Harvard-MIT Division of Health Sciences and Technology
HST.523J: Cell-Matrix Mechanics
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Linear viscoelastic behavior

1. The constitutive equation depends on load history.
2. Diagnostic testing for time dependence.
3. Mechanical models.
4. Mechanical memory.
5. The Boltzmann equation.
6. Weeping memory of cartilage.

1. The constitutive equation depends on the load history.

Time-dependent behavior of tissues

- Weak time dependence of mechanical behavior in tendon, ligaments and bone. Stronger time dependence in skin, and cartilage.

- Can one still use elastic theory anyway

  or is the predicted behavior going to be grossly wrong?

- Need a diagnostic test for time dependence: Elastic or viscoelastic?

Study of torsional displacement of skin in vivo

Photo removed for copyright reasons.

Barbenel et al.

Stress relaxation of skin: twist skin and measure resulting time-dependent torque

Diagrams removed for copyright reasons.

disc and guard ring

Torque relaxation from each displace-ment cycle

Barbenel et al.

Graph “Stress relaxation behavior of skin...”
removed for copyright reasons.

Barbenel et al.

Stress relaxation study of the medial collateral ligament shows a clear drop in the stress measured at various constant strain levels. Notice use of logarithmic time in abscissa.

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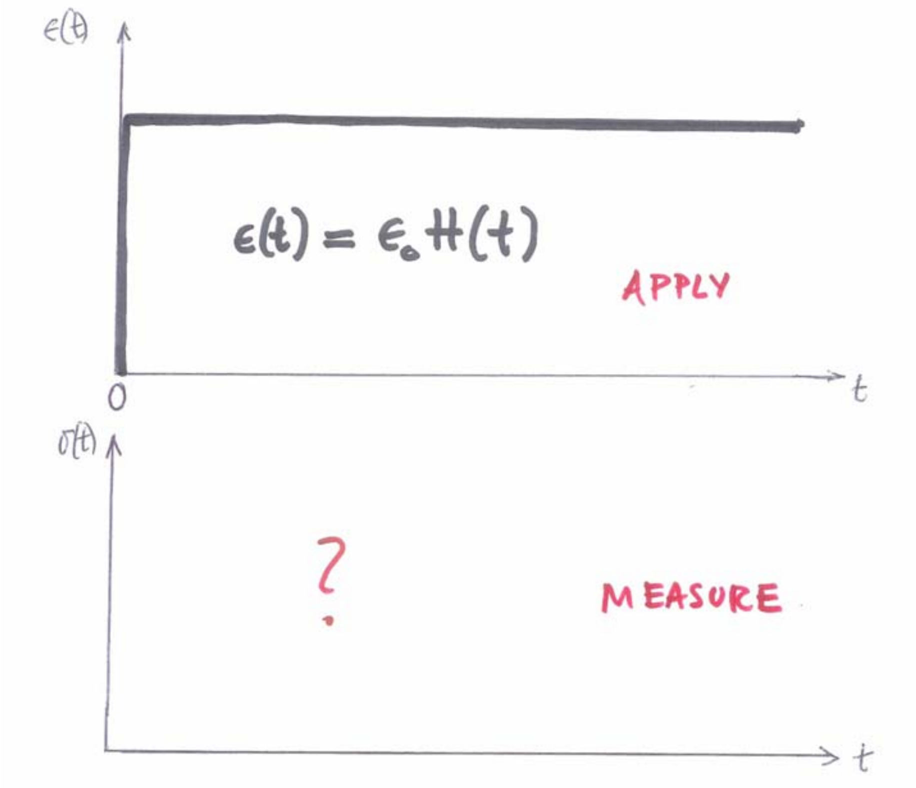
Provenzano et al., 2001

