Harvard-MIT Division of Health Sciences and Technology
HST.523J: Cell-Matrix Mechanics
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Elements of linear elastic mechanics (LEM). Outline of topics

A. Basic rules of LEM.

B. Modes of deformation and elastic constants. Tension, compression, shear, hydrostatic pressure.

C. The stress and strain tensors. Simple stress fields.

D. Constitutive equations for a linear, elastic isotropic solid.

E. Pure bending of a beam.

F. Torsion of cylindrical shaft.

Text: Crandall et al. *An Introduction to the Mechanics of Solids* (on reserve), esp. Chs. on "Stress and strain", "Torsion", "Stresses due to bending".

A. Basic rules of LEM

- **<u>Matter is a continuum.</u>**
- **<u>Linearity</u>. Directly proportional relation between stress and strain. No terms with exponent higher than 1.**
- **<u>Elasticity</u>. Time-independent mechanical behavior. Loading and deformation independent of time.**
- **<u>Isotropy</u>. The elastic constants are independent of loading direction.**
- **<u>Conservation of mass</u>.**

Three steps used in solving structural problems of solid mechanics (Crandall's triad).

1. <u>Force equlibrium</u>: The sum of all external forces and moments that act on the body, and on every part of the body, is zero.
2. <u>Geometric compatibility</u>: Body deforms in such a way that a displacement of a given point takes only one value, so that adjacent parts of the body remain connected.
3. <u>Stress-strain laws (constitutive equations)</u>: Forces related to deformations through material properties (if "linear elastic" behavior, use elastic constants E, G, K or ν).

“Structures” in solid mechanics are houses, planes, bridges, hip prostheses

- In solving a <u>structure</u>, consider the geometry and especially the <u>symmetry</u> (tube? membrane? sheet?); the <u>load distribution</u> that, combined with the geometry, determines the “stress field” (e.g., single vs multiple load); and the <u>material</u> used (rigid? flexible?).

- Structure = geometry + loading pattern + + material properties

B. Modes of deformation and elastic constants.

- <u>Elastic constants</u>, such as Young's modulus, E, the shear modules, the compressibility modulus, are material properties that relate stress to strain. Another constant, Poisson's ratio, relates strain along two orthogonal axes.
- These constants can be defined by referring to certain simple <u>stress fields</u>. The simplest fields are simple tension and compression, shear and hydrostatic pressure.

Diagram defining simple tension removed for copyright reasons.

Simple tension

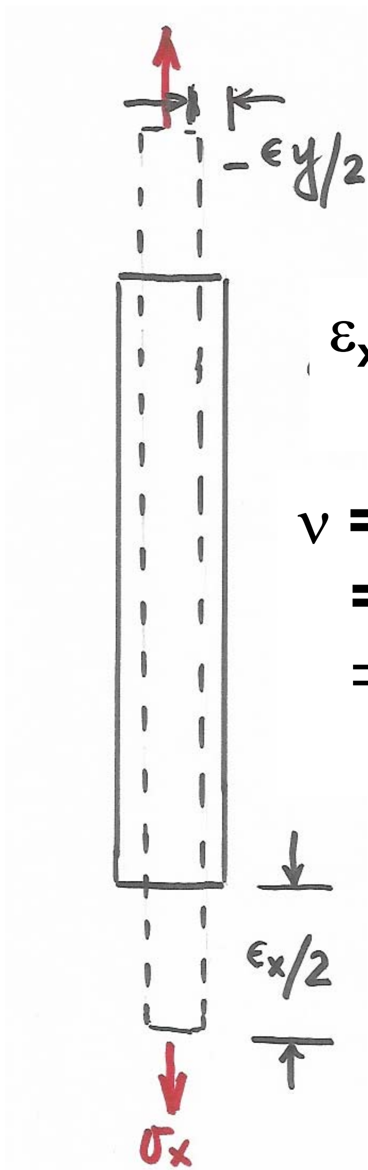
from Ashby textbook

Diagram defining simple compression removed for copyright reasons.

Simple compression

from Ashby textbook

Simple tension



$\varepsilon_x \cong (L-L_o)/L_o$. Compression described by reversing signs.

ν = lateral strain/longitudinal strain =
= tensile strain/contractile strain =
= $-\varepsilon_y/\varepsilon_x$

$0 \leq \nu \leq 0.5$

Typical values of Poisson's ratio, ν (a measure of volume change) :

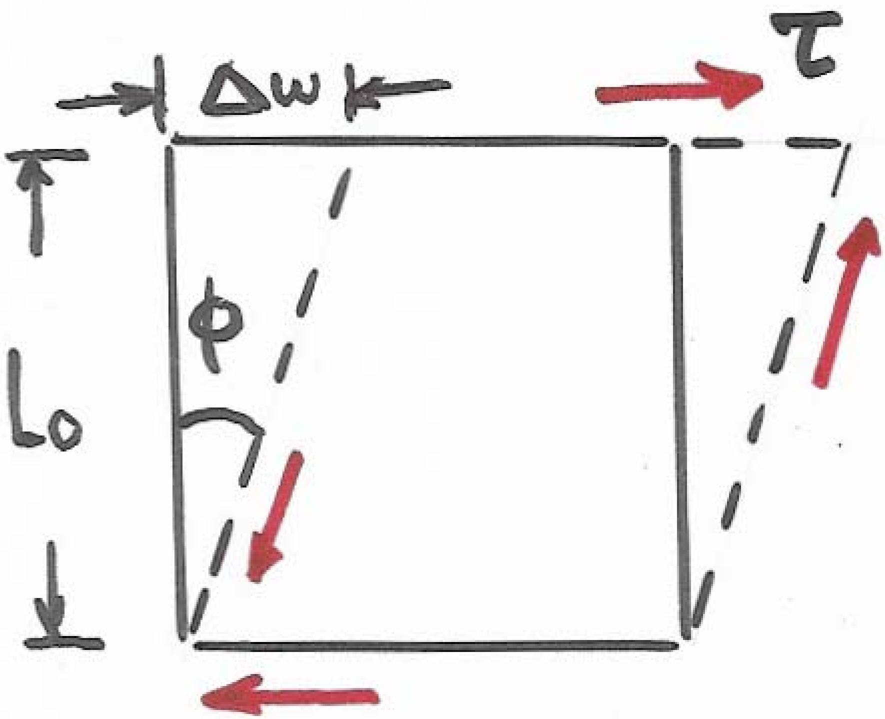
rubber, tissues $\cong 0.5$
aluminum, 0.33
stainless steel, 0.27

Diagram defining pure shear
removed for copyright reasons.

Pure shear

**From
Ashby textbook**

Pure shear



shear stress = τ

shear strain = $\gamma = \Delta w/L_o =$

$= \tan\varphi \cong \varphi$

shear modulus = G = $\tau/\gamma \cong \tau/\varphi$

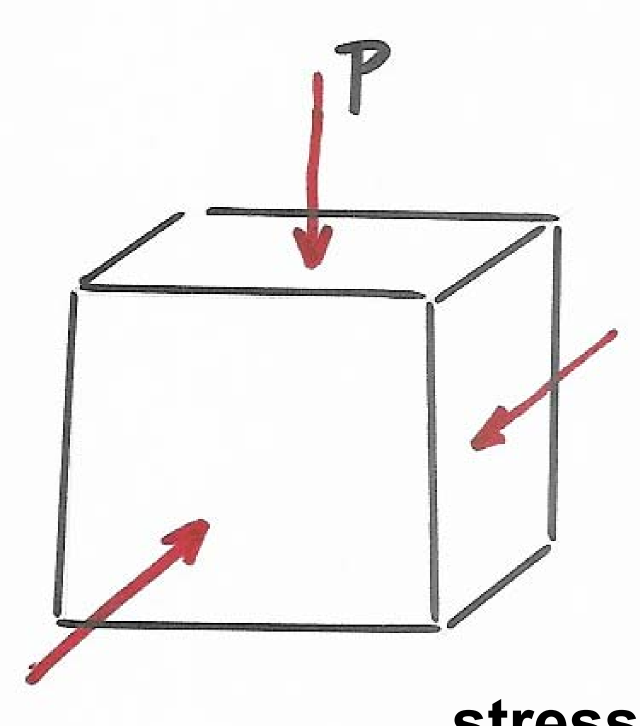
G is a measure of change in shape only (not size change)

Diagram defining hydrostatic pressure removed for copyright reasons.

Hydrostatic pressure

From Ashby textbook

Hydrostatic pressure



stress = P = $(\sigma_x + \sigma_y + \sigma_z)/3$

strain = $-\Delta V/V_o$

bulk modulus K = $-(\sigma_x + \sigma_y + \sigma_z)/(3\Delta V/V_o)$

K is a measure of change in size only (not shape)

Relations among elastic constants. Shape vs. size changes

$G = E/[2(1+\nu)]$. As $\nu \to 0.5$ tissues), $G \to E/3$ (rubber, many tissues)

$K = E/[3(1-2\nu)]$. As $\nu \to 0.5$ (tissues), $K \to \infty$ (incompressible material)

Generally, deformation = Δ(shape) + Δ(size).

$\Delta(\text{size}) = \Delta V/V_o = \varepsilon_x(1-2\nu)$.

If $\nu = 1/2$, the material is 'incompressible'.

