_{jk} = (\omega^2 A_{jk} - i\omega B_{jk} - C_{jk})\zeta_k$$

A_{jk} is the added mass for forces in the j direction due to motion in the k direction.
B_{jk} is the damping coefficient for forces in the j direction due to motion in the k direction.
C_{jk} is the hydrostatic "spring constant".



$$A_{33} = \int_L a_{33}d\xi - \frac{U}{\omega^2}b_{33}^A \qquad B_{33} = \int_L b_{33}d\xi + Ua_{33}^A$$

$$A_{35} = -\int_L \xi a_{33}d\xi - \frac{U}{\omega^2}B_{33}^o + \frac{U}{\omega^2}x_A b_{33}^A - \frac{U^2}{\omega^2}a_{33}^A$$

$$B_{35} = -\int_L \xi b_{33}d\xi + UA_{33}^o - Ux_A a_{33}^A - \frac{U^2}{\omega^2}b_{33}^A$$

A_{33}^o and B_{33}^o are the speed independent parts of the respective coefficients.

$$A_{53} = -\int_L \xi a_{33} d\xi + \frac{U}{\omega^2} B_{33}^o + \frac{U}{\omega^2} x_A b_{33}^A$$

$$B_{53} = -\int_L \xi b_{33} d\xi - U A_{33}^o - U x_A b_{33}^A$$

$$A_{55} = -\int_L \xi^2 a_{33} d\xi + \frac{U^2}{\omega^2} A_{33}^o - \frac{U}{\omega^2} x_A^2 b_{33}^A + \frac{U^2}{\omega^2} x_A a_{33}^A$$

$$B_{55} = \int_L \xi^2 b_{33} d\xi + \frac{U^2}{\omega^2} B_{33}^o - U x_A^2 a_{33}^A + \frac{U^2}{\omega^2} x_A b_{33}^A$$

$$C_{33} = \rho g \int_L b d\xi = \rho g A_{WP}$$

b is the beam of each section and A_{WP} is the waterplane area.

$$C_{35} = C_{53} = -\rho g \int_L \xi b d\xi = -\rho g M_{WP}$$

$$C_{55} = \rho g \int_L \xi^2 b d\xi = \rho g I_{WP}$$

M_{WP} is the moment of area of the waterplane and I_{wp} is the moment of inertia of the waterplane.

$$F_3 = \rho\alpha \int_L (f_3 + h_3) d\xi + \rho\alpha \frac{U}{i\omega} x_A h_3^A$$

$$F_5 = -\rho\alpha \int_L \left[\xi(f_3 + h_3) + \frac{U}{i\omega} h_3 \right] d\xi - \rho\alpha \frac{U}{i\omega} x_A h_3^A$$

α is the wave amplitude. $f_3 = F_3 e^{i\omega t}$ is the Froude-Krilov force.

ST3C

$$F_3(x) = ge^{-ikx\cos\beta} \int_{C_x} N_3 e^{iky\sin\beta} e^{kz} d\ell$$

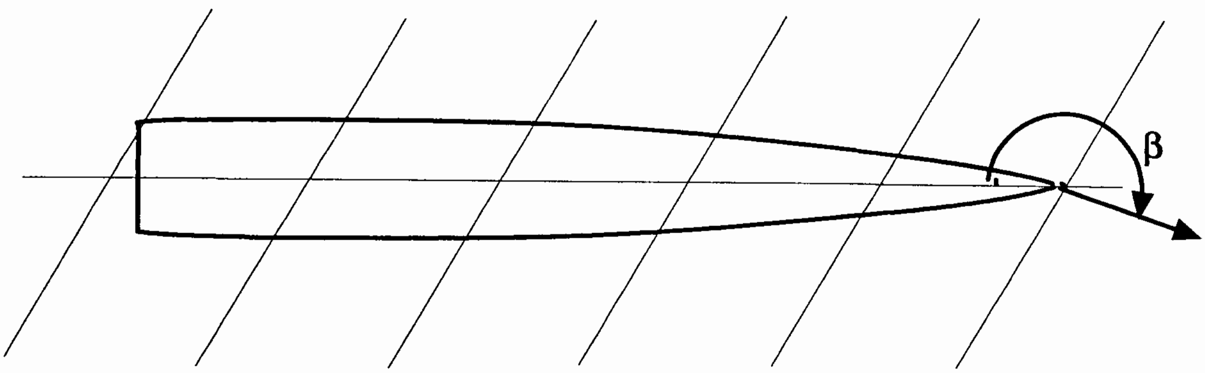
N_3 is the vertical component of the 2D normal to the section. N_2 is the horizontal component of the 2D normal. β is the wave propagation angle. $d\ell$ is the element of arc length around the section.

h_3 is the sectional diffraction force.

$$h_3(x) = \omega_o e^{-ikx\cos\beta} \int_{C_x} (iN_3 + N_2 \sin\beta) e^{iky\sin\beta} e^{kz} \psi_3 d\ell$$

$$\omega_o = \sqrt{gk} \qquad \omega_o = \omega + kU\cos\beta$$

ψ_3 is the velocity potential for a 2D cylinder of shape C_x oscillating in heave. ψ_3 is the solution to $\nabla^2 \psi_3 = 0$ subject to the boundary condition for heave motion. It can be obtain by several ways including panel methods.



Thus, to do the the longitudinal integrals ($d\xi$), one must know the 2D hydrostatic terms and the 2D added mass, damping, and velocity potential for heave.

SWAY, ROLL AND YAW EQUATIONS

$$\left[-\omega^2(A_{22}+M)+i\omega B_{22}\right]\zeta_2+\left[-\omega^2(A_{24}-Mz_c)+i\omega B_{24}\right]\zeta_4+\left[-\omega^2 A_{26}+i\omega B_{26}\right]\zeta_6=F_2$$

$$\begin{aligned}\left[-\omega^2(A_{42}-Mz_c)+i\omega B_{42}\right]\zeta_2+\left[-\omega^2(A_{44}+I_4)+i\omega B_{44}+C_{44}\right]\zeta_4&\\ +\left[-\omega^2(A_{46}-I_{46})+i\omega B_{46}\right]\zeta_6 &= F_4\end{aligned}$$

$$\left[-\omega^2 A_{62}+i\omega B_{62}\right]\zeta_2+\left[-\omega^2(A_{64}-I_{46})+i\omega B_{64}\right]\zeta_4+\left[-\omega^2(A_{66}+I_6)+i\omega B_{66}\right]\zeta_6=F_6$$

All the coefficients can be determined from the 2D sectional sway and roll added mass and damping, the 2D sectional potentials for sway and roll and the hydrostatic roll restoring force.

For all the five motions considered, the response at the resonant frequency is largely controlled by the wave generation damping (B coefficients) except for roll where the damping at resonance is dominated by the viscous damping. Therefore, for strip theory to give accurate results for roll, an estimate for the viscous damping coefficient must be added to B_{55}.

SIMULATIONS OF SHIP MOTIONS IN RANDOM SEAS

The complete problem includes effects of waves coming from all directions. Here, for simplicity and clarity we will consider long-crested random waves coming from one direction.

The "system functions", $\xi_j(\omega)$, are complex numbers, dependent on frequency. For each one, the magnitude is the ratio of the sinusoidal motion amplitude to the wave amplitude. The phase is the phase lead of the motion with respect to the wave elevation at the origin of the coordinate system.

Suppose we have a wave spectrum, $S_e(\omega)$. The wave elevation at the origen of the ship, or offshore structure, at the origin of the coordinate system, can be simulated as:

$$\zeta_w(t) = \sum_{n=-N}^{N} Z_n e^{i(n\delta\omega)t}$$

$$\text{where:} \quad Z_n = e^{i\alpha_n}\sqrt{\frac{1}{2}S_e(n\delta\omega)\delta\omega}$$

Within the restrictions of linear theory, each ship motion can be simulated in the specified random wave field as:

$$\eta_j(t) = \sum_{n=-N}^{N} \xi_j(n\delta\omega) Z_n e^{i(n\delta\omega)t}$$

ST4b

Added Resistance and Drift Forces

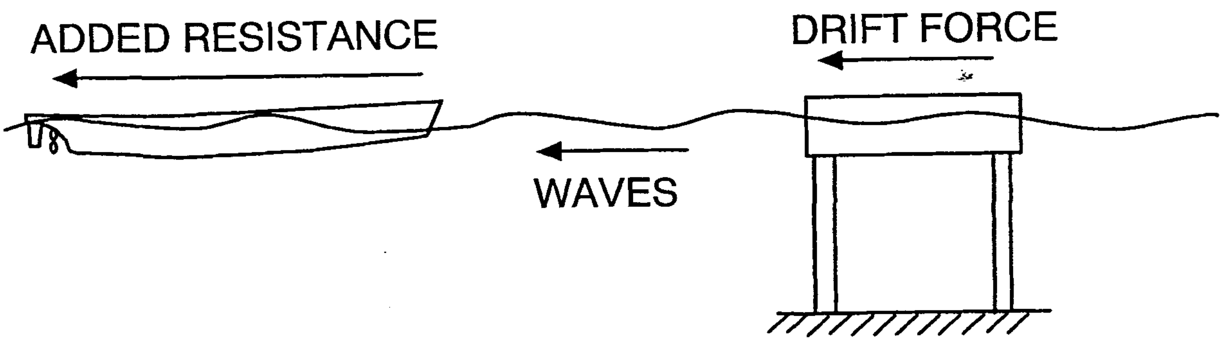
An important "second order" effect is the average force an oscillating wave field can imoise on an object. These forces are typically small in comparison of oscillating forces and spring-like resorting forces so the horizontal mean forces are most important. For a ship, this force is called the added resistance and for an offshore structure it is called the drift force.