Stress relaxation at constant strain

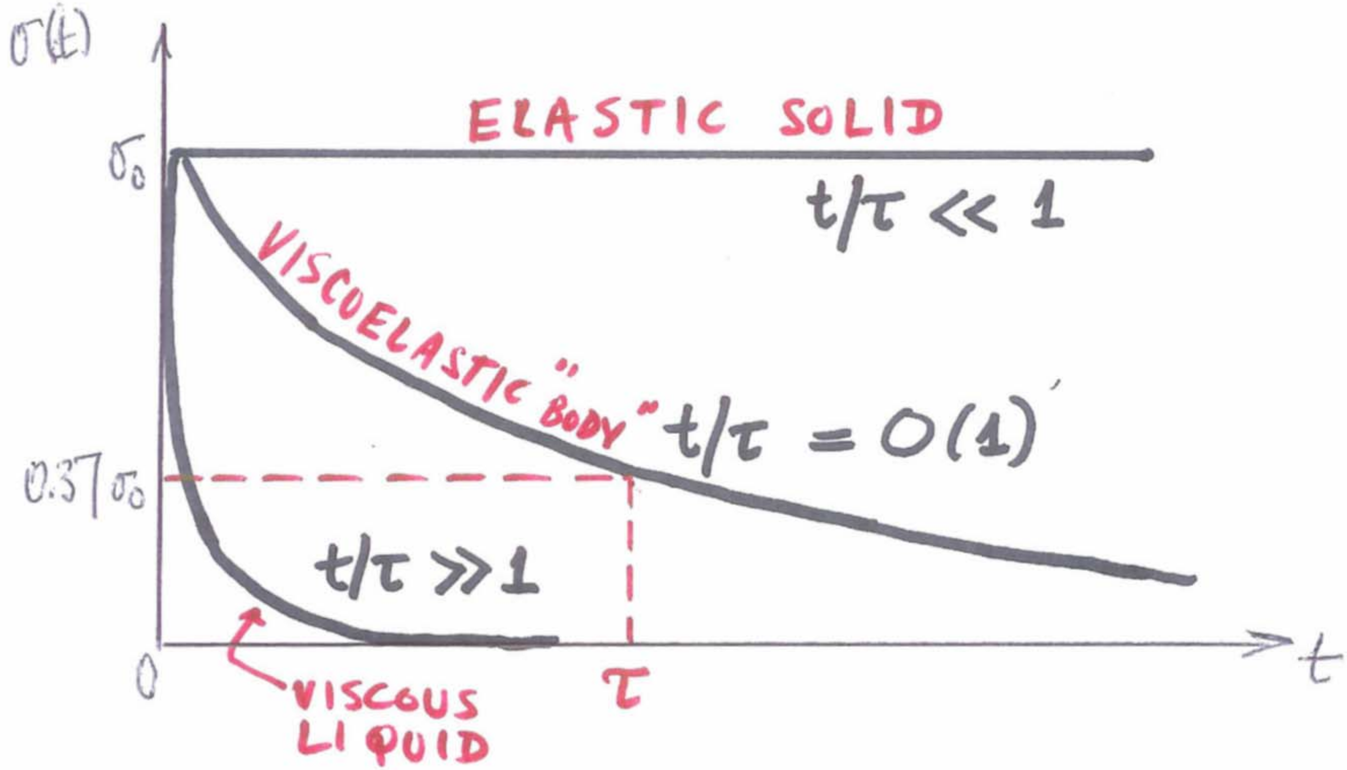
- **Test sequence: Suddenly apply a <u>simple</u> deformation and measure the time-dependent relaxation in the stress.**
- **Tensor notation typically not required for the simple stress fields used in testing. Below apply $\varepsilon_x(t) = \varepsilon\ (t)$ and measure $\sigma_x(t) = \sigma$(t.**
- **Strain applied suddenly to a constant level ε_0. Use the Heaviside function to describe "switching on" or "off" of the strain.**

  $H(t) = 0$ at $t < 0$ and $H(t) = 1$ at $t \geq 0$.
- **Strain "history" shows what happened to the strain: $\varepsilon(t) = \varepsilon_0 H(t)$.**

<u>Stress relaxation test</u>. Apply constant strain suddenly and maintain it at constant level while measuring the time-dependent stress. A material relaxes at a rate set by its intrinsic time constant, the relaxation time, τ.