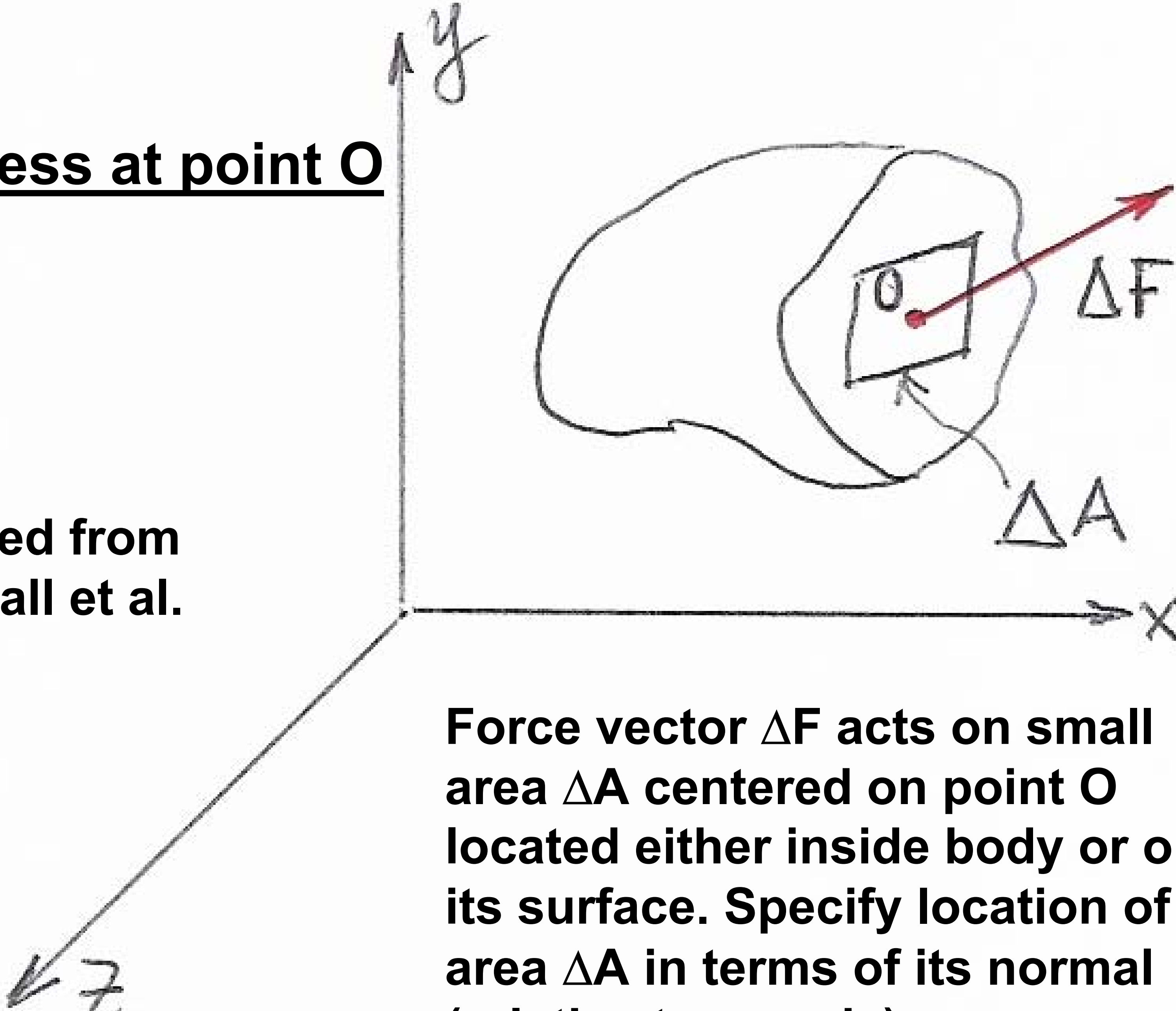
Δ(shape) = depends on geometry. Certain modes of loading lead to change of shape with no change in size (pure torsion); the reverse can also occur (hydrostatic compression).

C. The stress and strain tensors. Simple stress fields.

- **Unlike forces that are represented as <u>vectors</u> (3 components acting along each of three axes), stress and strain are represented as <u>tensors</u> (9 components). The reason for the additional complexity arises from the need to describe not only the <u>magnitude</u> (scalar) and <u>direction</u> (vector describes both magnitude and direction) of a stress component but also the plane in which it is <u>located</u> inside the body. A tensor co,mpoenent informs about <u>magnitude</u> and <u>direction</u> at a specific <u>location</u> inside a medium.**
- **Stress and strain tensors are symmetric. They summarize valuable information about the mechanical conditions at a point O, located either inside or at the surface of a body.**

Stress at point O

Adapted from Crandall et al.

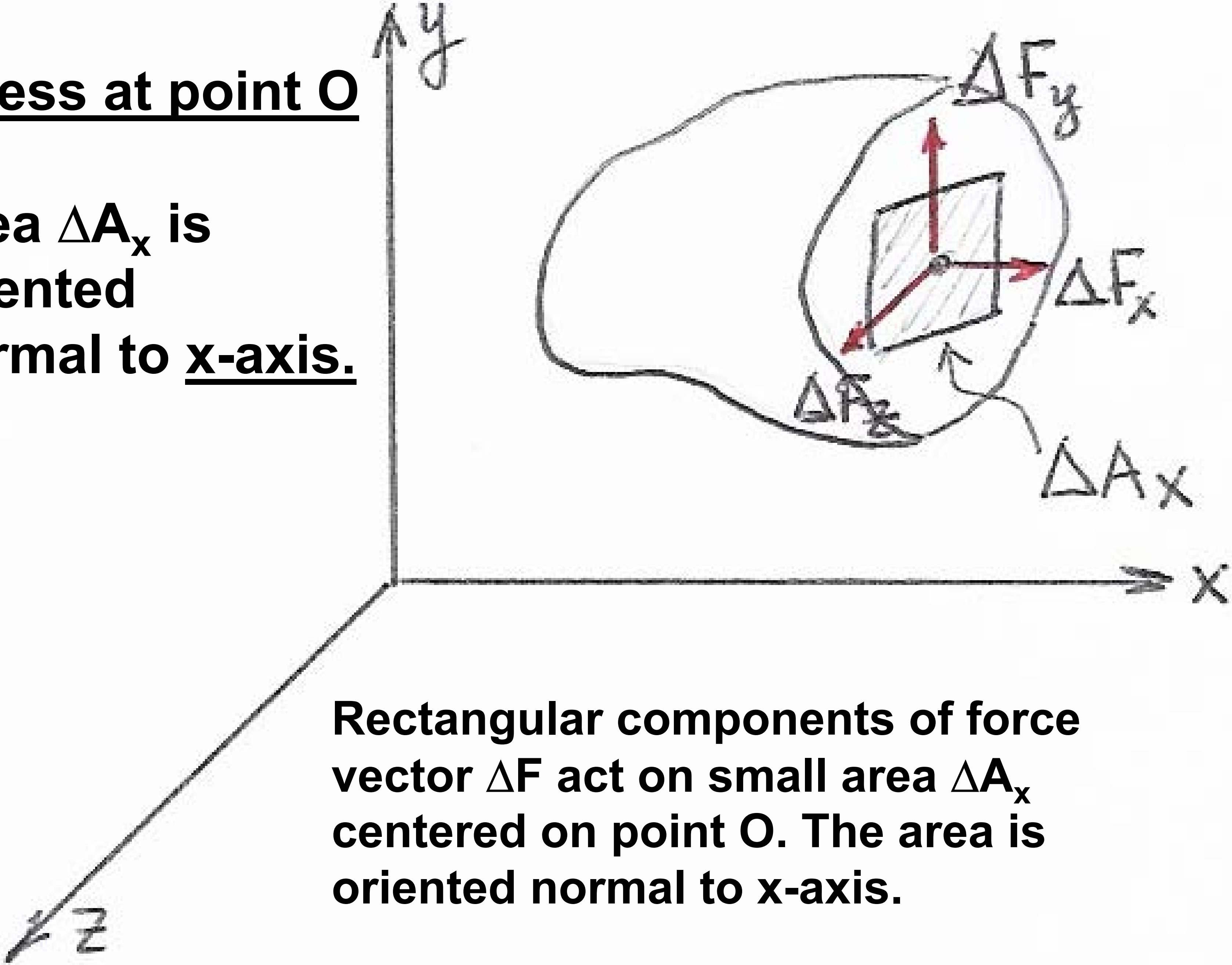


Force vector ΔF acts on small area ΔA centered on point O located either inside body or on its surface. Specify location of area ΔA in terms of its normal (relative to an axis).

Sketches from Crandall et al.

Stress at point O

Area ΔA_x is oriented normal to x-axis.



Rectangular components of force vector ΔF act on small area ΔA_x centered on point O. The area is oriented normal to x-axis.

Sketches from Crandall et al.

From force vector to stress

In the limit, as $\Delta A_x \to 0$, $\Delta F_x/\Delta A_x$ becomes a stress component acting at point O. It is located in the plane normal to the x-axis and acts along the x-axis as well.

$$\sigma_{xx} = \lim_{\Delta A_x \to 0} \Delta F_x/\Delta A_x$$

The first subscript of the tensor notation describes the location of the stress component in the plane that is defined by its normal to the x-axis.

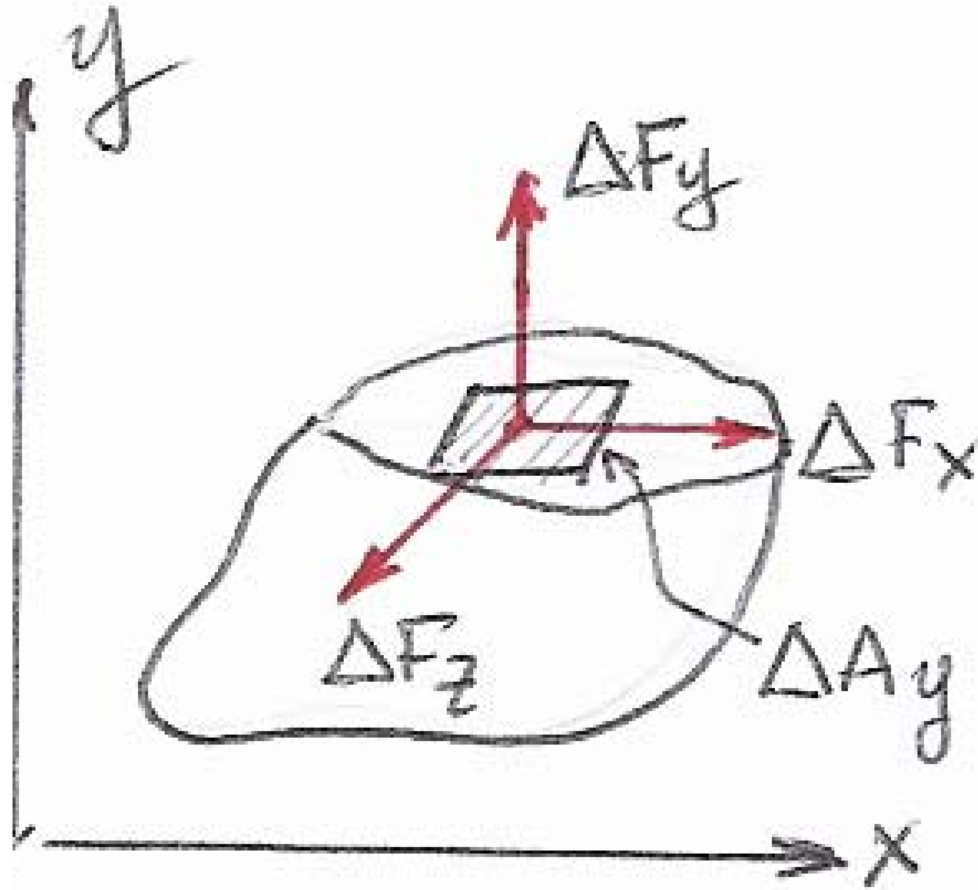
The second subscript describes the direction of the stress component, defined simply (as with vector notation) by the x-axis along which it acts.

If we know the stress components for all possible orientations of faces through point O, we say that we know the "state of stress at that point".

Stress at point O

This time the area ΔA_y is oriented normal to the y-axis

Define stress components located in plane normal to y-axis: σ_{yy}, τ_{yx}, τ_{yz} **etc.**



Rectangular components of force vector ΔF act on small area ΔA_y centered on point O. The area is oriented normal to y-axis.

Sketches from Crandall et al.