One again, for simplicity and clarity we will consider waves propagating in a single direction. If a sinusoidal wave has amplitude A, the dominant mean force has the form $r_a(\omega)A^2$. $r_a(\omega)$ is called the added resistance operator. It is found by solving the second order hydrodynamic problem. However, Some first order effects contribute.

Consider the term in Euler's equation $(\vec{V} \cdot \nabla)\vec{V}$ When $\vec{V}$ is sinusoidal, this term will contribute zero frequency terms.

In the presence of a wave spectrum $S(\omega)$, the total added reesistance is:

$$R_{added} = \int_0^\infty 2r_a(\omega)S(\omega)d\omega$$



ST462

Gerritsma and Beukelman Theory for Added Resistance

The "exact" formulation for the added resistance operator, $r_a(\omega_e)$ requires solution of the complicated 3-D second order problem. However, Gerritsma and Beukelman [1] developed a semi-empirical formulation for $r_a(\omega_e)$ based on strip theory that is remarkably accurate. The basis of their theory was as follows:

1. Each section of the ship encounters a relative vertical velocity that depends on the wave, and the heave and pitch of the ship.
2. This relative vertical motion generates waves which carry energy away from the ship.
3. Equating this radiated energy, per unit time, to the added resistance times the ship speed provides a formula for the added resistance.

This formula is:

$$r_a(\omega_e) = \frac{k}{2\omega_e} \int_0^{\mathcal{L}} \left[N(x) - V \frac{dm(x)}{dx} \right] V_z^2(x) dx$$

where the ship extends from 0 to $\mathcal{L}$ and k = wavenumber of the wave,
$N(x)$ = heave damping coefficient per unit length of ship at position x,
V = forward speed of ship,
$m(x)$ = added mass per unit length of ship cross section at position x,
$V_z(x)$ = relative vertical water velocity amplitude at position x.

$$V_z = \dot{z} - x\dot{\theta} + V\theta - \zeta_a$$

where $\dot{z}$ is the heave velocity of the ship at $x = 0$, θ is the pitch angle of the ship, and ζ_a is the average of the velocity of the fluid motion in the wave over the width of the ship cross section at its local depth.

[1] Gerritsma, J., and Beukelman, W., "Analysis of the Increase in Resistance in Waves of a Fast Cargo Ship", Technical report 169 s, Netherlands Ship Research Centre TNO, April, 1972

Nonlinear Wave Force Calculations

Second order wave forces arise both from the second order potential, which increases the accuracy with which the nonlinear terms in the free surface boundary condition are met, and also from the influence of the first order solution on the nonlinear term in Euler's equation. For example, consider the term: $u\frac{\partial u}{\partial x}$. As an example, a field of two sinusoidal waves is :

$$u = A_1 \sin(k_1 x - \omega_1 t) + A_2 \sin(k_2 x - \omega_2 t)$$

$$\frac{\partial u}{\partial x} = A_1 k_1 \cos(k_1 x - \omega_1 t) + A_2 k_2 \cos(k_2 x - \omega_2 t)$$

At $x = 0$:

$$u = -A_1 \sin(\omega_1 t) - A_2 \sin(\omega_2 t)$$

$$\frac{\partial u}{\partial x} = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)$$

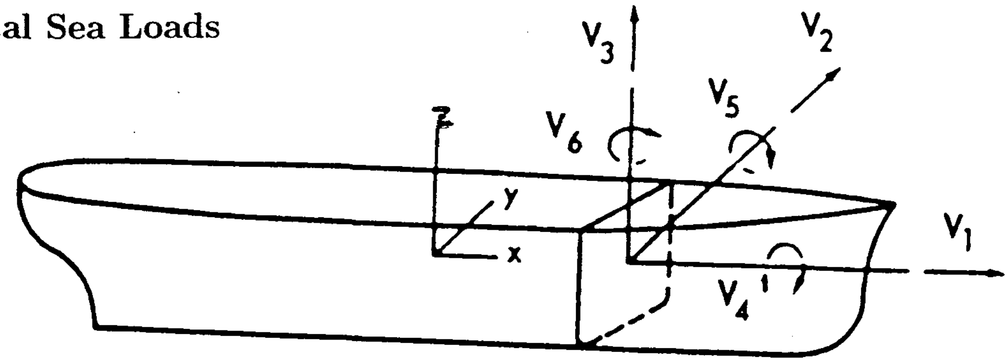
$$\begin{aligned} u\frac{\partial u}{\partial x} &= -A_1^2 k_1 \sin(\omega_1 t)\cos(\omega_1 t) - A_1 A_2 k_2 \sin(\omega_1 t)\cos(\omega_2 t) \\ &\quad -A_1 A_2 k_1 \sin(\omega_2 t)\cos(\omega_1 t) - A_2^2 k_2 \sin(\omega_2 t)\cos(\omega_2 t) \end{aligned}$$

Consider just one of the four terms:

$$\begin{aligned} \left(u\frac{\partial u}{\partial x}\right)_1 &= -A_1 A_2 k_2 \sin(\omega_1 t)\cos(\omega_2 t) \\ &= -\frac{A_1 A_2 k_2}{2}\sin(\omega_1 + \omega_2)t - \frac{A_1 A_2 k_2}{2}\sin(\omega_1 - \omega_2)t \end{aligned}$$

If ω_1 and ω_2 are only slightly different, $\omega_1 + \omega_2$ is a comparatively high frequency and $\omega_1 - \omega_2$ is a very low frequency.

Vertical Sea Loads



V_1 = compression force
V_2 = horizontal shear force
V_3 = vertical shear force
V_4 = torsional moment
V_5 = vertical bending moment
V_6 = horizontal bending moment

The complete strip theory sea loads are considered in the reference "Ship Motions and Sea Loads", by Salvesen, Tuck and Faltinsen. Here we consider the vertical loads which lead to the shear force, V_3' and the longitudinal vertical bending moment, V_5'. Sinusoidal forces and motions are considered. For example,

$V_j' = V_j e^{i\omega t}$ where the real part of all complex expressions is implied

$$\text{Likewise, } \eta_j = \zeta_j e^{i\omega t}$$

The fluid forces are separated into hydrostatic forces R_j, sea wave exciting forces E_j, and hydrodynamic forces resulting from unsteady ship motions D_j. I_j is the inertial component of the j^{th} structural force due to motions of the ship. Then, the structural loads can be expressed symbolically as:

$$V_j = I_j - R_j - E_j - D_j$$

We are concerned with terms having subscripts 3 and 5. All the longitudinal integrals in the following are over the portion of the ship forward of the section under consideration. We denote these integrals as:

$$\int_{L_f} ....d\xi$$

$$I_3 = \int_{L_f} -\omega^2 m(\xi)\ [\zeta_3 - \xi\zeta_5]\ d\xi$$

5T56

$$I_5 = -\int_{L_f} -\omega^2 m(\xi)\,[\xi - x]\,[\zeta_3 - \xi\zeta_5]\,d\xi$$

$$R_3 = -\rho g \int_{L_f} b(\xi)\,[\zeta_3 - \xi\zeta_5]\,d\xi$$

$$R_5 = \rho g \int_{L_f} b(\xi)\,[\xi - x]\,[\zeta_3 - \xi\zeta_5]\,d\xi$$

$$E_3 = \rho\alpha \left\{ \int_{L_f} (f_3 + h_3) d\xi + \left(\frac{U}{i\omega} h_3 \right)_{\xi = x} \right\}$$

$$E_5 = \rho\alpha \int_{L_f} \left\{ (\xi - x)(f_3 + h_3) + \frac{U}{i\omega} h_3 \right\} d\xi$$

$$\begin{aligned} D_3 \;\; = \;\; & -\int_{L_f} \left\{ \omega^2 a_{33}(\zeta_3 - \xi\zeta_5) + i\omega b_{33}(\zeta_3 - \xi\zeta_5) + U b_{33}\zeta_5 + i\omega U a_{33}\zeta_5 \right\} d\xi \, - \\ & - \left\{ i\omega a_{33} U(\zeta_3 - \xi\zeta_5) + U b_{33}(\zeta_3 - \xi\zeta_5) + U^2 a_{33}\zeta_5 - \frac{iU^2}{\omega} b_{33}\zeta_5 \right\}_{\xi = x} \end{aligned}$$

$$\begin{aligned} D_5 \;\; = \;\; & \int_{L_f} a_{33}(\xi - x) \left\{ -\omega^2(\zeta_3 - \xi\zeta_5) + i\omega b_{33}(\zeta_3 - \xi\zeta_5\,) \right\} d\xi \, + \\ & \int_{L_f} \left\{ i\omega U a_{33}(\zeta_3 - x\zeta_5\,) + U b_{33}(\zeta_3 - x\zeta_5\,) + U^2 a_{33}\zeta_5 - \frac{iU^2}{\omega} b_{33}\zeta_5 \right\} d\xi \end{aligned}$$

APPENDIX

Further Material on Panel Methods and Strip Theory

Panel Methods

13.024 Numerical Marine Hydrodynamics

Alexis Mantzaris

During the last decade, advancements in computer technology have made possible the development of new classes of three-dimensional numerical tools for analyzing problems in Naval Architecture, such as ship wave resistance and motions.