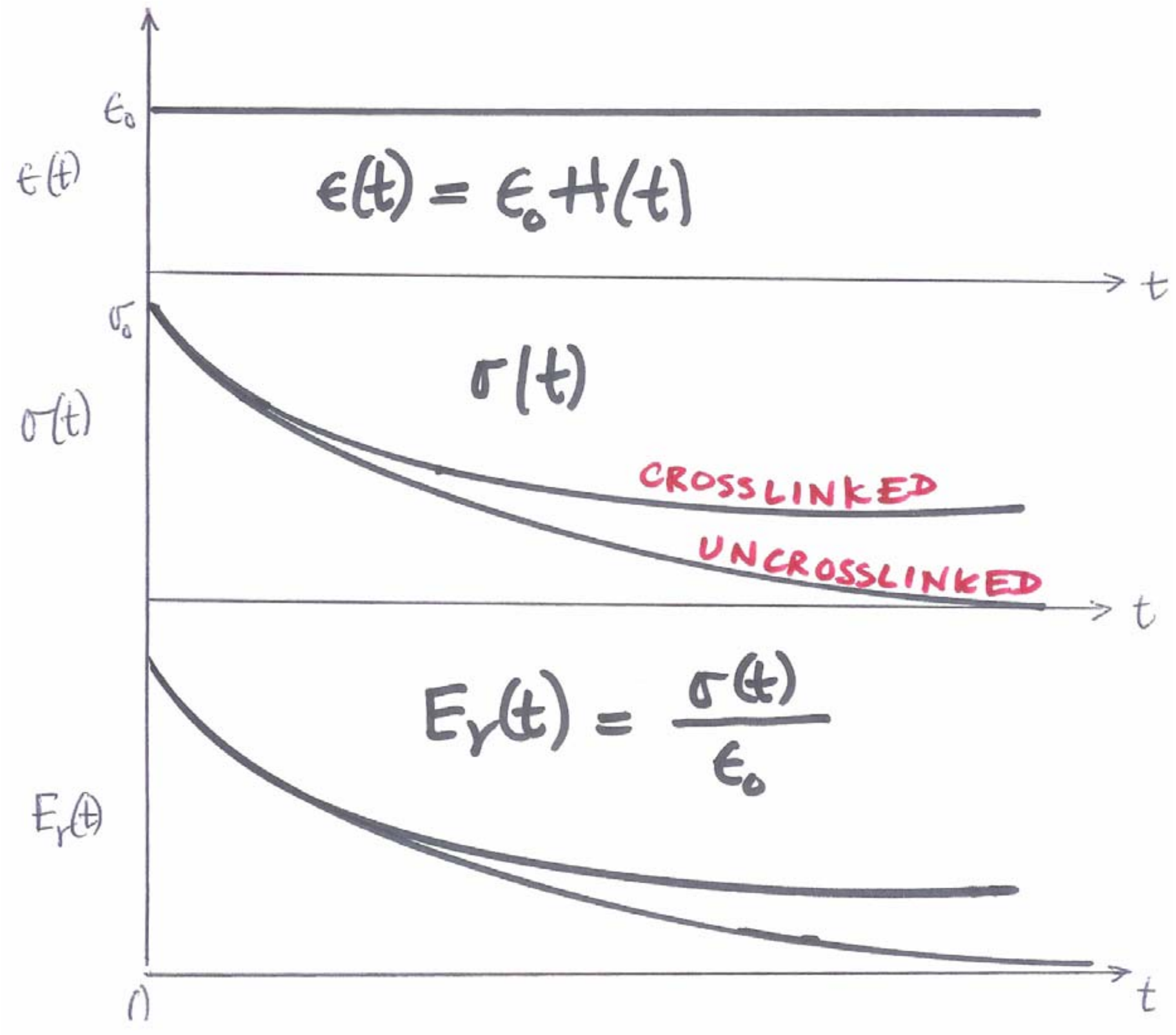
Three classes of materials described below. Elastic solids "never" relax ($\tau \to \infty$). Viscoelastic bodies relax during the (arbitrary) experimental timescale, t (τ = intermed.). Viscous liquids relax "very fast" ($\tau \to 0$). Use dimensionless (t/τ) to diagnose "rheological" (flow) state of unknown material.



Define the stress relaxation modulus, $E_r(t)$

- **Apply $\varepsilon_0 H(t)$.**
- **Measure $\sigma(t)$.**
- **Define $E_r(t) = \sigma(t)/\varepsilon_0$.**
- **Plot data along an axis of logarithmic time since relaxation often proceeds over very long time (20 decades for amorphous polymers).**
- **Stress eventually relaxes to zero when the material (typically a polymer or tissue) is uncrosslinked. If crosslinked, stress relaxes to constant value (true of skin data).**