Normal and shear stress components of the stress tensor

- **A <u>normal</u> stress component acts along the <u>same</u> axis as that which defines its location. There are only three normal components: σ_{xx}, σ_{yy} , σ_{zz}. They are abbreviated as σ_x, σ_y, σ_z.**
- **A <u>shear</u> stress component acts along an axis that is <u>different</u> than that which defines its location. There are six shear components: τ_{xy}, τ_{xz}, etc.**
- **The stress tensor comprises 3 normal + 6 shear components = 9 total components. However, since the diagonal components $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, etc., these three equalities reduce the number of independent components of the tensor to 6.**

Set up the symmetric stress and strain tensors as follows:

1. **The tensor is a 3x3 matrix. Start by setting up the normal stresses along the diagonal:**

$$\begin{matrix} \sigma & & \\ & \sigma & \\ & & \sigma \end{matrix}$$

2. **Write τ for the shear stresses in all other positions:**

$$\begin{matrix} \sigma & \tau & \tau \\ \tau & \sigma & \tau \\ \tau & \tau & \sigma \end{matrix}$$

Set up stress and strain tensors (cont.):

3. Take advantage of the symmetry by inserting the <u>first</u> subscripts with x's in first row, y's in second row, etc.:

$$\begin{matrix} x & x & x \\ y & y & y \\ z & z & z \end{matrix}$$

4. Now insert the <u>second</u> subscripts with x, y, z in each row:

$$\begin{matrix} x & y & z \\ x & y & z \\ x & y & z \end{matrix}$$

A similar procedure yields the strain tensor.

Stress and strain tensors

stress tensor

$$\begin{matrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{matrix}$$

strain tensor

$$\begin{matrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_z \end{matrix}$$

- **These two tensors define the state of stress and strain <u>at a point</u> in an isotropic linear elastic body. Only 6 components are independent.**

Examples

Tensorial representation of certain simple stress fields:

- Tendon and ligament
- Dermis and cornea
- Femoral bone

a. Tendon and ligament: simple tension (one-dimensional beam; both size and shape changes). Stress tensor is:

$$\begin{matrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

Tendon connects muscle to bone and supports axial forces

Diagram illustrating extension and flexion removed for copyright reasons.

From Vander et al. *Human Physiology*, McGraw-Hill, 1970, p. 230

b. <u>Dermis and cornea</u>: plane stress (two dimensional sheet/membrane; both size and shape changes).

$$\begin{matrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{matrix}$$

sphere of radius r and thickness t ($t \leq 0.1r$)

$\sigma_r = 0$

$\sigma_\theta = pr/2t$

$\sigma_z = pr/2t$

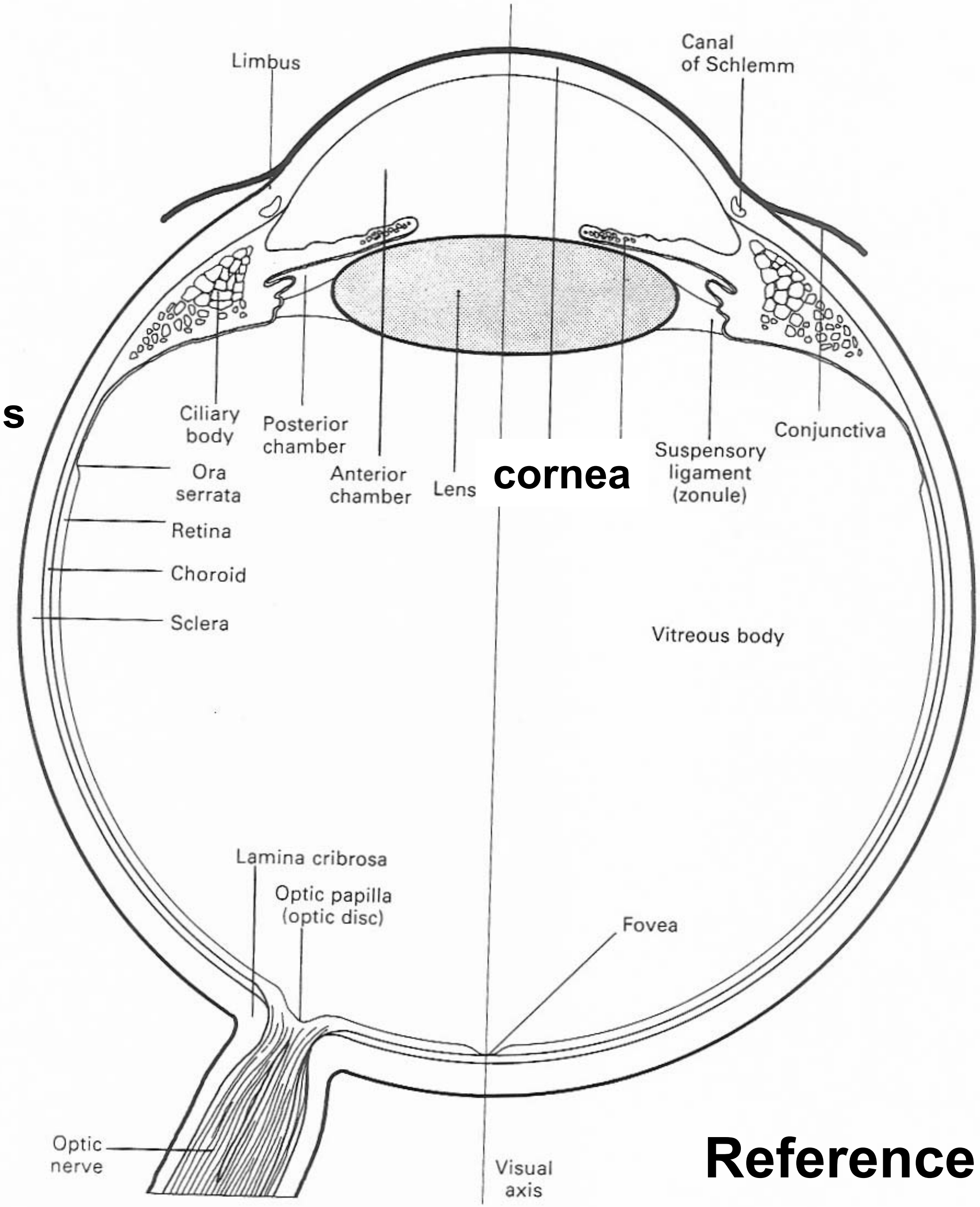
shear stresses are zero (symmetry)

Diagram defining biaxial tension removed for copyright reasons.

From Ashby textbook

The eye

the cornea is a very thin membrane that is subjected to a plane stress field



Reference not recalled

CORNEA. Membrane protects curved eye surface from tangential forces. <u>Planar</u> orientation of collagen fibers.

Photo removed for copyright reasons.

Tadpole cornea

Gross article from *Sci. Amer*.

SKIN

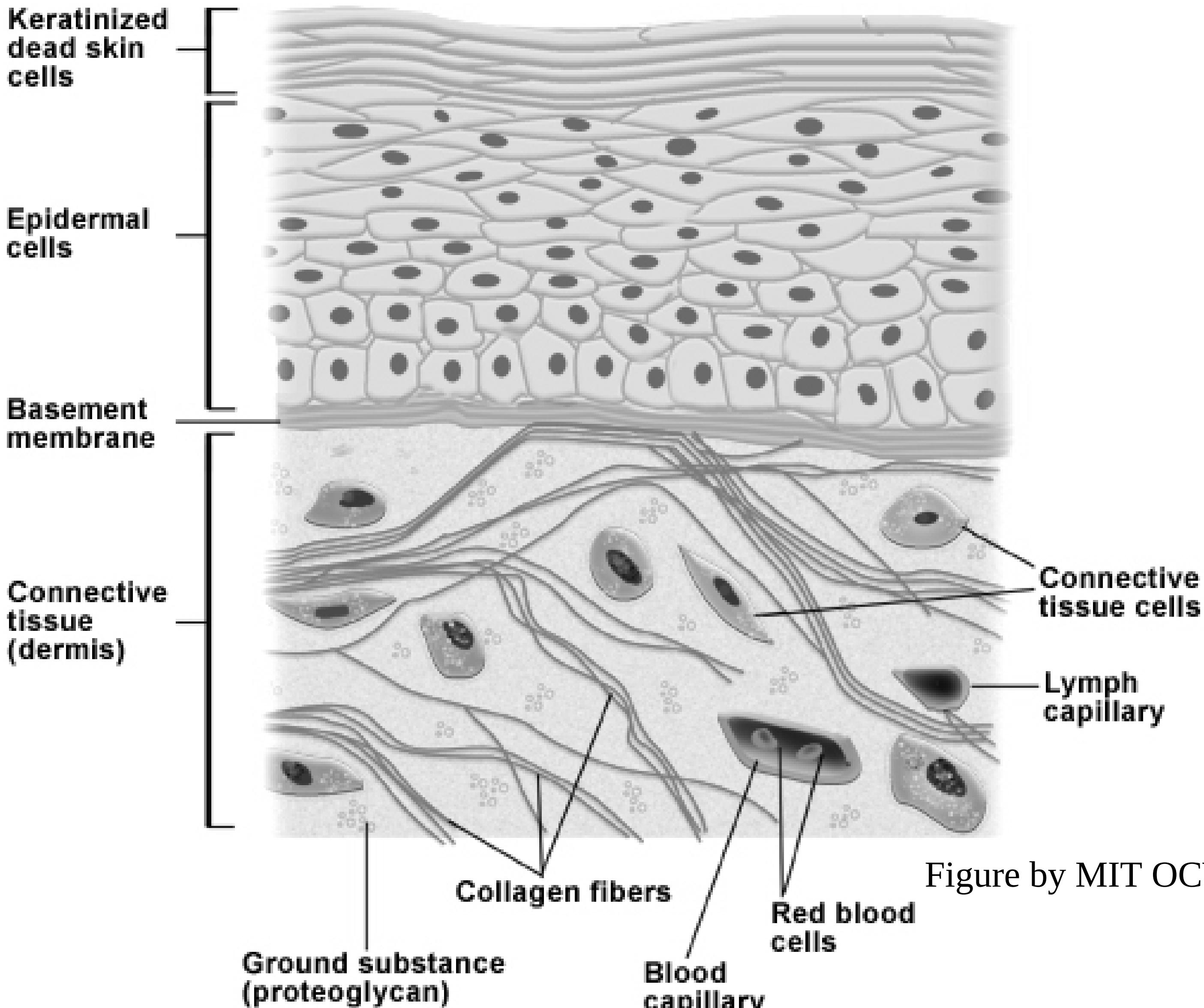


Figure by MIT OCW.