Early attempts to model ships in potential flow focused on variations of slender body and strip theory to study simplified body geometries and free surface conditions.

As computing power increased, so did the development of three-dimensional methods. Of these, considerable attention has been received by boundary element or panel methods.

Panel Methods at a Glance

- Distribute sources and dipoles on body
- Discretize
- Green's Theorem gives system of equations for singularity strength on each panel in terms of boundary conditions
- Forces on body found from flow solution

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Panel methods attempt to solve the Laplace equation in the fluid domain by distributing sources and dipoles on the body and, in some methods, on the free surface.

These surfaces are divided into panels, each one associated with a source and dipole distribution of unknown strength.

Green's theorem relates the source and dipole distribution strength to the potential and normal velocity on each panel.

The boundary conditions to be applied to the problem are often linearized and they determine either the potential or the normal velocity on each panel.

Having solved for the unknown source and dipole strengths, Green's theorem may be used to find the potential at any point in the fluid domain.

Hydrodynamic forces are found from pressure integration and are used with Newton's Law to determine motions.

Cases to be Examined

- Unbounded fluid flows
  - Steady motion through unsteady flow field
- Lifting flows
  - Forced motion in free stream
- Wave flows
  - Ship under steady motion in calm water
  - Free motions of buoy in waves

Panel methods can ultimately solve complex problems involving free motions of forward moving vessels with lifting surfaces in incident waves. Jumping right in to the formulation of such a problem, however, would be rather overwhelming.

We will, therefore, start by formulating a simple problem and move on to progressively more complex cases.

Flow in Unbounded Fluid

Non-lifting body in unsteady flow

Unsteady incident

Total potential: $$\Phi = \varphi + \phi - Ux$$

Perturbation

Free stream

Body advancing with speed U in an unsteady flow field

Body-fixed coordinate system.

Unit normal n, to body surface S_B, pointing out of fluid.

Reasons to solve this problem:

- Get pressure distribution on body
- Determine added mass
- Introduce techniques to be used with more complex problems

Separate total potential into:

- free stream potential
- incident potential excluding free stream
- perturbation from incident potential

Body Boundary Condition

(No flux condition)

$$\frac{\partial \Phi}{\partial n} = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial \varphi}{\partial n} = \left(U\hat{i} - \nabla \phi\right) \cdot \vec{n}$$

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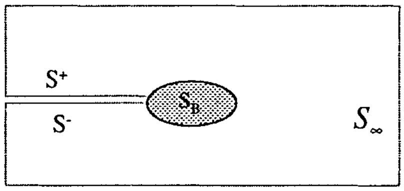
The normal velocity of fluid must be zero in the body-fixed coordinate system.

Using the decomposition of total potential into its components, it follows that the normal velocity of the perturbation flow must be equal and opposite to the the incident flow, which is given.

Boundary Integral Equation

Green's Theorem for field points on body surface, S_B



$$\varphi = \frac{1}{2\pi}\iint_{S_B}\left[G\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial G}{\partial n}\right]dS$$

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The application of Green's second identity transforms the boundary value problem stated previously into a boundary integral equation. This facilitates the numerical solution as the entire fluid volume does not have to be discretized.

The integral over the part of the control surface at infinity vanishes because the flow sufficiently far from the body is undisturbed. The integrals over the connecting surfaces S^+ and S^- cancel each other out. So what is left defines the potential on the body in terms of a source (G) and dipole (dG/dn) distribution on the body surface. The strength of the source distribution is given by the magnitude of the normal velocity on the body, while the strength of the dipole distribution is equal to the magnitude of the potential on the body.

For this particular problem, $d\varphi/dn$ is given from the boundary condition, while φ is unknown.

Numerical Solution

Discretize integral equation and substitute body BC

$$2\pi\varphi_i + \sum_j \varphi_j \frac{\partial G_{ij}}{\partial n_j} = \sum_j G_{ij}\left(Un_{jx} - \frac{\partial \phi_j}{\partial n_j} \right)$$

System of linear equations for unknown φ_i

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The body is discretized into *n* panels, each of area A_j.

The singularity distribution on each panel can be constant, or of higher order. In any case, G_{ij} is the potential at the control point of panel *i*, due the source distribution on panel *j*.

Having a higher order distribution on each panel results in less panels needed for convergence and leads to a more robust way of calculating the tangential velocities on the body, if needed. More on higher order distributions later.

In the above system of *n* linear equations, the RHS is known from the body boundary condition. The potential on each panel may thus be found by a standard linear solver. The flow is hence completely specified.

Hydrodynamic Forces and Moments

$$\vec{F} = -\rho \sum_i \left(\frac{\partial \Phi_i}{\partial t} + \frac{1}{2} \nabla \Phi_i \cdot \nabla \Phi_i \right) \vec{n}_i A_i$$

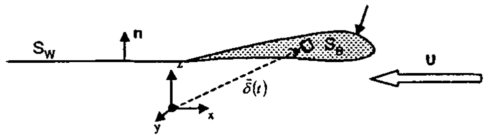
$$\vec{M} = -\rho \sum_i \left(\frac{\partial \Phi_i}{\partial t} + \frac{1}{2} \nabla \Phi_i \cdot \nabla \Phi_i \right) (\vec{x}_i \times \vec{n}_i) A_i$$

From the potential and its gradient on the body, it is straightforward to determine the pressure distribution and hence the hydrodynamic forces and moments.

These forces are often linearized by assuming small perturbations about the free stream.

Lifting Flows

Forced motions in steady free stream



Lifting surfaces common in Naval Architecture include hydrofoils, rudders, control fins, sailboat keels and sails, and catamaran hulls.

A special treatment is needed for such lift-producing bodies because the potential flow solution to the problem as previously formulated would include infinite velocities at the sharp trailing edge of the foil under angle of attack.

In order to ensure a smooth flow at the trailing edge, which in real life is attained due to the presence of viscosity, the wake shed from the hydrofoil must be modeled.

The problem examined here involves a hydrofoil performing small motions about a steady forward motion. So in addition to the effect of lift, there is a new element in the formulation of the boundary value problem. The body now moves with respect to the coordinate system, which is translating with a steady velocity, U.

Wake Model

Thin free vortex sheet

Across wake:

- Continuous normal velocity
- Discontinuous potential (jump=$\Delta\Phi$)
- Zero pressure jump:

$$\frac{\partial \Delta\varphi}{\partial t} - U\frac{\partial \Delta\varphi}{\partial x} = 0$$

(Bernoulli, linearized about free stream)

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The wake is modeled as a free vortex sheet shed downstream from the trailing edge of the foil. Mathematically this may be defined as a dipole distribution on a surface S_W, of zero thickness.

The operator Δ denotes a jump in a quantity from one side of the wake to the other.

Across the wake we have continuity of normal velocity and a jump in potential. The pressure on both sides of the wake should be equal because otherwise we would have infinite particle acceleration since the wake is infinitesimally thin.

The wake can be shed either straight back, following the free stream, or it could have each point follow the total velocity induced at its location by both the foil and the rest of the wake. In general, however, the additional computational load and stability problems do not justify the slight increase in accuracy achieved by tracking the exact position of the wake.

Kutta Condition

- Requires zero pressure jump at trailing edge of foil.
- This is ensured by continuity of potential, together with the condition of zero pressure jump across the wake

$$\Phi_{TE:body} = \Phi_{TE:wake}$$

The flow past a lifting body cannot be uniquely determined unless some additional condition is specified which sets the amount of circulation produced by the foil. As previously formulated, (without the wake) there would be no circulation and the velocity at the sharp trailing edge of the foil would be infinite.

This situation may be avoided if a Kutta condition requiring tangential velocities at the trailing edge is enforced. An alternate way of enforcing this condition is to require continuous pressure at the trailing edge. In fact, this condition is preferred here because it can be easily linearized about the free stream.

The requirement of zero pressure jump at the trailing edge in the wake is already satisfied as we saw before. Thus, by also requiring continuity of potential from the body into the wake at the trailing edge, the Kutta condition of zero pressure jump on the body is automatically satisfied.