Definition of $E_r(t)$



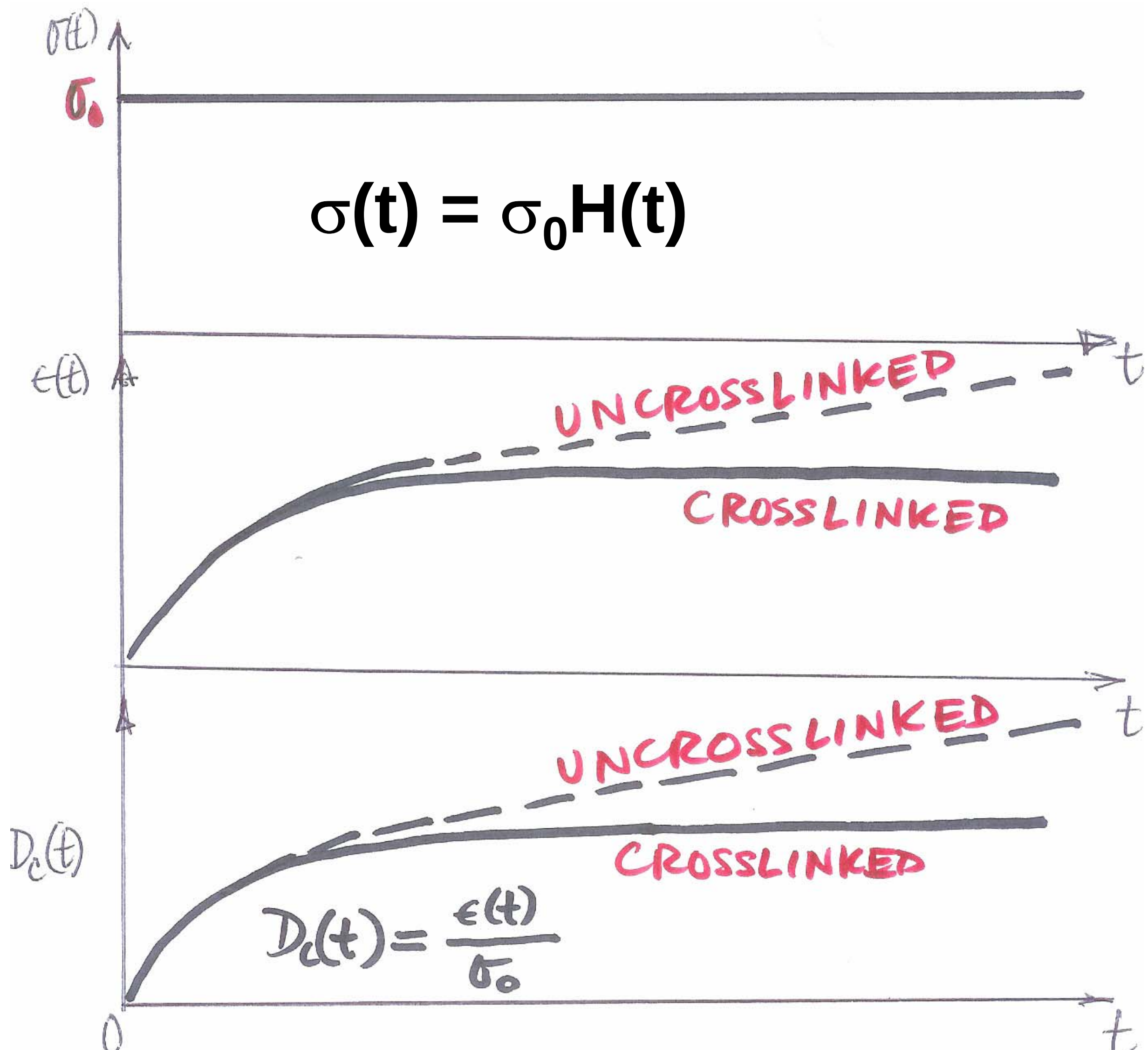
Creep at constant stress

- **This test is a mirror image of stress relaxation at constant strain.**
- **Apply suddenly a constant load, σ_0:**

$$\sigma(t) = \sigma_0 H(t).$$

- **Measure the resulting time-dependent strain, $\varepsilon(t)$.**
- **With crosslinked materials, the strain eventually reached an asymptote. If material is uncrosslinked, the strain rises to very high values until the material becomes mechanically unstable: $\varepsilon(t) = 0$.**

Define the creep compliance: $D_c(t) = \varepsilon(t)/\sigma_0$



The uterine cervix before, during and after parturition (childbirth)

- **To be born, the fetus must pass through the cervix, a canal made up of rather stiff tissues, that hardly allows easy passage of the head (the largest obstacle to easy passage).**
- **During parturition, the cervix undergoes a dramatic change in its mechanical behavior, the result of degradation of the stiff collagen fiber network. It changes from a solid to a liquid viscoelastic tissue, similar to that undergone when a crosslinked polymer network becomes degraded.**

Uterine cervix. Deformation following application of constant load (rat). During parturition (birth process) cylindrical cervix can deform indefinitely due to reversible loss of crosslinking