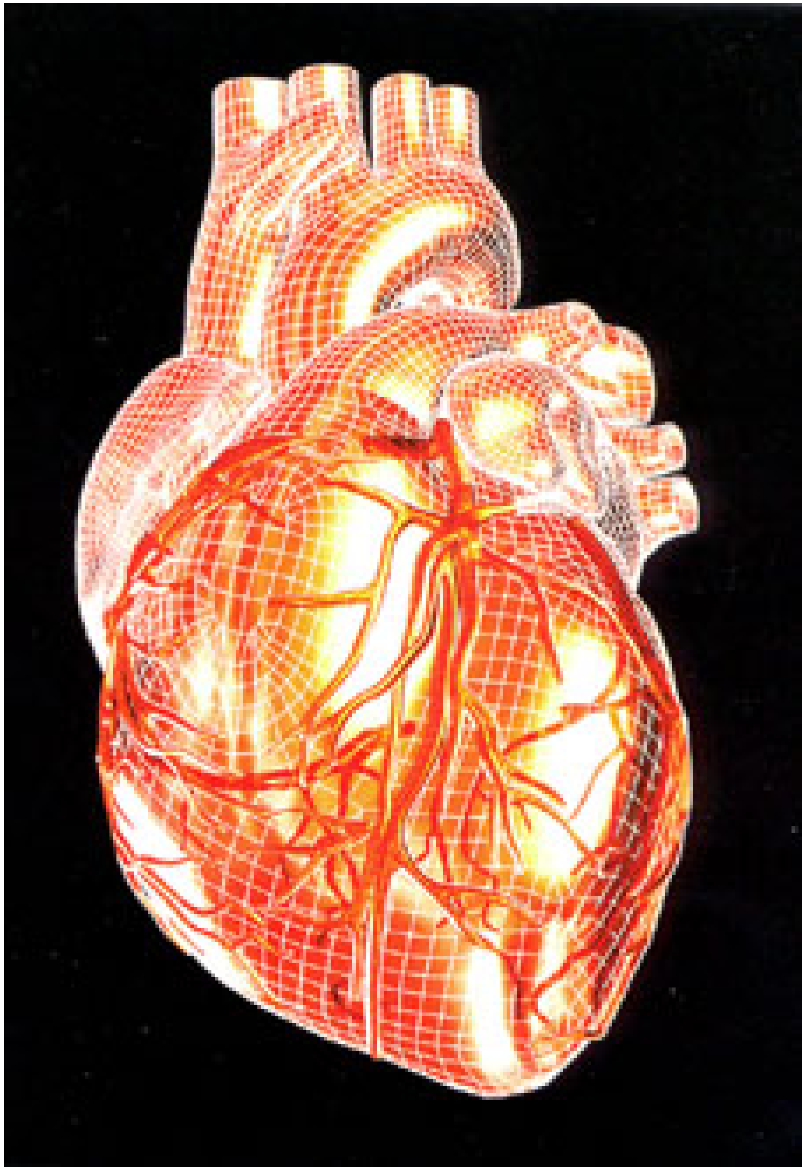


Image courtesy of Lawrence Berkeley National Laboratory.

<u>Example from current research</u>
Identify stress field that appears to block nerve regeneration

Rat sciatic nerve model of nerve regeneration

Diagram removed for copyright reasons.

From The Guinea Pig book

NERVE: Transection (cut through) leads to neuroma formation. The endoneurium does not regenerate.

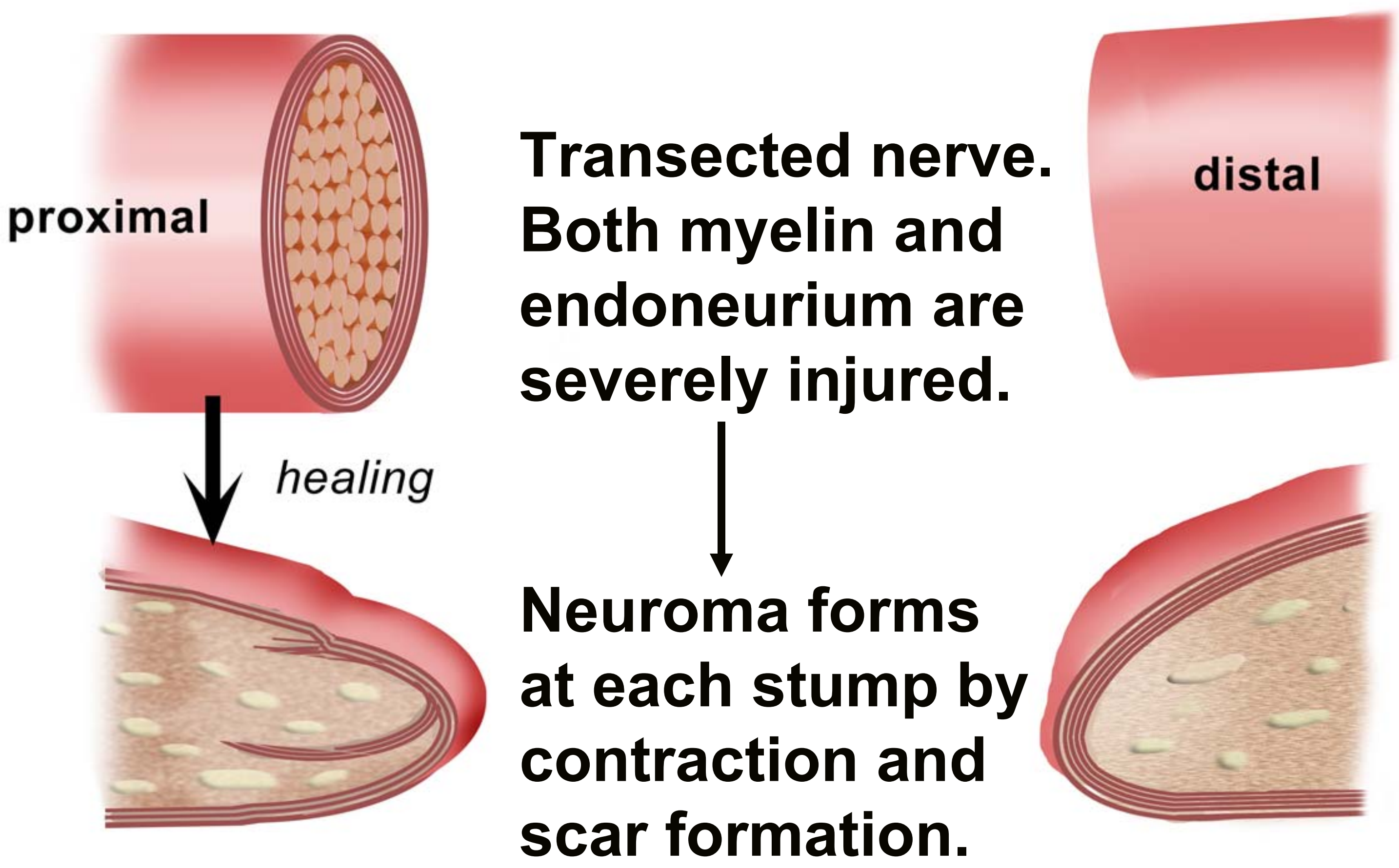
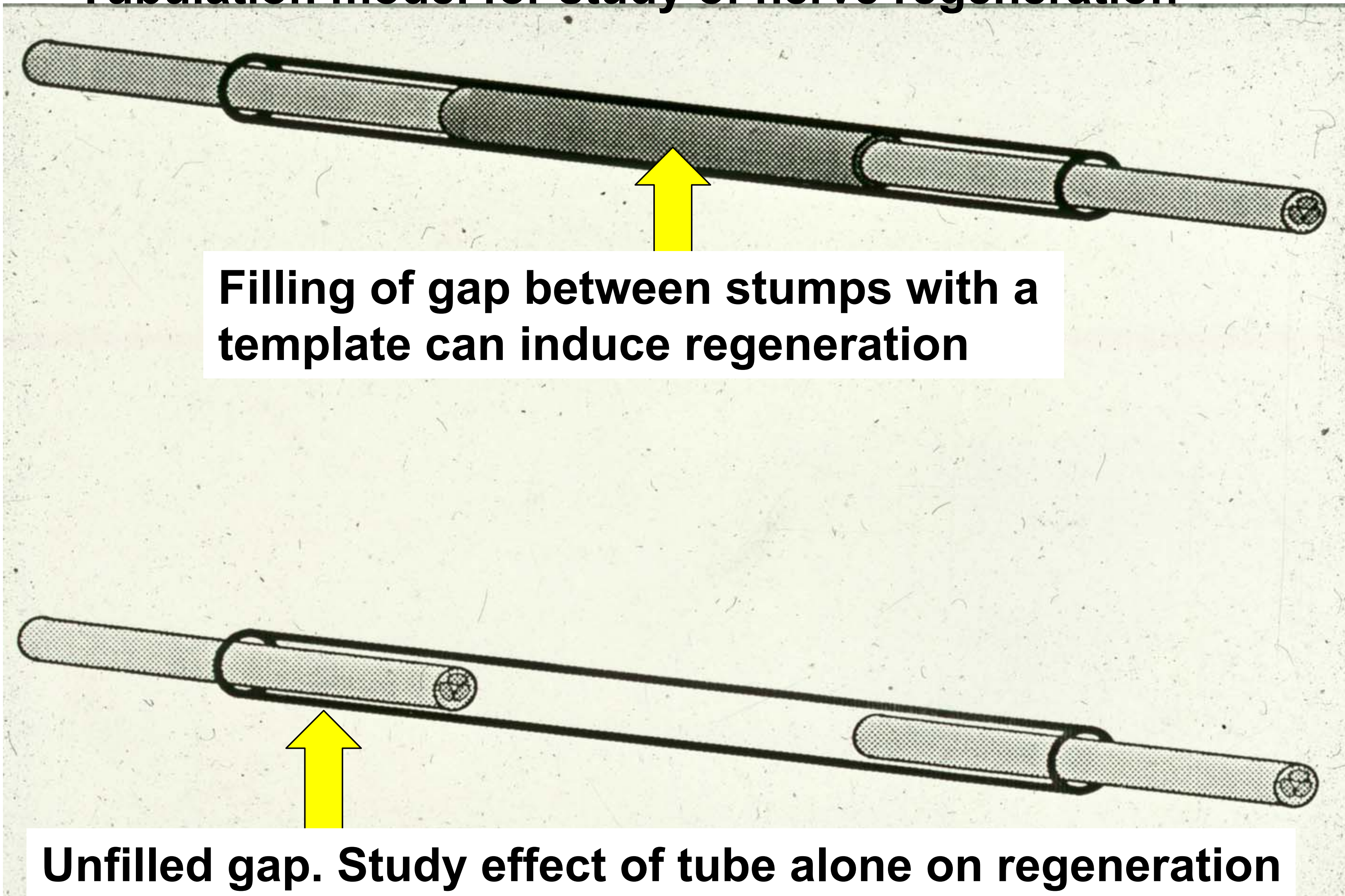


Figure by MIT OCW. After Yannas 2001.

Tubulation model for study of nerve regeneration



Filling of gap between stumps with a template can induce regeneration

Unfilled gap. Study effect of tube alone on regeneration

unpublished

Diagram removed due to copyright considerations.
See Figure 10.7 top left in [Yannas 2001]: Yannas, I. V. *Tissue and Organ Regeneration in Adults*. New York: Springer, 2001.

normal sciatic nerve of rat

Jenq and Coggeshall, 1985

<u>Classical data</u>: Cell capsule is always present around regenerated nerve. Circular perimeter of capsule. Kinks in capsule perimeter.

Diagram removed for copyright reasons.
See Figure 10.7 in [Yannas 2001].

Regenerated nerve

Intact nerve

Jenq and Coggeshall, 1985

Regenerated nerve. Close view of capsule surrounding it.