Forced Motions

Displacement about frame of reference due to translation and rotation:

$$\vec{\delta}(\vec{x},t)=\vec{\xi}_T(t)+\vec{\xi}_R(t)\times\vec{x}$$

$$\vec{\xi}_T=(\xi_1,\xi_2,\xi_3) \qquad \vec{\xi}_R=(\xi_4,\xi_5,\xi_6)$$

Since for this problem the body is not fixed with respect to the coordinate system, we need to define its motions.

The rigid body motions in six degree of freedom can be fully described by a translation and a rotation vector.

The displacement of any point on the body with respect to its original position may be described in terms of these two vectors and its original displacement from the origin.

Body Boundary Conditions

Applied at exact body surface:

$$\frac{\partial \varphi}{\partial n} = \left(U\hat{i} + \frac{\partial \vec{\delta}}{\partial t} \right) \cdot \vec{n}$$

Linearized and applied at mean position of body (assuming small body motions)

$$\frac{\partial \varphi}{\partial n} = \frac{d\vec{\xi}_T}{dt} \cdot \vec{n} + \frac{d\vec{\xi}_R}{dt} \cdot (\vec{x} \times \vec{n}) + U(\xi_5 n_z - \xi_6 n_y + n_x)$$

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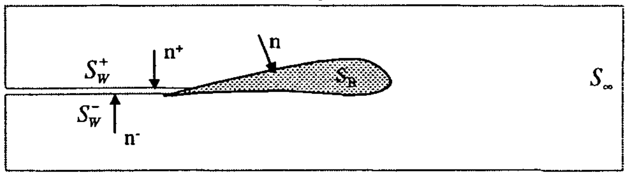
The boundary condition is defined in terms of the body motions in order to preserve the zero normal flux requirement.

This condition should, strictly, be applied to the exact position of the body surface. This would require the re-discretization of the body surface at each time step. Although some panel methods take this approach to the problem, the solution becomes much easier numerically if the motions can be assumed small and can be linearized about the mean position of the body.

The linear body boundary condition is derived by applying a Taylor expansion about the mean body position and retaining linear terms.

Boundary Integral Equation

Green's Theorem for field points on the body



$$2\pi\varphi + \iint_{S_B} \varphi \frac{\partial G}{\partial n} dS + \iint_{S_W} \Delta\varphi \frac{\partial G}{\partial n} dS = \iint_{S_B} G \frac{\partial \varphi}{\partial n} dS$$

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Green's theorem is used once more to derive the integral equations.

As with the non-lifting case, the integral over the control surface at infinity vanishes. The integral over the connecting surfaces that run over the wake do not completely cancel each other, however, due to the discontinuity in potential across the wake. Instead, we get a term involving an integral over S_W of the potential jump multiplied by the dipole potential.

The problem can no longer be solved by placing panels only on the body. The wake also needs to be discretized.

Numerical Solution

Discretize and Fourier Transform integral equation, using body BC

$$2\pi\varphi_k + \sum_{j=1}^{N_{body}} \varphi_j \frac{\partial G_{kj}}{\partial n_j} + \sum_{j=N_{body}+1}^{N_{wake}} \Delta\varphi_j \frac{\partial G_{kj}}{\partial n_j} =$$

$$= \sum_{j=1}^{N_{body}} G_{kj} \left\{ i\omega\left[\vec{\xi}_T \cdot \vec{n}_j + \vec{\xi}_R \cdot \left(\vec{x}_j \times \vec{n}_j\right)\right] + U\left(\xi_5 n_{zj} - \xi_6 n_{yj}\right)\right\}$$

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Due to the wake shed downstream, this problem has memory and thus the solution depends on the flow at previous time instances. This means that the solution has to be evolved in time, or needs to be solved by tanking the Fourier transform and solving for each frequency component present.

The integral equation shown above is in the frequency domain, if the forced motions are sinusoidal. Solution in the time domain would require the numerical evaluation of the time derivatives.

The system of equations shown above are simply the integral equation at each panel, with the body boundary condition substituted at the RHS.

There are, however, more unknowns than integral equations due to the extra panels of unknown potential jump in the wake. The extra equations to close the problem are derived from the wake condition of zero pressure jump.

Numerical Solution

Wake Condition:

To be solved simultaneously with integral equation

at T.E.... $\Delta\varphi_{wake} = \left[\varphi_{upper} - \varphi_{lower}\right]_{body}$

in wake... $i\omega\,\Delta\varphi_k - U\dfrac{\partial\Delta\varphi_k}{\partial x} = 0$

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For a solution in the frequency domain, the wake condition is discretized and yields one extra equation for each panel in the wake.

If the problem is solved in the time domain, the potential jump can be expressed exclusively in terms of the potential jump at panels during the previous time step. Instead of unknown potential jump on the entire set of wake panels, the only extra unknowns would thus be a strip of panels immediately downstream of the trailing edge of the foil. This reduces the size of the matrix to be solved, at the cost of having to evolve the solution in time.

The wake condition involves the evaluation of spatial derivatives of the potential jump. We will examine methods for doing this later.

Hydrodynamic Forces and Moments

$$\vec{F} = -\rho \sum_{i=1}^{N_{body}} \left(i\omega\varphi_i - U\frac{\partial\varphi_i}{\partial x} + \frac{1}{2}\nabla\varphi_i \cdot \nabla\varphi_i \right) \vec{n}_i A_i$$

$$\vec{M} = -\rho \sum_{i=1}^{N_{body}} \left(i\omega\varphi_i - U\frac{\partial\varphi_i}{\partial x} + \frac{1}{2}\nabla\varphi_i \cdot \nabla\varphi_i^{\bullet} \right) (\vec{x}_i \times \vec{n}_i) A_i$$

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After having solved the flow, the calculation of the hydrodynamic forces and moments is again a matter of integrating the pressure distribution over the surface of the body.

For steady flows it is also possible to determine the lift and drag based on a Trefftz plane integration:

$$D = \frac{1}{2}\rho \int_{-s/2}^{s/2} \Delta\Phi \frac{\partial\Phi}{\partial z} dy$$

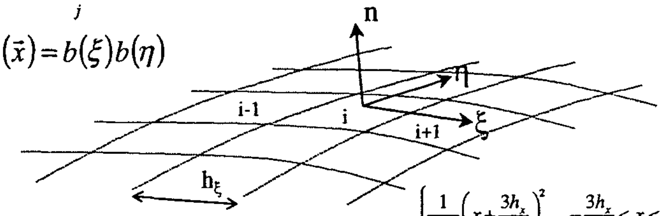
$$L = \rho U \int_{-s/2}^{s/2} \Delta\Phi \, dy$$

where s is the span of the foil.

Higher Order Potential Distribution

$$\varphi(\vec{x}) = \sum_j c_j \, B_j(\vec{x})$$

$$B(\vec{x}) = b(\xi)\, b(\eta)$$



Bi-quadratic B-Spline: $$b(x)=\begin{cases} \dfrac{1}{2h_x^2}\left(x+\dfrac{3h_x}{2}\right)^2, & -\dfrac{3h_x}{2}<x<-\dfrac{h_x}{2} \\ \dfrac{1}{h_x^2}\left(-x^2+\dfrac{3h_x^2}{4}\right) & -\dfrac{h_x}{2}<x<\dfrac{h_x}{2} \\ \dfrac{1}{2h_x^2}\left(-x+\dfrac{3h_x}{2}\right)^2, & \dfrac{h_x}{2}<x<\dfrac{3h_x}{2} \end{cases}$$

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In general, the potential distribution on each panel is not constant.

A B-spline representation, for example, represents the potential as a summation with weight c_j of all basis functions B_j centered on each panel *j*. A second order basis function is shown above. Note that the field point needs to be converted into local panel coordinates for the evaluation of the basis functions.

The spline coefficients c_j, determine the amount of contribution from each panel, and become the unknowns in the integral equations. Due to the overlap of the basis functions in determining the potential at the center of several panels, however, the unknown spline coefficients are still one per panel.

A consequence of higher order B-spline distributions is that end conditions need to be specified at the edges of the spline sheets, so that the spline coefficients may be uniquely determined.

Higher order singularity distributions require fewer panels to achieve numerical flow convergence. Note that similarly, geometrical convergence may be achieved faster if the surface is described not in terms of flat quadrilateral panels, but by B-spline surfaces.

Evaluation of tangential derivatives

constant distribution: $$\frac{\partial \varphi_i}{\partial \xi} = \frac{\varphi_{i+1} - \varphi_{i-1}}{h_{i\xi}}$$

higher order: $$\frac{\partial \varphi}{\partial \xi} = \sum_j c_j \frac{\partial b_j(\xi)}{\partial \xi} b_j(\eta)$$

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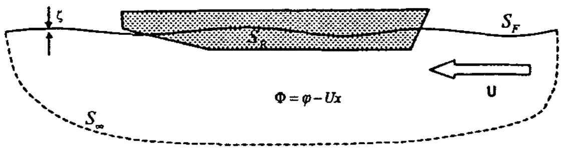
As already seen, there is a need for the calculation of the tangential velocity on a panel. This may be done by finite differences, and one example is shown above. Of course, any other finite differences scheme could be used, provided that it does not make the overall method unstable.