From Harkness and Harkness

deformation

during parturition

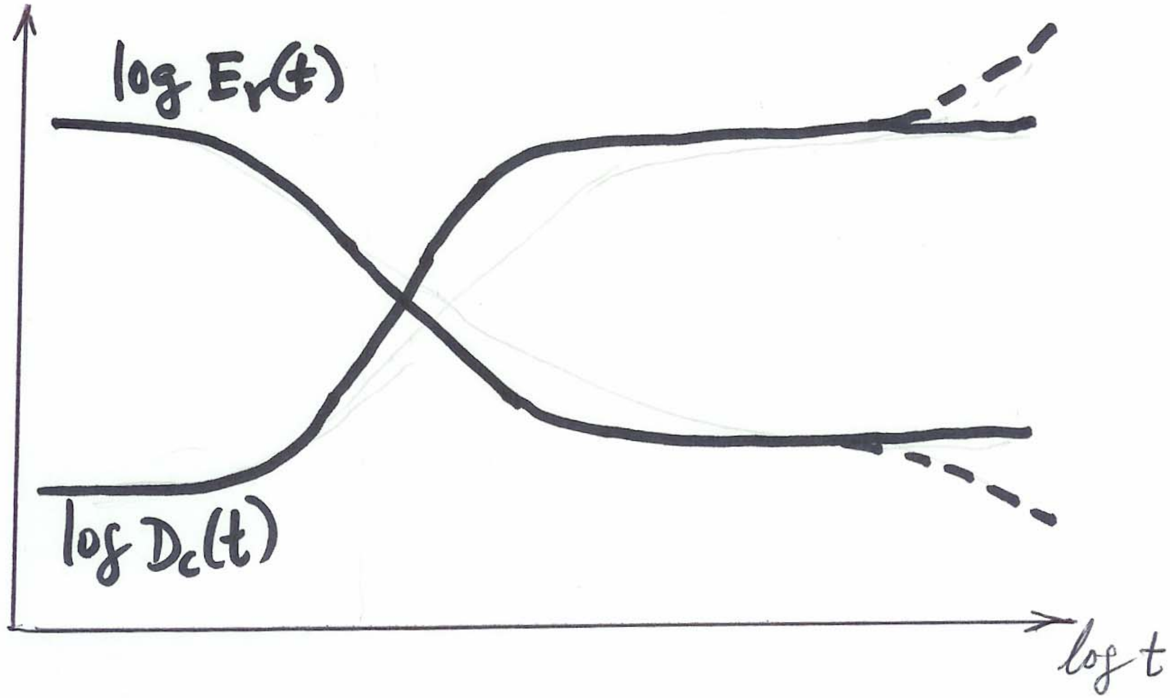
Graph removed for copyright reasons.

after

before

time

The complete functions



Data show the evolution of $E_r(t)$ and $D_c(t)$ for an amorphous synthetic polymer over about 20 decades of logarithmic time. These "master curves" can be obtained in a morning using time-temperature superposition.

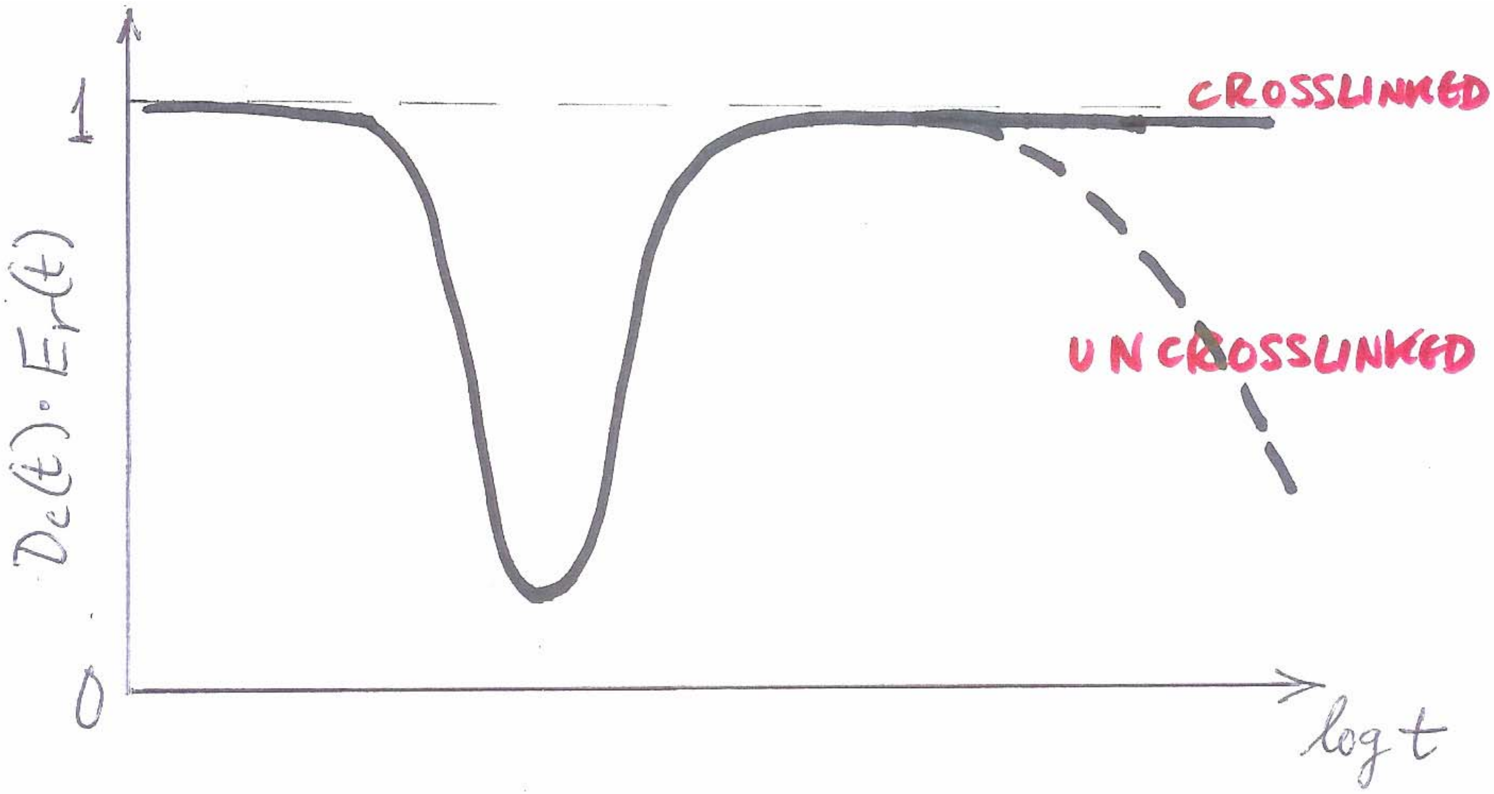
2. Diagnostic testing for time dependence.