Diagram removed for copyright reasons.
See Figure 10.7 top right in [Yannas 2001].

Jenq and Coggeshall, 1985

Capsule of contractile cells surrounds regenerating nerve

Photo removed for copyright reasons.

regenerated nerve

contractile cells

original stump surface

Spilker and Seog, 2000

silicone tube yields low quality regenerated nerve

collagen tube yields a regenerated nerve of much higher quality

Photo removed for copyright reasons.

Photo removed for copyright reasons.

thick capsule of contractile cells (arrows)

thin capsule

Chemical composition of tube has dramatic effect on nerve regeneration

Chamberlain et al., 1998

Use the data to set up a mathematical model of nerve regeneration

<u>The pressure cuff hypothesis</u>. A cuff of contractile cells surrounding the regenerating nerve blocks its regeneration. Model the effect of a contractile cell capsule surrounding a regenerating nerve as if it were a cylinder filled with gas under pressure:

$$\sigma_r = 0$$
$$\sigma_\theta = pr/t$$
$$\sigma_z = pr/2t$$

Calculate the radial strain $\varepsilon_r = \Delta R/R_o$

$$\Delta R/R_o = (1/E)(1- \nu)\sigma_\theta \text{ where } \sigma_\theta = \sigma_{cell} t$$

Brau, 2002

c. <u>Femoral bone</u>: pure bending of beam (shear stresses zero; only one normal stress nonzero; geometry alone determines deformation; no change in <u>shape</u>)

$$\begin{matrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

Normal bones

Collagen fibers in bones lack crosslinking

Photos removed for copyright reasons.

Gross article in Sci. American

d. <u>Knee joint</u> in basketball injury: simple torsion of shaft (all normal stresses zero; only one shear stress nonzero; geometry alone determines deformation; no change in <u>size</u>)

$$\begin{matrix} 0 & \tau_{\theta z} & 0 \\ \tau_{\theta z} & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

D. Constitutive equations of linear elastic mechanics (Hooke's law).

Six equations expressing each component of the strain tensor as a function of one or more components of stress tensor:

$$\varepsilon_x = (1/E)\,[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = (1/E)\,[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = (1/E)\,[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \tau_{xy}/G$$

$$\gamma_{yz} = \tau_{yz}/G$$

$$\gamma_{zx} = \tau_{zx}/G$$

Constitutive equations of LEM (cont.)

- **Exactly how does the mathematical notation describe these bodies as being "linear"? What in the math makes them "elastic"? Which experiments can diagnose a material as being linear? Elastic?**
- **The constitutive equations can alternately be written in terms of the <u>stress</u> components as well as in <u>indicial notation</u>.**

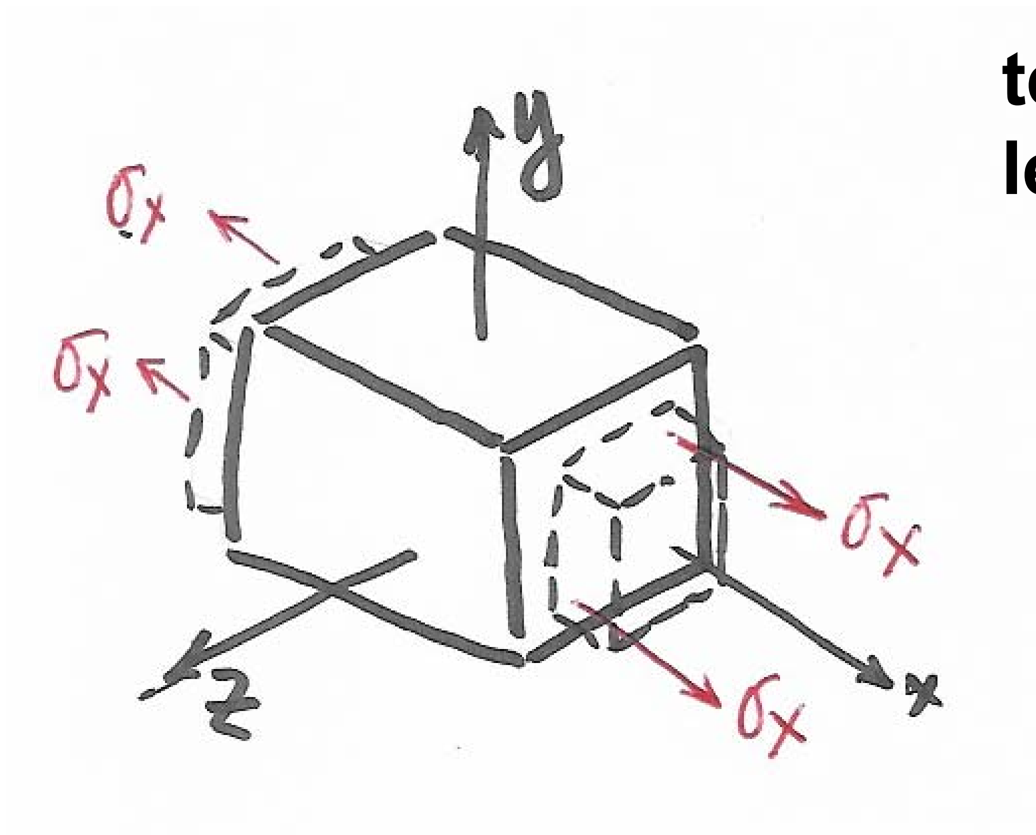
  (see Crandall et al. Chap. “Stress and strain”.)

Derive the constitutive equations by identifying stress components of the tensor one by one

1. Add the first stress σ_x. It causes strain ε_x .
2. Assume linear elastic isotropic material.

Uniaxial stress σ_x acting alone.

$$\sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$$

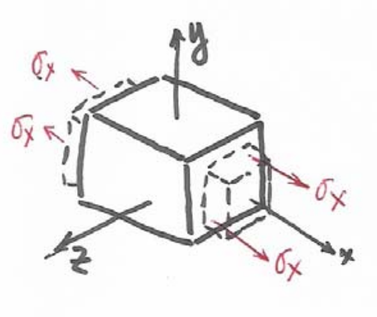


tension along x-axis leads to tensile strain

$$\varepsilon_x = \sigma_x/E$$

Sketches adapted from Crandall et al.

2. <u>Lateral strains resulting from first stress σ_x</u>. The normal stress σ_x also causes lateral contraction along y- and z- axes. Lateral strains due to contraction along y-axis and z-axis must be equal because neither the material (isotropic) nor the mode of stressing favors either direction.

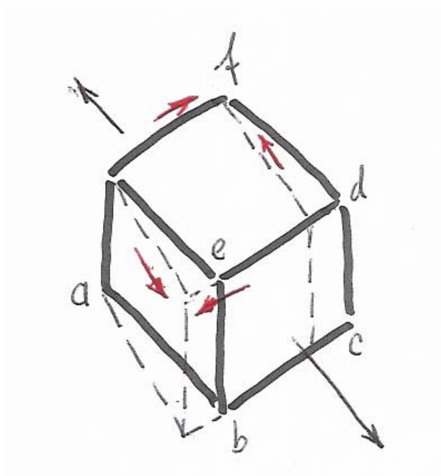


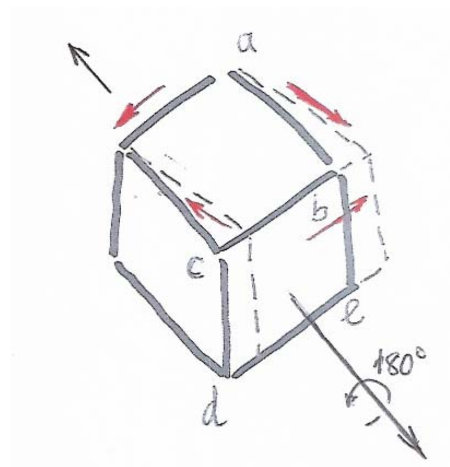
$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\sigma_x/E$$

Sketches adapted from Crandall et al.

3. <u>Do normal stresses produce shear strains</u>? Consider possibility that shear strains result from normal stress σ_x . If a shear strain was present (left, broken lines), rotation by 180^o about the x-axis would give shear strain in the opposite sense (right). However, since the material is isotropic, the stress-strain behavior should be independent of a 180^o rotation. Avoid this contradiction only if shear strain due to normal stress becomes zero.

before 180^o rotation





after 180^o rotation

Sketches adapted from Crandall et al.

$\therefore$A normal stress produces only normal strains. No shear strains produced from normal stress.

4. <u>Strains from adding a second normal component of stress, σ_y</u>.

Because of the linearity of the σ-ε relation an increment of stress produces the same increment of strain regardless of the level of stress present before the increment was later added. Therefore, the strains resulting from adding σ_y are linearly related to σ_y and are directly <u>additive</u> to the strains due to σ_x .

$$\varepsilon_y = \sigma_y/E$$

Furthermore, due to isotropy, the lateral strains ε_x and ε_z due to σ_x are <u>equal</u>.

$$\varepsilon_x = \varepsilon_z = -\nu\varepsilon_y = -\nu\sigma_y/E$$

5. Add a third normal component of the stress, σ_z.

Analogous results obtained for strains due to σ_z:

$$\varepsilon_z = \sigma_z/E$$

$$\varepsilon_x = \varepsilon_y = -\nu\varepsilon_z = -\nu\sigma_z/E$$

6. Can a <u>normal</u> strain component ε_y be due to a <u>shear</u> stress component τ_{zx} ?

A rotation would change the sign of the shear stress component τ_{zx} but the sign of the hypothetical ε_y would be unchanged. However, linearity of the material requires consistent changes of sign for a proportionality between ε_y and τ_{zx} to exist. Contradiction avoided if the normal strain ε_y is zero. Similar arguments lead to conclusion:

$\therefore$ Each shear stress component produces only its corresponding shear strain component.

7. Shear strains caused by shear stresses. **Since linearity requires proportionality between stress and strain, and isotropy requires that the constant of proportionality G (shear modulus) remain independent of orientation, we get:**

$$\gamma_{zx} = \tau_{zx}/G$$
$$\gamma_{xy} = \tau_{xy}/G$$
$$\gamma_{yz} = \tau_{yz}/G$$

Add up all the contributions from preceding steps 1-7

$\varepsilon_x = (1/E)\,[\sigma_x - \nu(\sigma_y + \sigma_z)]$

$\varepsilon_y = (1/E)\,[\sigma_y - \nu(\sigma_z + \sigma_x)]$

$\varepsilon_z = (1/E)\,[\sigma_z - \nu(\sigma_x + \sigma_y)]$

$\gamma_{xy} = \tau_{xy}/G$

$\gamma_{yz} = \tau_{yz}/G$

$\gamma_{zx} = \tau_{zx}/G$

Example of simple tension. Calculate components of stress and strain tensors.