If the potential distribution is of higher order then the tangential velocities can be found analytically from direct differentiation of the basis functions.

Note that the above derivatives are given in panel local coordinates. Since the derivatives are usually required with respect to the global coordinate system, a transformation is needed for each panel.

Free Surface Flows

Steady ship motion in calm water



Total potential: $\Phi = \varphi - Ux$

Perturbation — Free stream

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The evaluation of the steady wave resistance of a ship has always been of great importance to Naval Architects. Three dimensional panel methods have the ability to estimate this quantity without resorting to expensive towing tank testing.

We will formulate the problem of a ship advancing steadily through calm water, linearizing the solution about the free stream. The total flow is therefore broken down into a free stream and a small perturbation flow components.

Free Surface Boundary Conditions

Dynamic:

zero total pressure on free surface

$$p_a = -\rho\left(\frac{\partial\varphi}{\partial t} - U\frac{\partial\varphi}{\partial x} + \frac{1}{2}\nabla\varphi\cdot\nabla\varphi + g\zeta\right) = 0$$

Kinematic:

particle on free surface remains there

$$\left(\frac{\partial}{\partial t} + \nabla\Phi\cdot\nabla\right)[z-\zeta] = 0$$

ζ = wave elevation

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Boundary conditions are required to determine the behavior of the flow near the free surface, and hence uniquely determine the solution.

The dynamic condition requires the pressure at the free surface to be equal to the atmospheric pressure, which will be taken arbitrarily to be equal to zero. The condition is thus expressed by the Bernoulli equation above.

The kinematic condition requires that a particle on the free surface remains on the free surface forever. This means that the material derivative of its vertical distance from the free surface should be zero.

Linearization about Free Stream

Kelvin boundary conditions

$$\frac{\partial \phi}{\partial t} - U \frac{\partial \phi}{\partial x} = -g\zeta$$

$$\frac{\partial \zeta}{\partial t} - U \frac{\partial \zeta}{\partial x} = \frac{\partial \phi}{\partial z}$$

The free surface boundary conditions previously stated are non-linear and are to be applied at the exact position of the free surface, which is unknown. The numerical solution algorithm becomes much simpler and computationally efficient if these conditions can be linearized and applied to a known surface.

The above linear conditions, also known as the Kelvin free surface boundary conditions, were derived using a Taylor expansion about z=0 for small wave elevations and slopes, and ignoring higher order terms.

The wave elevation can, of course, be eliminated by combining the two equations, resulting in a condition involving only the perturbation potential and its temporal and spatial derivatives.

Boundary Integral Equation

$$2\pi\varphi = \iint_{S_B+S_F}\left[G\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial G}{\partial n}\right]dS$$

Take Kelvin wave source as the Green function
As shown in hydrodynamics review of this course,
waterline integral replaces integral over free surface:

$$2\pi\varphi = \iint_{S_B}\left[G\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial G}{\partial n}\right]dS+\frac{U^2}{g}\oint_{wl}\left[G\frac{\partial\varphi}{\partial x}-\varphi\frac{\partial G}{\partial x}\right]\frac{n_x}{\cos\gamma}dl$$

γ = flare angle

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Green's theorem is used again to derive the integral equation, only this time an integral over the free surface is needed, as well as over the body. Using a Green function that satisfies the linear free surface conditions, however, the free surface integral may be collapsed into a waterline integral. This means that no panels are needed on the free surface.

One difficulty is that the first derivative of the very complicated Green function is required, but this can be done numerically.

Note that this formulation of the integral equation relies on the use of the linearized Kelvin boundary conditions. This is because Green functions satisfying any other free surface linearization are not readily obtainable.

Numerical Solution

Discretize Integral Equation and Substitute Body BC

$$2\pi\varphi_k + \sum_j \varphi_j \frac{\partial G_{kj}}{\partial n_j} + \frac{U^2}{g}\sum_{j\in WL} \varphi_j \frac{\partial G_{kj}}{\partial x}\frac{n_j}{h_j \cos\gamma_j} =$$

$$= U\sum_j G_{kj} n_{jx} + \frac{U^3}{g}\sum_{j\in WL} G_{kj}\frac{n_{jx}^2}{h_j \cos\gamma_j}$$

Linear system of equations for φ_k

Discretizing the integral equation, a system of linear equations is obtained for the potential on each panel, as before.

Care must be taken in evaluating the waterline integral, since the value of the potential on the free surface needs to be estimated from the potential on the body, which is often discretized only below the z=0 plane.

Wave Resistance

From Momentum Conservation

$$R_W = \iint_{S_B} p\, n_x dS - \frac{\rho g}{2} \oint_{WL} \zeta^2 \frac{n_x}{\cos\gamma} dl$$

(Proof follows)

The wave resistance of a ship may be found from the above formula after the flow has been solved.

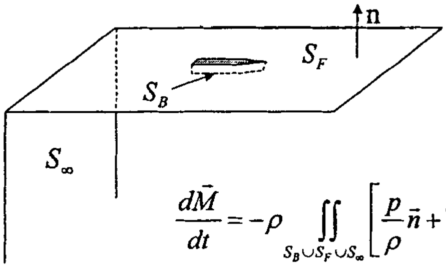
The pressure should *include* the quadratic term in Bernoulli's equation, even though terms of comparable magnitude have been omitted in the linearization of the free surface boundary conditions.

If the quadratic terms are omitted, the resistance of full-shaped vessels is over-predicted. The reason for this is that close to such bluff bodies, which is where we are performing the pressure integration, the perturbation potential is actually of the same order as the free stream potential, so the linearization is not accurate.

Linearizing about a double-body basis flow, as we will see later, solves this problem and the quadratic terms are not as important.

Wave Resistance

Conservation of Momentum



$$\frac{d\bar{M}}{dt} = -\rho \iint_{S_B \cup S_F \cup S_\infty} \left[\frac{p}{\rho} \vec{n} + \nabla\Phi \left(\nabla\Phi \cdot \vec{n} \right) \right] dS = 0$$

The proof of the formula given for the wave resistance follows from the application of the momentum conservation principle inside an appropriately chosen control volume of fluid.

Within the enclosed volume, the rate of change of fluid momentum vanishes.

Note that because of the radiation condition, the only surface at infinity where the integrand does not vanish is far downstream of the body, at S_∞.

Control Volume at Exact Position of Fluid Surfaces

Using boundary conditions:

$$R_W = \iint_{S_B} p\, n_x \, dS = -\rho \iint_{S_\infty} \left[\frac{p}{\rho} n_x + \frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial n} \right] dS$$

From radiation condition and Bernoulli:

$$R_W = -\frac{\rho g}{2} \int_{C_d} \zeta^2 dy - \frac{\rho}{2} \iint_{S_d} \left[-\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \right] dS$$

C_d is the intersection of S_∞ with z=0
S_d is the part of S_∞ lying below z=0

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Taking the exact wetted surface of the hull, the exact position of the free surface, and S_∞, as the control surfaces (all at rest with respect to the body) and using the body and free surface boundary conditions, the only terms that do not vanish are the pressure integration over the body wetted surface (which is defined as the non-linear wave resistance), and the momentum flux and pressure integration at infinity.

The fluid velocity in the x and z directions may be found from the Kelvin free surface boundary conditions, and the fluid pressure from Bernoulli.

The resulting expression is an exact representation of the wave resistance in terms of far-field quantities.

Control volume at linearized position of fluid surfaces

Using Kelvin boundary conditions and Bernoulli:

$$\iint_{\bar{S}_B} p\, n_x\, dS - \rho g \iint_{\bar{S}_F} \zeta \frac{\partial \zeta}{\partial x} dS + \frac{\rho}{2} \iint_{S_d} \left[-\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \right] dS = 0$$

After application of Stokes theorem:

$$\iint_{\bar{S}_B} p\, n_x\, dS - \frac{\rho g}{2} \oint_{WL} \zeta^2 \frac{n_x}{\cos\gamma} dl = -\frac{\rho g}{2} \int_{C_d} \zeta^2 dy - \frac{\rho}{2} \iint_{S_d} \left[-\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \right] dS$$

RHS is equal to wave resistance, R_W, from momentum conservation in control volume bound by exact surfaces

Repeating the same procedure for a control volume bound by the linearized free surface (z=0 plane), the body below z=0, and the same surface at infinity, a similar expression is derived. This time, the momentum flux across the free surface does not vanish because the normal fluid velocity at z=0 is not zero. The Kelvin conditions are used to express the fluid velocities on the free surface in terms of the wave elevation.

Finally, an application of Stokes theorem transforms the surface integral over the z=0 plane to a pair of line integrals at the body and at infinity.

The line integral along with the surface integral at infinity are recognized as the wave resistance as previously derived using a control volume bound by the exact free surface.

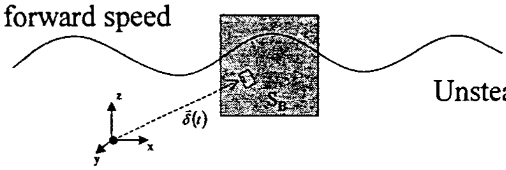
An expression is therefore derived for wave resistance in terms of near-field quantities, starting from the principle of momentum conservation. Comparing this expression to the one derived from pressure integration, we observe that they are similar, but the waterline integral terms have the opposite sign!!