Two simple diagnostic tests:

- **Define experimental time, t, and relaxation time, τ.**

  <u>Test 1</u>: viscoelastic if $t/\tau \cong 1$; elastic if $t/\tau \ll 1$.
- **Creep and stress relaxation experiments. Define time-dependent creep compliance, $D_c(t)$, and time-dependent stress relaxation modulus, $E_r(t)$.**
- **<u>Test 2</u>: viscoelastic if $D_c(t) \cdot E_r(t) < 1$; elastic if $D_c(t) \cdot E_r(t) \cong 1$.**

When $D_c(t) \cdot E_r(t) < 1$ the behavior is time dependent. Elastic behavior when $D_c(t) \cdot E_r(t) = 1$. This is consistent with $D \cdot E = 1$ (by definition for elastic behavior).



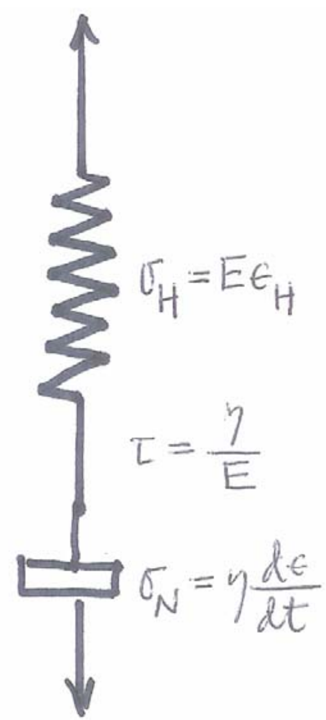
3. Mechanical models of viscoelastic behavior.

Model stress relaxation behavior using a spring and dashpot in series

- In series arrangement of spring (elastic) and dashpot (viscous).
- The spring has an elastic (Hookean) modulus

  $E = \sigma_H/\varepsilon_H$.
- The dashpot has a Newtonian viscosity

  $\eta = \sigma_N/(d\varepsilon_N/dt)$
- The stresses are equal;
- the strains are additive.



<u>Maxwell model</u>. Derivation of equation of motion (constitutive equation for stress relaxation)

$$\sigma = \sigma_H = \sigma_N$$

$$\varepsilon = \varepsilon_H + \varepsilon_N$$

$$\frac{d\varepsilon}{dt} = \frac{d\varepsilon_H}{dt} + \frac{d\varepsilon_N}{dt} = \frac{1}{\mathrm{E}}\frac{d\sigma}{dt} + \frac{\sigma}{\eta}$$