$\sigma_x = \varepsilon_x E$; $\sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{zy} = 0$

$\varepsilon_x = (1/E)[\sigma_x - \nu(\sigma_y + \sigma_z)] =$

$= (1/E)[\sigma_x - \nu(0 + 0)] = = (1/E)\sigma_x$

$\varepsilon_y = (1/E)[\sigma_y - \nu(\sigma_x + \sigma_z)] = (1/E)[0 - \nu(\sigma_x + 0)] =$

$= - \nu\sigma_x/E$

$\varepsilon_z = (1/E)[\sigma_y - \nu(\sigma_x + \sigma_z)] =$

$= (1/E)[0 - \nu(\sigma_x + 0)] = - \nu\sigma_x/E$

Example of simple tension. Set up stress and strain tensors

stress tensor

$$\begin{matrix} \varepsilon_x E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

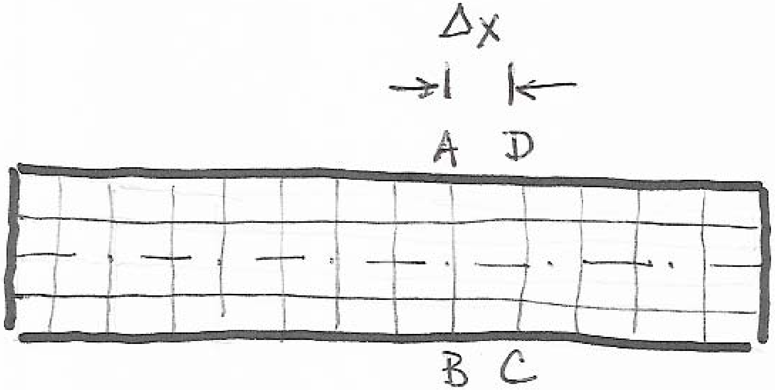
strain tensor

$$\begin{matrix} \sigma_x/E & 0 & 0 \\ 0 & -\nu\sigma_x/E & 0 \\ 0 & 0 & -\nu\sigma_x/E \end{matrix}$$

E. Pure bending of a beam.

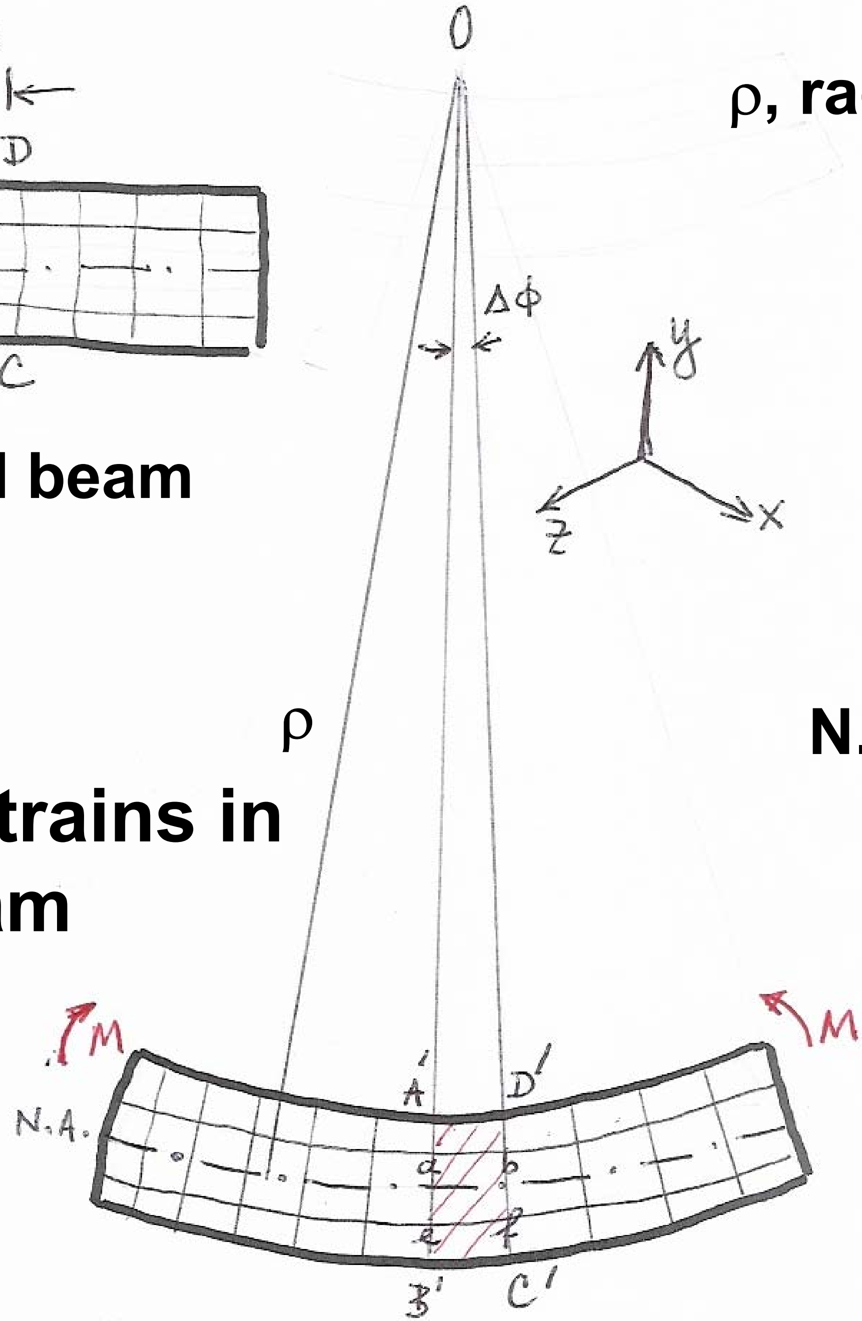
- **What is a beam? A slender member (L/W>10) that is subjected to transverse loading.**
- **In "pure bending" a beam transmits a constant bending moment. Result: In a plane of symmetry, plane cross sections remain plane. Planes do not bulge or buckle!**
- **<u>Procedure</u>:**

  1. Calculate the strains. 2. Calculate the stresses. 3. Derive the constitutive equations.



undeformed beam

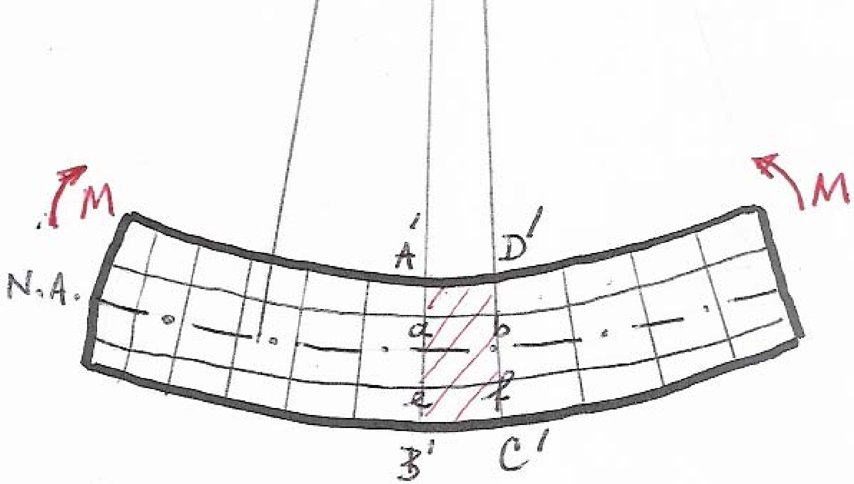
ρ, radius of curvature

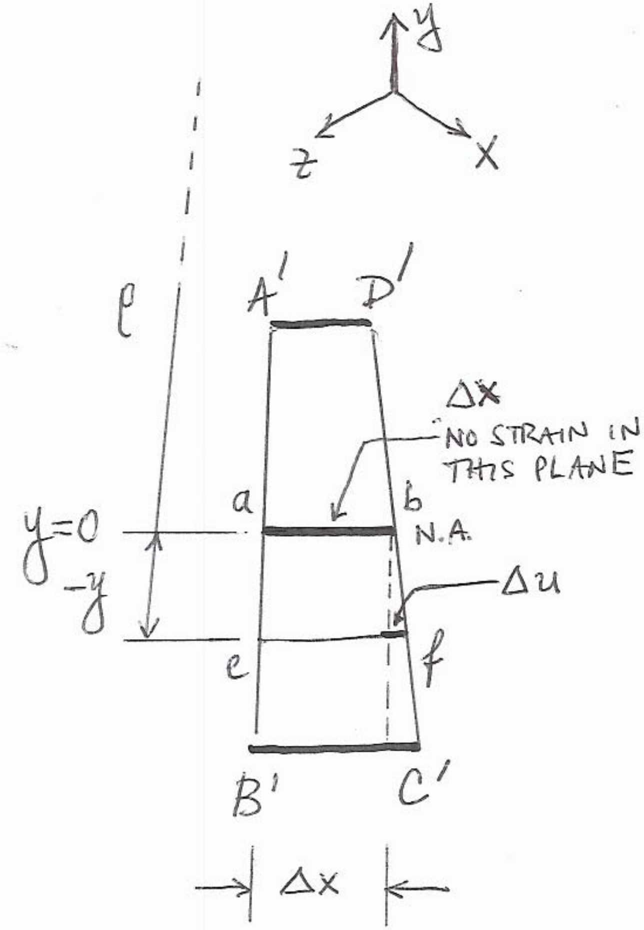


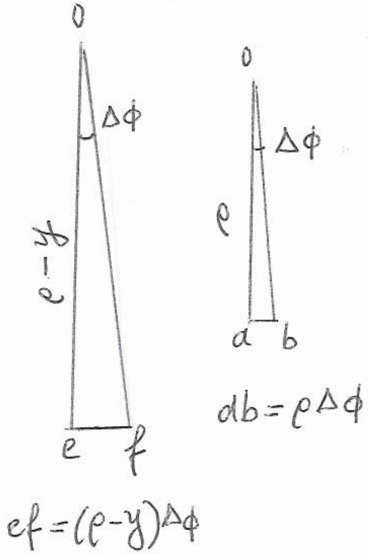
N.A., neutral axis

1. Calculate strains in deformed beam

Sketches adapted from Crandall et al.





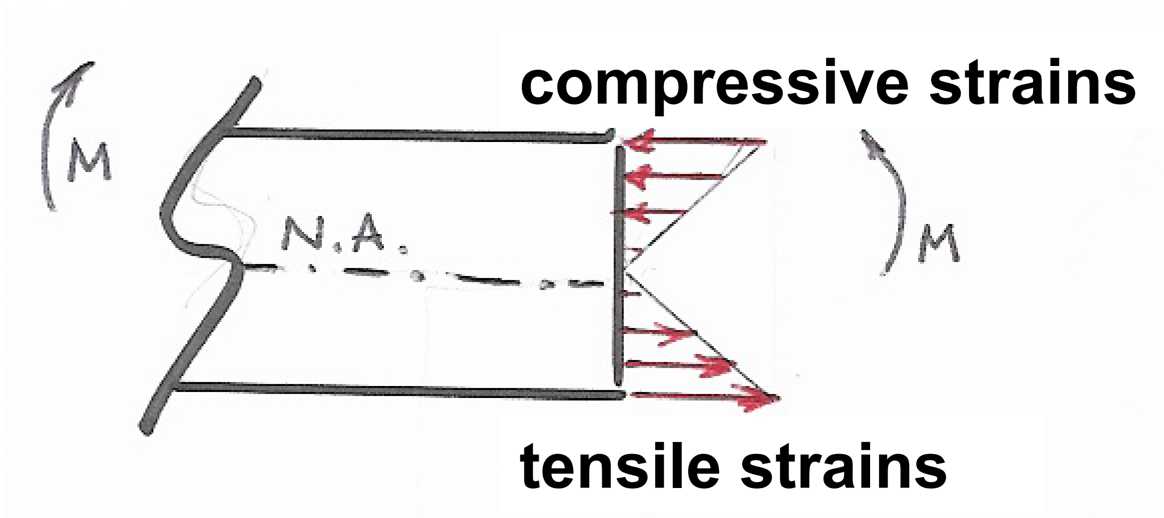


$$\epsilon_x = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \cong \frac{ef-ab}{ab} = \frac{(\rho-y)\Delta\phi - \rho\Delta\phi}{\rho\Delta\phi} = -\frac{y}{\rho}$$

∴ In pure bending, the strains increase linearly away from N.A.