This paradox is due to the inconsistency of retaining second order terms in the definition of wave resistance, but omitting them from the free surface linearization, as previously mentioned. As the beam of the ship approaches zero, the waterline integral term vanishes and the two definitions are in agreement.

Free Surface Flows

Free Motions of Body in Waves

Zero forward speed



Unsteady incident

Total potential: $\Phi = \varphi + \phi$

Perturbation

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The determination of the motions of floating bodies in waves is another problem of interest to ocean engineers. Here we will examine the panel method solution of a buoy in incident monochromatic waves. Solutions to more complex problems with forward speed and multiple frequencies can be easily obtained by a simple extension of this problem and the previous one examined.

Since there is no forward speed in this problem, the total potential is divided into the incident wave and perturbation potentials.

Solution Method

System of Equations:

- Boundary Integral Equation
- Body Boundary Condition
- Equations of Motion

Unknowns:

- Potential on each panel
- Normal Velocity on each panel
- Body motions

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So far the motions of the body have been prescribed, which resulted in the body boundary conditions being completely specified. For freely floating bodies in waves, however, the body boundary condition is a function of the motions, which are unknown. The motions are connected to the hydrodynamic forces through the equations of motion to close the problem.

System of Equations

Boundary integral equation

$$2\pi\varphi = \iint_{S_B}\left[G\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial G}{\partial n}\right]dS + \frac{U^2}{g}\oint_{wl}\left[G\frac{\partial\varphi}{\partial x} - \varphi\frac{\partial G}{\partial x}\right]\frac{n_x}{\cos\gamma}dl$$

Body boundary condition

$$\frac{\partial\varphi}{\partial n} = \left(\frac{\partial\vec{\delta}}{\partial t} - \nabla\phi\right)\cdot\vec{n}$$

As before, the body boundary condition may be substituted into the integral equation. This time, however, the body motions are unknown, so the integral needs to be solved simultaneously with the equations of motion.

Equations of Motion

$$M\ddot{\vec{\xi}} + C\vec{\xi} = -\rho \iint_{S_B} \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla\Phi \cdot \nabla\Phi \right] \vec{n}\, dS$$

$$\vec{\delta} = \vec{\xi}_T + \vec{\xi}_R \times \vec{x}$$
$$\vec{\xi}_T = (\xi_1, \xi_2, \xi_3)$$
$$\vec{\xi}_R = (\xi_4, \xi_5, \xi_6)$$
$$\vec{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$$

$$(n_1, n_2, n_3) = \vec{n}$$
$$(n_4, n_5, n_6) = \vec{x} \times \vec{n}$$
$$M = inertia\ matrix$$
$$C = matrix\ of\ restoring\ coefs$$

The equations of motion balance the inertia forces and the hydrostatic restoring forces with the hydrodynamic forces obtained from the flow solution.

A notation is adopted that merges the translation and rotation vectors so that the equations of motion become a six-dimensional matrix equation, balancing both forces and moments.

Rankine Panel Methods

- Panels both on body and free surface
- Boundary integral equation becomes:

$$2\pi\varphi = \iint_{S_B+S_F}\left[G\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial G}{\partial n}\right]dS$$

- Use Rankine source ($G=1/4\pi r$) as elementary singularity
- Boundary conditions determine potential and normal velocity on free surface
- Linearize about basis flow (double body)

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Rankine panel methods distribute panels on both the body and the free surface. They thus have a greater freedom in the free surface boundary conditions that they can apply. This comes at the expense of introducing extra errors due to the discretization of the free surface.

The integral equation retains the free surface term (without collapsing it into a waterline integral as with Neumann-Kelvin methods) and thus has extra unknown source and dipole distributions associated with the free surface panels. These are found from the dynamic and kinematic free surface boundary conditions.

Another advantage of Rankine panel methods is that they do not have to have their solution linearized about the free stream, which is rather poor especially near the ends of the vessel. Instead, they can linearize the solution about a double-body basis flow, which produces more accurate results.

Discretization Issues

- Distortion of the free surface
  - Dispersion
  - Damping
- Stability
  - Spatial
  - Temporal
- Radiation condition
  - Truncation errors and domain size sensitivity

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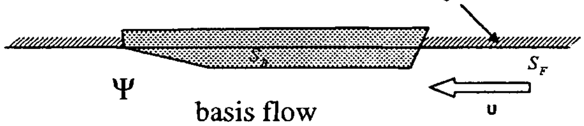
A discrete free surface has a different dispersion relation than a continuous one.. For finite panel sizes spurious wavelengths smaller than five panel lengths are supported and need to be filtered out. Any damping of the numerical method (i.e. Rayleigh viscosity), so that the radiation condition may be satisfied, also affects the numerical dispersion relation.

For a convergent numerical algorithm the numerical dispersion relation should approach the continuous dispersion relation in the limit of infinitesimally small panel sizes. The numerical dispersion relation results in stability criteria, from which required relations between quantities such as panel dimensions, Froude number, time step, can be derived.

Another difference between the continuous and numerical free surfaces is the truncation of the free surface. The condition at the edge of the computational domain should be such that the sensitivity of the solution to the size of the domain is minimized. One way of imposing the radiation condition so that reflected waves from the edge of the domain are minimized is to apply matching at some control volume around the fluid domain which contains a flow satisfying the radiation condition. An alternate (easier) way is to use a numerical beach where the kinematic boundary condition is modified to allow a mass flux through the free surface (Newtonian cooling), thus damping wavelengths less than about twice the extent of the beach.

Linearization about basis flow

$$\frac{\partial \Psi}{\partial n} = \frac{\partial \Psi}{\partial z} = 0 \quad \text{on } S_F$$



Ψ

basis flow

Total potential: $\Phi = \Psi + \varphi - Ux$

wave flow << Ψ

free stream

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The Neumann-Kelvin linearization assumes that the perturbation potential is small compared to the free stream. This assumption is not very good, especially near the bow and stern of a ship where the perturbation velocity is equal and opposite to that of the free stream.

A better linearization for ships with forward motion is to divide the total potential into the free stream, perturbation, and basis flow potentials. The basis flow is usually taken to be the solution past the hull with the free surface treated as rigid walls. Since this problem can be solved by taking a mirror image of the hull below the waterline, this basis flow is also known as the "double-body" flow.

Basis Flow Solution

After discretization, solve basis flow as shown for bodies in unbounded fluid.

Body Boundary Condition:

$$\frac{\partial \Psi}{\partial n} = U\, n_x$$

The double-body basis is a special case of a problem we have already seen. A stationary non-lifting body in an unbounded free stream is simpler than all the cases that we have examined thus far.

The solution is obtained after the body is discretized and before proceeding to the wave flow. Note that the panels on the free surface are not needed for the solution of the basis flow.

Wave Flow Body Boundary Conditions

$$\frac{\partial \varphi}{\partial n} = \frac{\partial \vec{\delta}}{\partial t} \cdot \vec{n}$$

linearized about basis flow, for small motions:

$$\frac{\partial \varphi}{\partial n} = \sum_{j=1}^{6} \left(\frac{d\xi_j}{dt} n_j + \xi_j m_j \right)$$

$$(n_1, n_2, n_3) = \vec{n} \qquad (m_1, m_2, m_3) = (\vec{n} \cdot \nabla)(\hat{i}U - \nabla\Psi)$$

$$(n_4, n_5, n_6) = \vec{x} \times \vec{n} \qquad (m_4, m_5, m_6) = (\vec{n} \cdot \nabla)[\vec{x} \times (\hat{i}U - \nabla\Psi)]$$

Since the forcing due to the free stream is accounted for in the basis flow, the body boundary condition for the wave flow component of the solution includes only the normal velocity due to the body unsteady motions.

Taking a Taylor expansion about the mean body position, and ignoring higher order terms, a linear body boundary condition is derived. As before, the combined translation/rotation vector is used to describe the body motions.

The m-terms provide a coupling between the basis flow and the unsteady wave solution, and their evaluation is important, especially near the ends of the ship.

Wave Flow Free Surface Boundary Conditions

linearized about Ψ and applied at z=0

Dynamic

$$\frac{\partial \varphi}{\partial t} - U\frac{\partial \varphi}{\partial x} + \nabla\Psi\cdot\nabla\varphi = -g\zeta + U\frac{\partial \Psi}{\partial x} - \frac{1}{2}\nabla\Psi\cdot\nabla\Psi$$

Kinematic

$$\frac{\partial \zeta}{\partial t} - U\frac{\partial \zeta}{\partial x} + \nabla\Psi\cdot\nabla\zeta = \frac{\partial^2\Psi}{\partial z^2}\zeta + \frac{\partial \varphi}{\partial z}$$

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The free surface boundary conditions are linearized assuming that the wave flow is small compared to the basis flow. As with the Kelvin condition, these linearized boundary conditions are applied at the z=0 plane.