IN STRESS RELAXATION EXPERIMENTS:

$$\frac{d\varepsilon}{dt} = 0$$

$$\frac{d\sigma}{dt} = \frac{\mathrm{E}\sigma}{\eta}$$

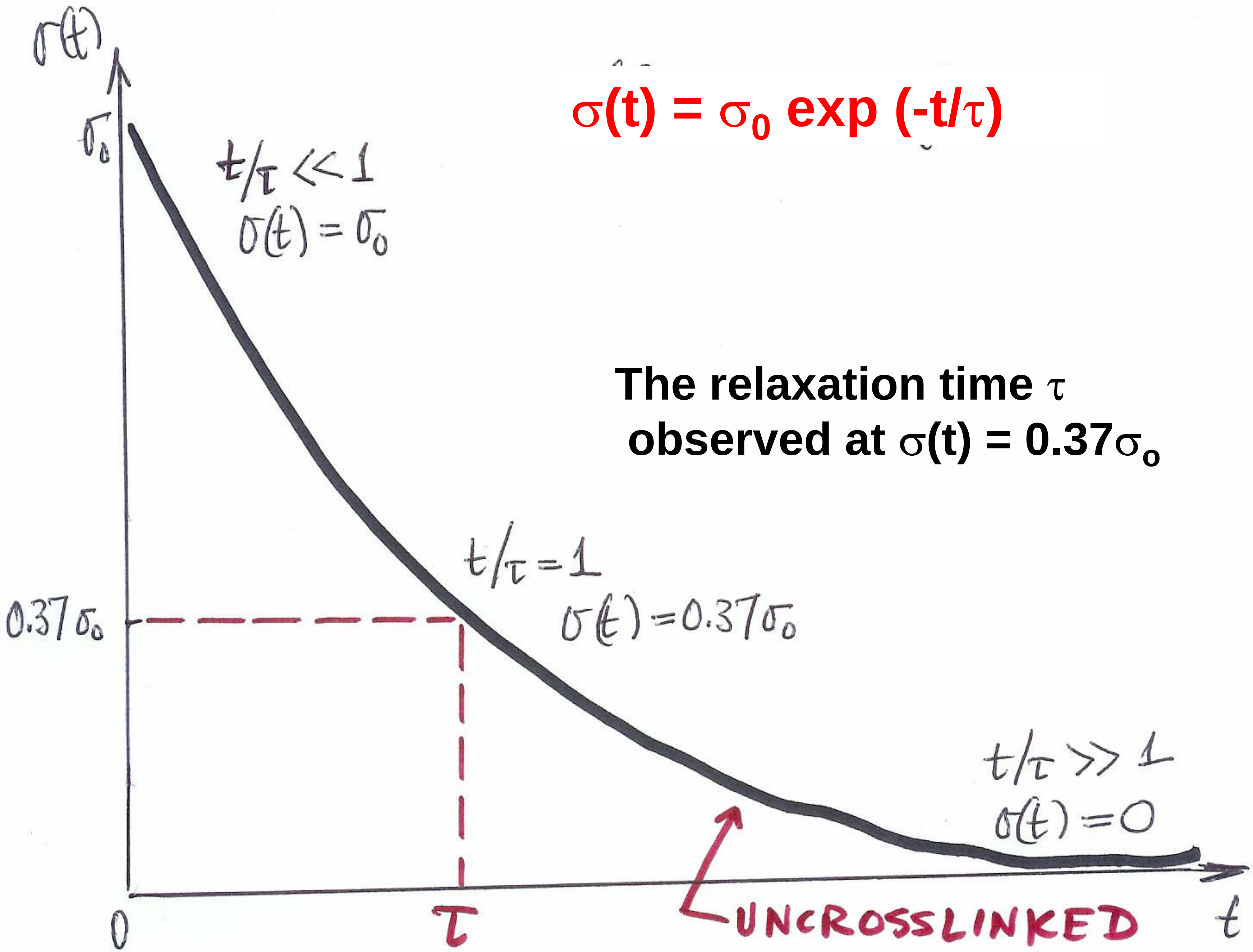
$$\int_{\sigma_O}^{\sigma(t)} \frac{d\sigma}{\sigma} = -\frac{\mathrm{E}}{\eta}\int_{o}^{t} dt$$

$$\ln\frac{\sigma(t)}{\sigma_O} = -\frac{\mathrm{E}t}{\eta}$$

$$\sigma(t) = \sigma_O \exp\left(-\frac{\mathrm{E}t}{\eta}\right) = \sigma_O \exp\left(-t/\tau\right)$$