Sketches adapted from Crandall et al.

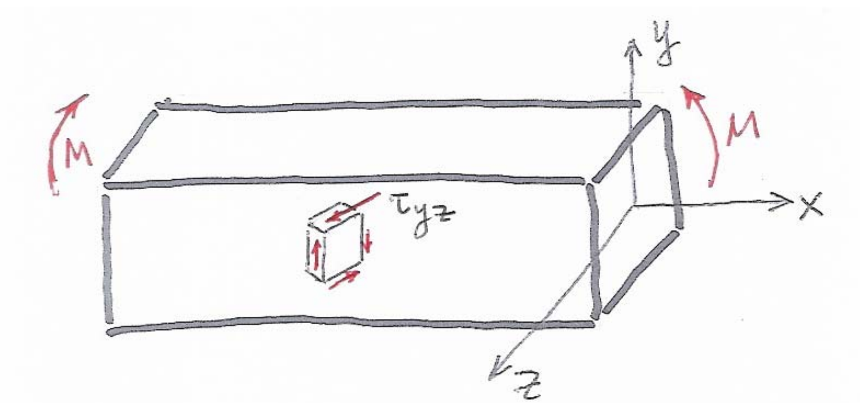
Since $\varepsilon_x = -y/\rho$, the normal strain increases linearly with -y. Strains are negative (compressive) when y>0 and positive (tensile) when y<0. The strain profile along the cross section is linear. At the neutral axis, the profile changes from tension to compression.

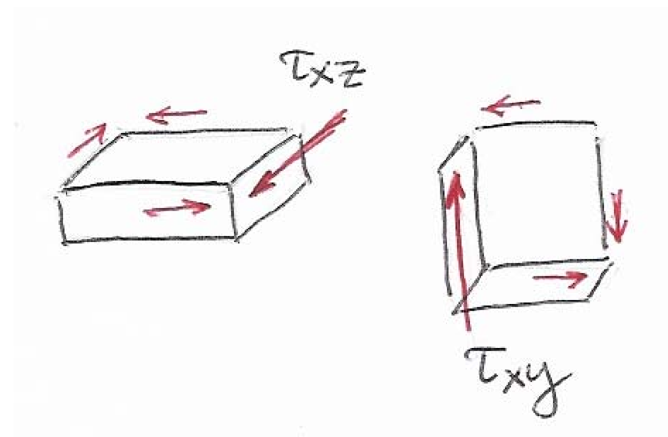


section of a beam subjected to pure bending with a bending moment M.

Since plane sections remain plane (rather than bend or buckle) in pure bending, the shear strains along the two orthogonal cross-sections of the beam (defined by planes xy and xz) must be zero.

$$\gamma_{xy} = 0$$
$$\gamma_{xz} = 0$$





Since $\gamma_{xy} = \tau_{xy}/G$, it follows that $\tau_{xy} = 0$. Also, $\tau_{xz} = 0$. At this point, there is still no information about τ_{yz}.

Sketches adapted from Crandall et al.

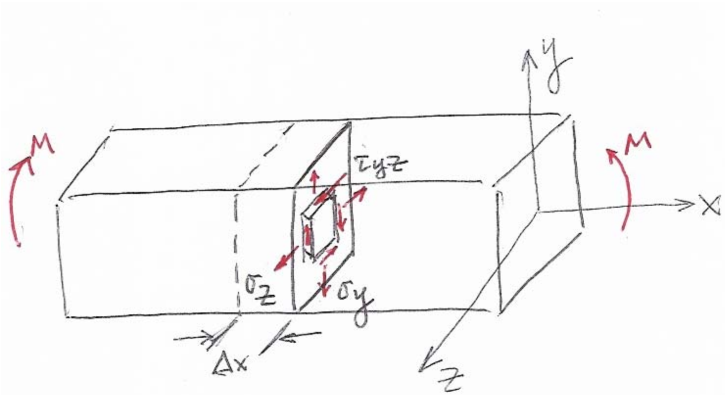
2. Calculate the stresses.

Due to the slendernesss of the beam (transverse dimensions are small compared to length) an assumption will be made about the transverse behavior. The external surfaces of an elemental slice of thickness Δx are free of normal and shear stresses. Due to slenderness assume that the stresses σ_y, σ_z and τ_{yz} shown in diagram below remain zero in the interior of the beam. We conclude that:

$$\sigma_y = \sigma_z = \tau_{yz} = 0$$

Since $\tau_{yz} = 0$ we also have $\gamma_{yz} = 0$. However, $\sigma_x \neq 0$ since $\varepsilon_x = -y/\rho$. Also

$$\varepsilon_x = (1/E)[\sigma_x - \nu(\sigma_y + \sigma_z)] =$$
$$= (1/E)[\sigma_x - \nu(0 + 0)] =$$
$$= (1/E)\sigma_x \text{ ; therefore, } \sigma_x = \varepsilon_x E =$$
$$= E(-y/\rho) = -Ey/\rho$$



Sketches adapted from Crandall et al.

3. Derive the constitutive equations for pure bending.

$\sigma_x = \varepsilon_x E = -E(y/\rho)$; $\sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{zy} = 0$

$\varepsilon_x = (1/E)[\sigma_x - \nu(\sigma_y + \sigma_z)] = (1/E)[\sigma_x - \nu(0 + 0)] =$
$= (1/E)\sigma_x = (1/E)[-E(y/\rho)] = -y/\rho$

$\varepsilon_y = (1/E)[0 - \nu(\sigma_x + 0)] = -\nu\sigma_x/E = -(\nu/E)(-yE/\rho) =$
$= \nu y/\rho$

$\varepsilon_z = (1/E)[0 - \nu(\sigma_x + 0)] = -\nu\sigma_x/E = -(\nu/E)(-yE/\rho) =$
$= \nu y/\rho$

Summary for pure bending.

In pure bending of a (slender) beam shear stresses are negligible; only tensile and compressive stresses are important. The stress and strain profiles along the cross section are linear, changing from tension to compression (at the neutral axis). Straight lines remain straight and plane cross sections remain plane.

stress tensor			strain tensor		
$-Ey/\rho$		0	0	$-y/\rho$	0	0
0	0	0	0	$\nu y/\rho$	0
0	0	0	0	0	$\nu y/\rho$

F. Torsion of cylindrical shaft.

- A slender (L/W>10) cylinder is subjected to a twisting moment.
- Mechanical power transmission over a long distance is based mostly on cylindrical shafts. This is analogous to electric power transmission over long distance via cable.
- Other examples are use of a torsion bar as a spring suspension for the front wheels of an automobile; and the torsion balance for weighing of small masses.

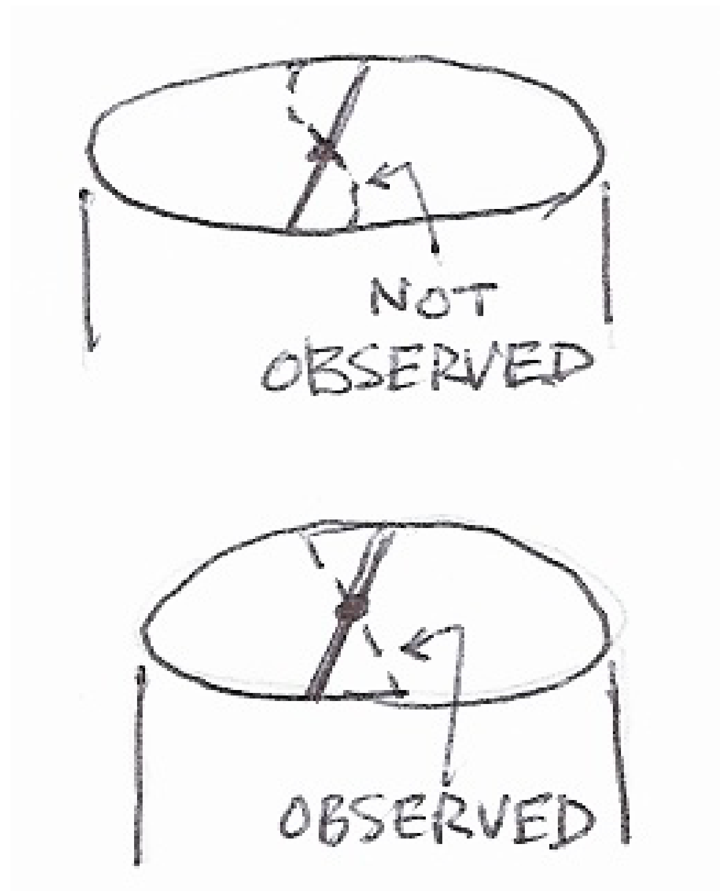
- When subjecting a slender cylinder to torsion we need to know the stresses in order to prevent damage to cylinder and be able to use cylinder over and over again.
- Use the symmetry of the cylinder to estimate the strains. Then calculate the stresses.

1. <u>Calculate the strains</u>.

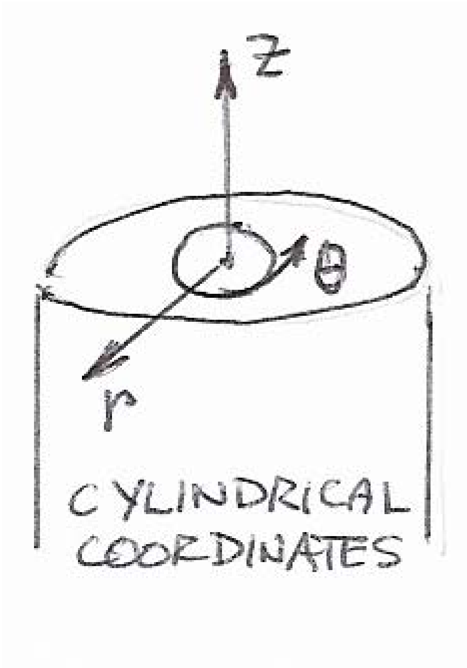
Cut a slice of the untwisted cylinder, originally with faces plane and normal to the axis of the shaft. Now cut the slice after twisting. Observe that the ends of the slice did not bulge or dish out: When a circular shaft is twisted, "plane cross sections remain plane". Also, there was no change in volume. It follows that the extensional strains are zero:

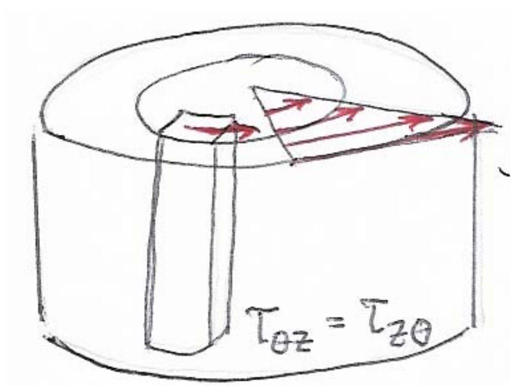
$$\varepsilon_x = \varepsilon_y = \varepsilon_z = 0$$

The original straight diameter (solid line) does not twist (broken line). "Diameters remain straight".