Numerical Solution

For wave flow, simultaneously solve:

- Boundary integral equation
- Kinematic FSBC
- Dynamic FSBC
- Body boundary condition
- Equations of motion

To obtain:

- Potential on body and free surface
- Normal Velocity on body and free surface
- Wave elevation
- Body motions

Rankine panel methods do not have the free surface boundary conditions satisfied automatically from the choice of Green function, and hence they need to be solved simultaneously with the integral equation, equations of motion, and body boundary condition.

Non-Linear Methods

- Higher order free surface boundary conditions
- Body-exact formulations
- Iterative linearization about a wave solution

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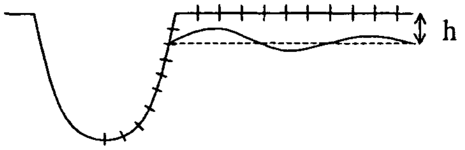
• There are cases when the linearization of the free surface conditions is not sufficient. Computation of higher order solutions is essential for some problems such as drift motions, slamming, etc. It is possible to use Rankine panel methods to solve the second order free surface boundary condition, but this would, in general, no longer involve a system of linear equations. The solution would therefore need to be found using some sort of non-linear solver.

• Body-exact methods discretize the body at its exact position at each time step, thus eliminating the error associated with the linearization of the free surface boundary conditions for large body motions. This can be very important, as seen from the inconsistencies that result when the body is only discretized below the z=0 plane.

• Taking the linearization about the double-body basis flow one step further, it is possible to obtain the linear solution and linearize the free surface conditions about that.solution. Linearizing the flow iteratively about the previous solution, the full non-linear free surface conditions should be satisfied when convergence is reached. This approach is practical only for the steady flow problem, but even for the unsteady problem several methods exist that linearize the flow about flows such as the steady wave solution or the incident wave. With these methods it is usually necessary to discretize the body and the free surface after each iteration, thus adding to the computational load. An exception is for raised panel methods, discussed later.

Raised Panel Methods

Panels above z=0 plane



- No free surface discretization necessary at each iteration
- Influence coefficients of free surface panels to body collocation points calculated only once
- Due to distance, h, the velocity field induced in the fluid domain from each panel is smoother

One successful way of implementing the body-exact iterative linearization about a basis wave flow is by using a "raised panel" method. Such methods place singularity distributions at a distance above the z=0 plane, with the collocation points still on the free surface.

The benefits of such methods are that the free surface panels do not have to be re-created at each iteration, and the free surface to body influence coefficients need only be calculated once. The method also has nice numerical properties since the infinite velocities which are self-induced on each free surface panel are no longer in the fluid domain. In addition, the process of linearizing the flow about the previous solution is made more straightforward since the flow field at the last iteration is always defined at the next estimation of the position of the free surface.


h

Strip Theory

Derivation of:

- Hydrodynamic Coefficients
- Exciting Force and Moment

Assumptions:

- Linear and harmonic motions
- Viscous effects negligible

Potential Flow Decomposition

- Time-independent and time-dependent components

$$\Phi(x, y, z;t) = [-Ux + \phi_S(x, y, z)] + \phi_T(x, y, z)e^{i\omega t}$$

- Incident, Diffraction, Radiation components

$$\phi_T = \phi_I + \phi_D + \sum_{j=1}^{6} \zeta_j \phi_j$$

$-Ux+\phi_S$ is the steady contribution with U the forward speed of the ship, ϕ_T is the complex amplitude of the unsteady potential, and w is the frequency of encounter in the moving reference frame. It is understood that the real part is to be taken in expressions involving $e^{i\omega t}$

ϕ_I is the incident wave potential, ϕ_D is the diffraction potential, and ϕ_j is the contribution to the potential from the j^{th} mode of motion (1=surge, 2=sway, 3=heave, 4=roll, 5=pitch, 6=yaw)

The decomposition of the potential into the above components is convenient for the linearization of the boundary conditions, as will be seen later.

Linearized Boundary Conditions

- Steady Perturbation Potential
  - Body BC:
    (applied at hull mean position)
    $$\frac{\partial}{\partial n}\left[-Ux+\phi_S\right]=0$$
  - Free Surface BC:
    (applied at z=0)
    $$U^2\frac{\partial^2\phi_S}{\partial x^2}+g\frac{\partial\phi_S}{\partial z}=0$$

In order to linearize the boundary conditions it is assumed that the geometry is such that the steady perturbation potential ϕ_S and its derivatives are small.

By assuming that the oscillatory motions are of small amplitude, the time-dependent component of the potential, ϕ_T, and its derivatives may also be considered small.

Under these assumptions the problem may be linearized by disregarding higher-order terms as well as cross-products on both ϕ_S and ϕ_T.

The above expressions for the linear boundary conditions were derived from the exact body and free surface conditions by including only linear terms and applying Taylor expansions about the mean hull position in the body BC and about the undisturbed free surface (z=0) in the free surface BC.

Linearized Boundary Conditions

- Incident and Diffracted Potentials
  - Body BC: (applied at hull mean position)

$$\frac{\partial \phi_I}{\partial n} + \frac{\partial \phi_D}{\partial n} = 0$$

  - Free Surface BC: (applied at $z=0$) where ϕ is ϕ_I or ϕ_D

$$\left[\left(i\omega - U\frac{\partial}{\partial x}\right)^2 + g\frac{\partial}{\partial z}\right]\phi = 0$$

Linearized Boundary Conditions

- Radiation Potentials
  - Body BC: (applied at hull mean position) $$\frac{\partial \phi_j^0}{\partial n} = i\omega n_j$$
  - Free Surface BC: (applied at z=0) $$\left(i\omega - U\frac{\partial}{\partial x}\right)^2 \phi_j^0 + g\frac{\partial}{\partial z}\phi_j^0 = 0$$

where: $\phi_j = \phi_j^0 \quad for \quad j = 1,2,3,4$

$$\phi_5 = \phi_5^0 + \frac{U}{i\omega}\phi_3^0$$
$$\phi_6 = \phi_6^0 - \frac{U}{i\omega}\phi_2^0$$

It can be shown that the radiation body BC is given by:

$$\frac{\partial \phi_j}{\partial n} = i\omega n_j + U m_j$$

The m-terms provide a coupling between the basis flow ϕ_B and the time-dependent potential.

$$(m_1, m_2, m_3) = -(\vec{n}\cdot\nabla)\frac{\nabla\phi_B}{U} \qquad (m_4, m_5, m_6) = -(\vec{n}\cdot\nabla)\left(\vec{x}\times\frac{\nabla\phi_B}{U}\right)$$

For our case, where: $\phi_B = -Ux$ we have: $(m_1, m_2, m_3) = \vec{0}$, $(m_4, m_5, m_6) = (0, n_3, -n_2)$

Let $\phi_j \equiv \phi_j^0 + \frac{U}{i\omega}\phi_j^U$, where ϕ_j^0 is speed independent and satisfies $\frac{\partial \phi_j^0}{\partial n} = i\omega n_j$ on the body, in addition to the Laplace equation and free-surface and infinity conditions.

It then follows that: $\phi_j^U = 0 \quad for \quad j = 1,2,3,4$

and
$$\phi_5^U = \phi_3^0$$
$$\phi_6^U = -\phi_2^0$$

Pressure Linearization

- From Bernoulli:

$$p = -\rho\left(i\omega - U\frac{\partial}{\partial x}\right)\phi_T e^{i\omega t} - \underbrace{\rho g(\zeta_3 + \zeta_4 y - \zeta_5 x)e^{i\omega t}}_{\text{buoyancy term ignored}}$$

(included in hydrostatic restoring coefficient)

Hydrodynamic force and moment:

$$H_j = -\rho\iint_S n_j\left(i\omega - U\frac{\partial}{\partial x}\right)\phi_T ds$$

(integration over the mean position of hull)

Similarly to the boundary conditions, the pressure is expanded as a Taylor series about the undisturbed position of the hull and the expression is linearized by neglecting quadratic and higher order terms in ϕ_S and ϕ_T.

The hydrodynamic forces and moments, H, include the exciting forces as well as the forces due to the ship motions (added mass and damping forces). They do dot include the hydrostatic restoring forces which are included elsewhere.

Hydrodynamic Forces

$$H_j = F_j + \sum_{k=1}^{6} T_{jk}\zeta_k$$

- Exciting Force & Moment

$$F_j = -\rho \iint_S n_j \left(i\omega - U\frac{\partial}{\partial x} \right)(\phi_I + \phi_D)ds$$

- Radiation Force & Moment

$$T_{jk} = -\rho \iint_S n_j \left(i\omega - U\frac{\partial}{\partial x} \right)\phi_k ds = \omega^2 A_{jk} - i\omega B_{jk}$$

- Need A, B, F to get equation of motion

T_{jk} is the hydrodynamic force in the j^{th} direction, due to a unit oscillatory displacement in the k^{th} direction.

The real and imaginary parts of this force is proportional to the added mass and damping coefficients respectively. These coefficients, along with the exiting force, will be expressed in terms of integrals of the sectional (2D) coefficients over the length of the hull.