Sketches adapted from Crandall et al.





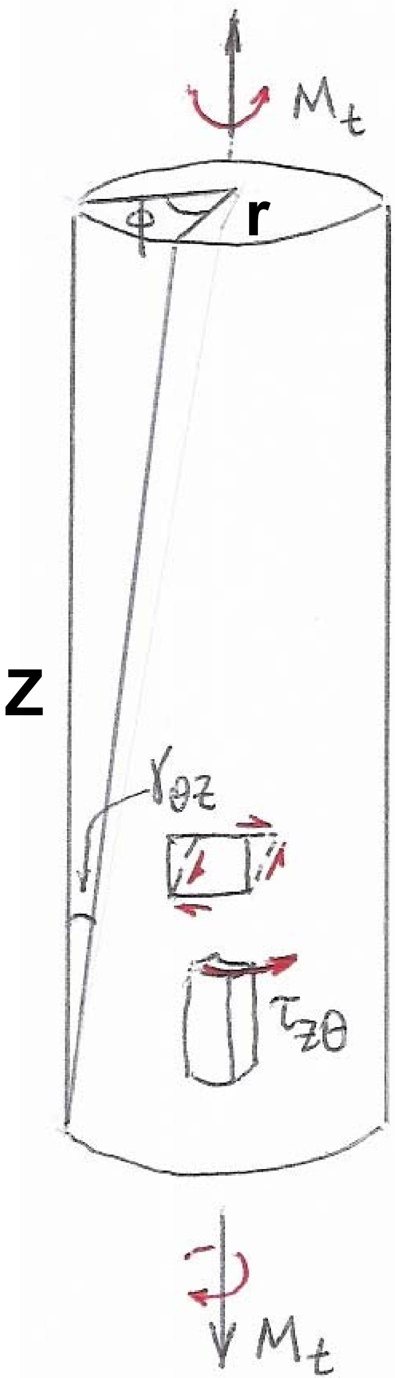
A twisted slice

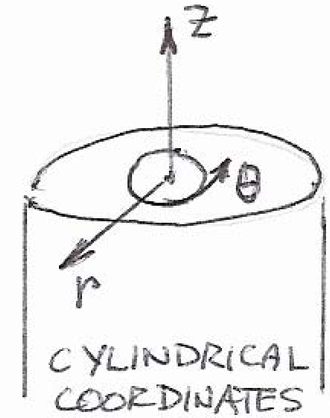
We conclude that the cross sections of the shaft remain undeformed and simply rotate with respect to one another.

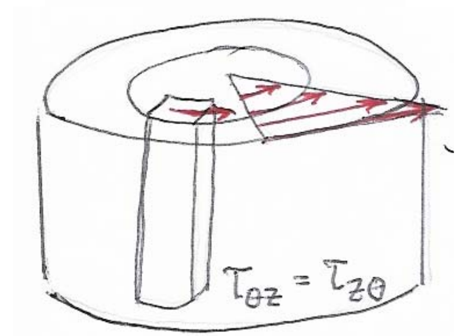
$$\gamma_{r\theta} = \gamma_{rz} = 0$$

but $\gamma_{\theta z} \neq 0$

Sketches adapted from Crandall et al.







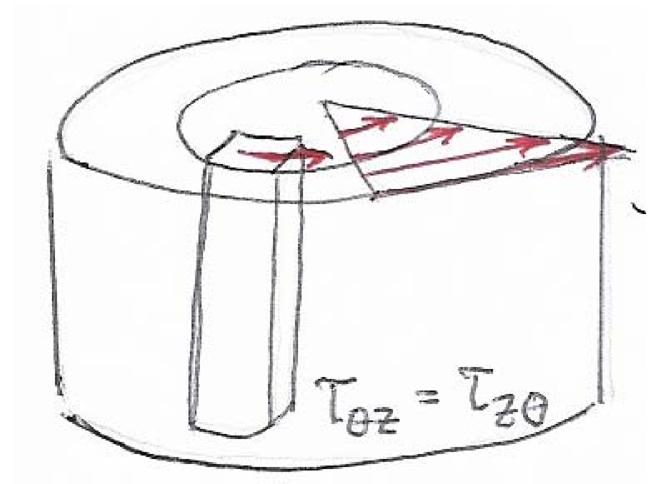
$$\gamma_{\theta z} = \lim_{\Delta z \to 0} r\Delta\phi/\Delta z = r(d\phi/dz)$$

The shear strain increases linearly with the radius

Sketches adapted from Crandall et al.

2. Calculate the stresses.

Use the values of the strains deduced above together with the constitutive equations of LEM to conclude that:



$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{r\theta} = \tau_{rz} = 0$$

$$\tau_{\theta z} = G\,\gamma_{\theta z} = Gr(d\phi/dz)$$

Sketches adapted from Crandall et al.

3. Derive the constitutive equations for pure torsion (use cylindrical coordinates).

$\sigma_r = \sigma_\theta = \sigma_z = \tau_{r\theta} = \tau_{rz} = 0$

$\tau_{\theta z} = G\gamma_{\theta z} = Gr(d\phi/dz)$

$\varepsilon_r = (1/E)[\sigma_r - \nu(\sigma_\theta + \sigma_z)] = 0$

$\varepsilon_\theta = \varepsilon_z = \gamma_{r\theta} = \gamma_{rz} = 0$

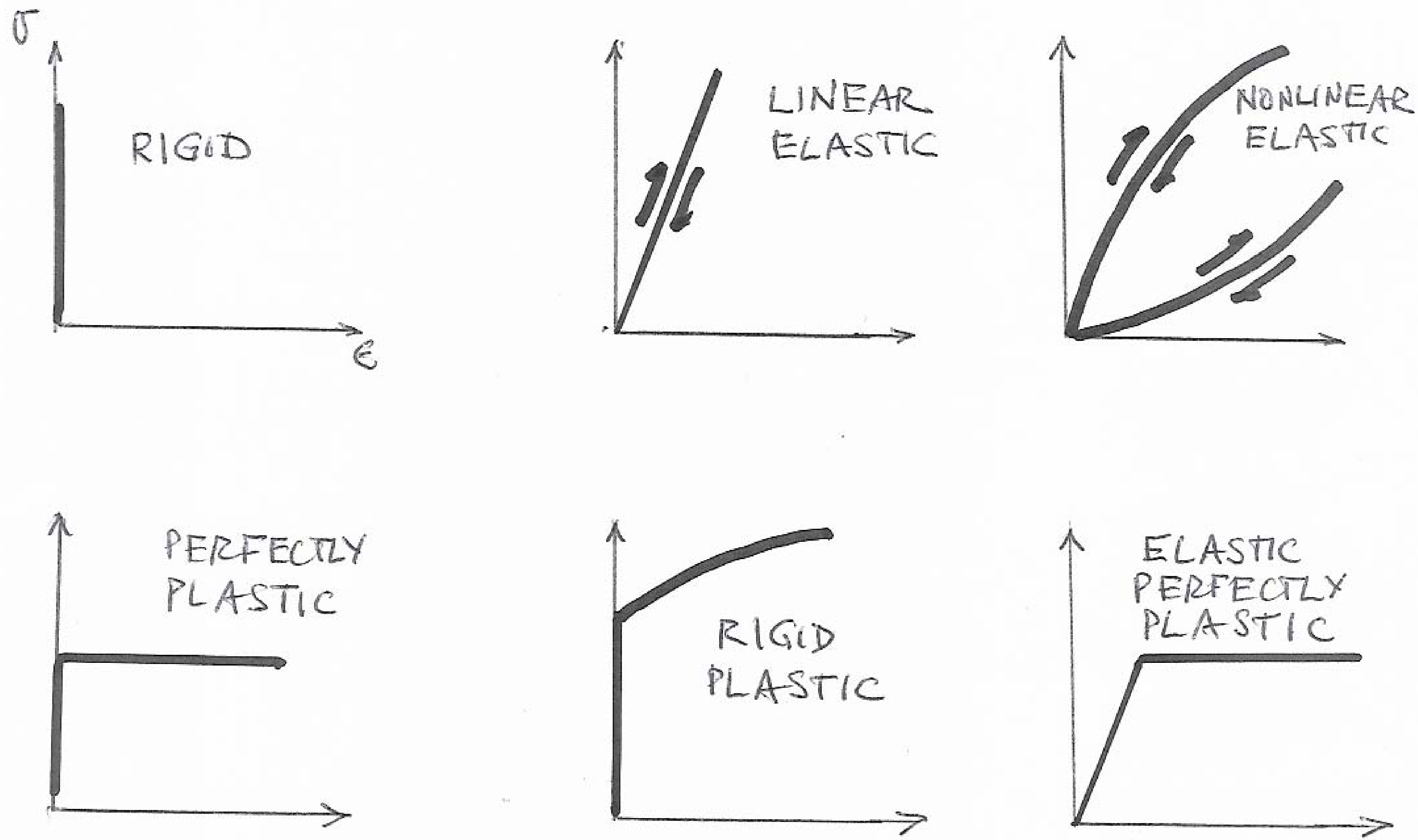
$\gamma_{\theta z} = r(d\phi/dz)$

Summary for pure torsion.

In pure torsion of a slender cylinder (shaft) plane cross sections remain plane and diameters remain straight. Normal and shear strains are negligible---except for the shear stress causing rotation of plane cross sections around the shaft axis which is nonzero. The shear stress and strain profiles are linear along (increase with) the radius.

<u>stress tensor</u>			<u>strain tensor</u>		
0	0	0	0	0	0
0	0	$G\gamma_{\theta z}$	0	0	$\gamma_{\theta z}$
0	$G\gamma_{\theta z}$	0	0	$\gamma_{\theta z}$	0

Various types of mechanical behavior



ERRORS

1. In derivation of constitutive equations, step 5, the equation should be:

   $\varepsilon_x = \varepsilon_y = -\nu\varepsilon_z = -\nu\sigma_z/E$

2. In "Example of simple tension", the strain tensor should be:

$$\begin{matrix} \sigma_x/E & 0 & 0 \\ 0 & -\nu\sigma_x/E & 0 \\ 0 & 0 & -\nu\sigma_x/E \end{matrix}$$

3. Femoral bone, delete "No change in shape".