The equation of motion of the ship will then be fully specified:

$$\sum_{k=1}^{6} \left[-\omega^2 (M_{jk} + A_{jk}) + i\omega B_{jk} + C_{jk} \right] \zeta_k = F_j$$

Radiation Forces

- Variant of Stokes' Theorem

$$\iint_S n_j U \frac{\partial \phi}{\partial x} ds = U \int_S m_j \phi \, ds - U \int_{C_A} n_j \phi \, dl$$

- From which:

$$T_{jk} = \underbrace{-\rho i \omega \int_S n_j \phi_k ds}_{T_{jk}^0} + U\rho \iint_S m_j \phi_k ds - U\rho \int_{C_A} n_j \phi_k dl$$

- Use the decomposition of the radiation potential to express T_{jk} in terms of speed-independent terms

In deriving the variant of Stokes' theorem, a small angle between the waterline and the x-axis is assumed. S is the hull surface forward of the cross section C_A.

As we did for the radiation potential, we can divide the hydrodynamic force into speed-independent and speed-dependent components. The speed-independent components are defined as follows:

$$T_{jk}^0 \equiv -\rho i \omega \int_S n_j \phi_k^0 ds \qquad t_{jk}^A = -\rho i \omega \int_{C_A} n_j \phi_k^0 ds$$

Then, using the properties of the radiation potential, we have:

for j,k=1,2,3,4:

$$T_{jk} = T_{jk}^0 + \frac{U}{i\omega} t_{jk}^A$$

for j=1,2,3,4:

$$T_{j5} = T_{j5}^0 + \frac{U}{i\omega} T_{j3}^0 + \frac{U}{i\omega} t_{j5}^A - \frac{U^2}{\omega^2} t_{j3}^A$$

$$T_{j6} = T_{j6}^0 - \frac{U}{i\omega} T_{j2}^0 + \frac{U}{i\omega} t_{j6}^A + \frac{U^2}{\omega^2} t_{j2}^A$$

for k=1,2,3,4:

$$T_{5k} = T_{5k}^0 - \frac{U}{i\omega} T_{3k}^0 + \frac{U}{i\omega} t_{5k}^A$$

$$T_{6k} = T_{6k}^0 + \frac{U}{i\omega} T_{2k}^0 + \frac{U}{i\omega} t_{6k}^A$$

and finally:

$$T_{55} = T_{55}^0 + \frac{U^2}{\omega^2} T_{33}^0 + \frac{U}{i\omega} t_{55}^A - \frac{U^2}{\omega^2} t_{53}^A$$

$$T_{66} = T_{66}^0 + \frac{U^2}{\omega^2} T_{22}^0 + \frac{U}{i\omega} t_{66}^A - \frac{U^2}{\omega^2} t_{62}^A$$

Strip Theory Approximations

- Length >> Beam, Draft

$$ds = d\xi dl \Rightarrow T_{jk}^{0} = -\rho i\omega \int_{L}\int_{C_x} n_j \phi_k^0 dl d\xi = \int_{L} t_{jk} d\xi$$

$$\frac{\partial}{\partial x} << \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$$

$$n_1 << n_2, n_3 \Rightarrow \begin{cases} n_j = N_j \quad (j = 2,3,4) \\ n_5 = -xN_3 \\ n_6 = xN_2 \end{cases}$$

N: 2D normal
n: 3D normal

$$\omega >> U\left(\frac{\partial}{\partial x}\right)$$

The above approximations are all consistent with the trip theory assumption of a long and slender ship.

The last condition, which states that the frequency of encounter is high, requires that the maximum wave length is of the same order as the ship's beam. This enables us to simplify the radiation potential free surface condition so that ϕ^0 is indeed speed-independent as assumed.

Under these assumptions, the 3D Laplace equation and the boundary conditions reduce to the 2D Laplace equation with the corresponding 2D boundary conditions.

Radiation Forces in terms of 2D hydrodynamic coefficients

$$T_{22}^0 = \int t_{22} d\xi \qquad T_{33}^0 = \int t_{33} d\xi \qquad T_{44}^0 = \int t_{44} d\xi$$
$$T_{26}^0 = T_{62}^0 = \int \xi t_{22} d\xi \qquad T_{35}^0 = T_{53}^0 = -\int \xi t_{33} d\xi \qquad T_{24}^0 = T_{42}^0 = \int t_{24} d\xi$$
$$T_{66}^0 = \int \xi^2 t_{22} d\xi \qquad T_{55}^0 = \int \xi^2 t_{33} d\xi \qquad T_{46}^0 = T_{64}^0 = \int \xi t_{24} d\xi$$

All the rest $T_{jk}^0=0$, for ships with lateral symmetry

where: $t_{jj} = -\rho i\omega \int_{C_x} N_j \phi_j^0 dl = \omega^2 a_{jj} - i\omega b_{jj} \quad for \quad j = 2,3,4$

$$t_{24} = -\rho i\omega \int_{C_x} N_2 \phi_4^0 dl = \omega^2 a_{24} - i\omega b_{24}$$

From the assumptions of strip theory, the two-dimensional radiation potential at each section, ψ_k, is equal to the three dimensional potential ϕ_k^0 for sway, heave and roll:

$$\phi_k^0 = \psi_k \quad for \quad k = 2,3,4$$

In addition, from the hull condition, we have for pitch and yaw:

$$\phi_5^0 = -x\psi_3 \quad and \quad \phi_6^0 = x\psi_2$$

and

$$\phi_1^0 << \phi_k^0 \quad for \quad k = 2\ldots6$$

The above relations were used in conjunction with the expressions for the sectional radiation forces and the strip theory approximations to get the zero-speed radiation forces in terms of the 2D forces.

So from all the above, and from the relation between the speed-independent and speed-dependent components of the radiation force, we have all we need in order to express the added mass and damping coefficients in terms of the sectional two-dimensional added mass and damping.

Incident Wave Exciting Forces

$$\left.\begin{aligned} F_j^I &= -\rho \iint_S n_j \left(i\omega - U\frac{\partial}{\partial x} \right) \phi_I \, ds \\ \phi_I &= \frac{ig\alpha}{\omega_0} e^{-ik(x\cos\beta - y\sin\beta)} e^{kz} \end{aligned}\right\} \Rightarrow F_j^I = -\rho i \omega_0 \iint_S n_j \phi_I \, ds$$

(Froude-Kriloff force and moment)

α: wave amplitude

k: wave number

β: heading angle

ω_0: wave frequency, related to frequency of encounter by $\omega_0 = \omega + kU\cos\beta$

Diffraction Forces

$$F_j^D = -\rho \iint_S n_j \left(i\omega - U \frac{\partial}{\partial x} \right) \phi_D ds$$

- Using:
  - Stokes' theorem
  - Hull BC
  - Green's 2nd identity

we get:

$$F_j^D = \rho \iint_S \left(\phi_j^0 - \frac{U}{i\omega} \phi_j^U \right) \frac{\partial \phi_I}{\partial n} ds + \frac{\rho U}{i\omega} \int_{C_A} \phi_j^0 \frac{\partial \phi_I}{\partial n} dl$$

The same form of Stokes' theorem that was used for the radiation forces earlier is applied to the diffraction forces.

The hull condition for the radiation potentials is then used to get products of potentials and normal velocities.

Green's identity, which involves such products, is then used to eliminate the radiation normal velocities (they get substituted by the diffraction normal velocities)

The hull boundary condition is then used to replace the diffraction normal velocities by the negative of the incident wave normal velocities which are known.

Use may then be made of the relations between the speed-independent and speed-dependent components of the radiation potentials to get an expression involving only the speed-independent components.

Finally, the incident wave potential, which is a known quantity, may be substituted.

Excitation Forces in terms of 2D sectional forces

$$F_1 << F_k \quad k = 2 \ldots 6$$

$$F_j = \rho\alpha \int_L (f_j + h_j) d\xi + \rho\alpha \frac{U}{i\omega} h_j^A \quad j = 2,3,4$$

$$F_5 = -\rho\alpha \int_L \left[\xi (f_3 + h_3) + \frac{U}{i\omega} h_3 \right] d\xi - \rho\alpha \frac{U}{i\omega} x_A h_3^A$$

$$F_6 = \rho\alpha \int_L \left[\xi (f_2 + h_2) + \frac{U}{i\omega} h_2 \right] d\xi + \rho\alpha \frac{U}{i\omega} x_A h_2^A$$

where:

$$f_j(x) = g e^{-ikx\cos\beta} \int_{C_x} N_j e^{iky\sin\beta} e^{kz} dl \quad (2D\ Froude-Kriloff)$$

$$h_j(x) = \omega_0 e^{-ikx\cos\beta} \int_{C_x} (iN_3 - N_2 \sin\beta) \times e^{iky\sin\beta} e^{kz} \varphi_j^0 dl \quad (2D\ diffraction)$$

The excitation forces in terms of the sectional Froude-Kriloff and diffraction forces are derived from the previously derived expressions by making use of the strip theory approximations.