$\sigma(t) = \sigma_0 \exp(-t/\tau)$

The relaxation time τ observed at $\sigma(t) = 0.37\sigma_o$



Model creep at constant stress behavior using a spring and dashpot in parallel

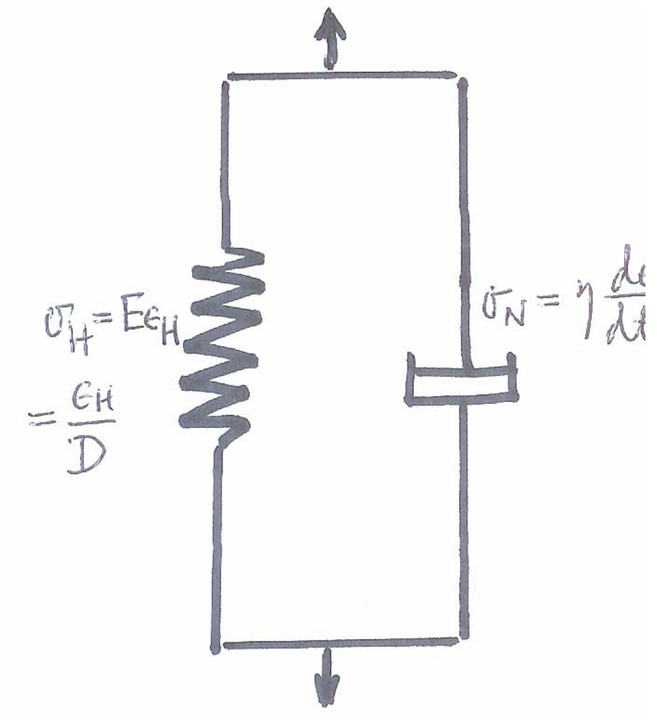
- In parallel arrangement of spring (elastic) and dashpot (viscous).
- The spring has an elastic (Hookean) modulus

$$E = \sigma_H/\varepsilon_H.$$

- The dashpot has a Newtonian viscosity

$$\eta = \sigma_N/(d\varepsilon_N/dt)$$

- The stresses are additive; the strains are equal.



<u>Kelvin-Voigt model</u>. Derivation of equation of motion (constitutive equation for creep at constant stress)

$$\sigma = \sigma_H + \sigma_N$$

$$\varepsilon = \varepsilon_H = \varepsilon_N$$

$$\sigma(t) = \sigma_O$$

$$\sigma_O = \frac{\varepsilon_H}{D} + \eta \frac{d\varepsilon_{\mathrm{H}}}{dt} = \frac{\varepsilon}{D} + \eta \frac{d\varepsilon}{dt}$$

$$\frac{d\varepsilon}{dt} = \frac{\sigma_O}{\eta} - \frac{\varepsilon}{\eta D}$$

$$d\varepsilon = \frac{\sigma_O}{\eta} dt - \frac{\varepsilon}{\eta D} dt = \left(\frac{\sigma_O}{\eta} - \frac{\varepsilon}{\eta D} \right) dt$$

$$\frac{d\varepsilon}{\dfrac{\sigma_O}{\eta} - \dfrac{\varepsilon}{\eta D}} = dt$$

$$\int_O^{\varepsilon(t)} \frac{d_{\mathrm{E}}}{\dfrac{\sigma_O}{\eta} - \dfrac{\mathrm{E}}{\eta D}} = \int_O^t dt$$

INTEGRAL OF FORM $$\int \frac{dx}{a - bx} = -\frac{1}{b} \ln(a - bx)$$

INTEGRAL OF FORM $\int \frac{dx}{a-bx} = -\frac{1}{b}\ln(a-bx)$

$$\left[-\eta D \ln\left(\frac{\sigma_O}{\eta} - \frac{\varepsilon}{\eta D}\right)\right]_O^{\varepsilon(t)} = [t]_o^t$$

$$-\eta D \ln\left(\frac{\sigma_O}{\eta} - \frac{\varepsilon(t)}{\eta D}\right) + \eta D \ln\left(\frac{\sigma_O}{\eta} - O\right) = t$$

$$-\eta D\left[\ln\left(\frac{\sigma_o}{\eta}-\frac{\varepsilon(t)}{\eta D}\right)-\ln\left(\frac{\sigma_o}{\eta}\right)\right]=t$$

$$\ln\left(\frac{\sigma_o}{\eta}-\frac{\varepsilon(t)}{\eta D}\right)-\ln\left(\frac{\sigma_o}{\eta}\right)=-\frac{t}{\eta D}$$

$$\ln\left[\frac{\frac{\sigma_o}{\eta}-\frac{\varepsilon(t)}{\eta D}}{\frac{\sigma_o}{\eta}}\right]=-\frac{t}{\eta D}$$

$$\frac{\frac{\sigma_o}{\eta}-\frac{\varepsilon(t)}{\eta D}}{\frac{\sigma_o}{\eta}}=\exp\left(-t/\eta D\right)$$

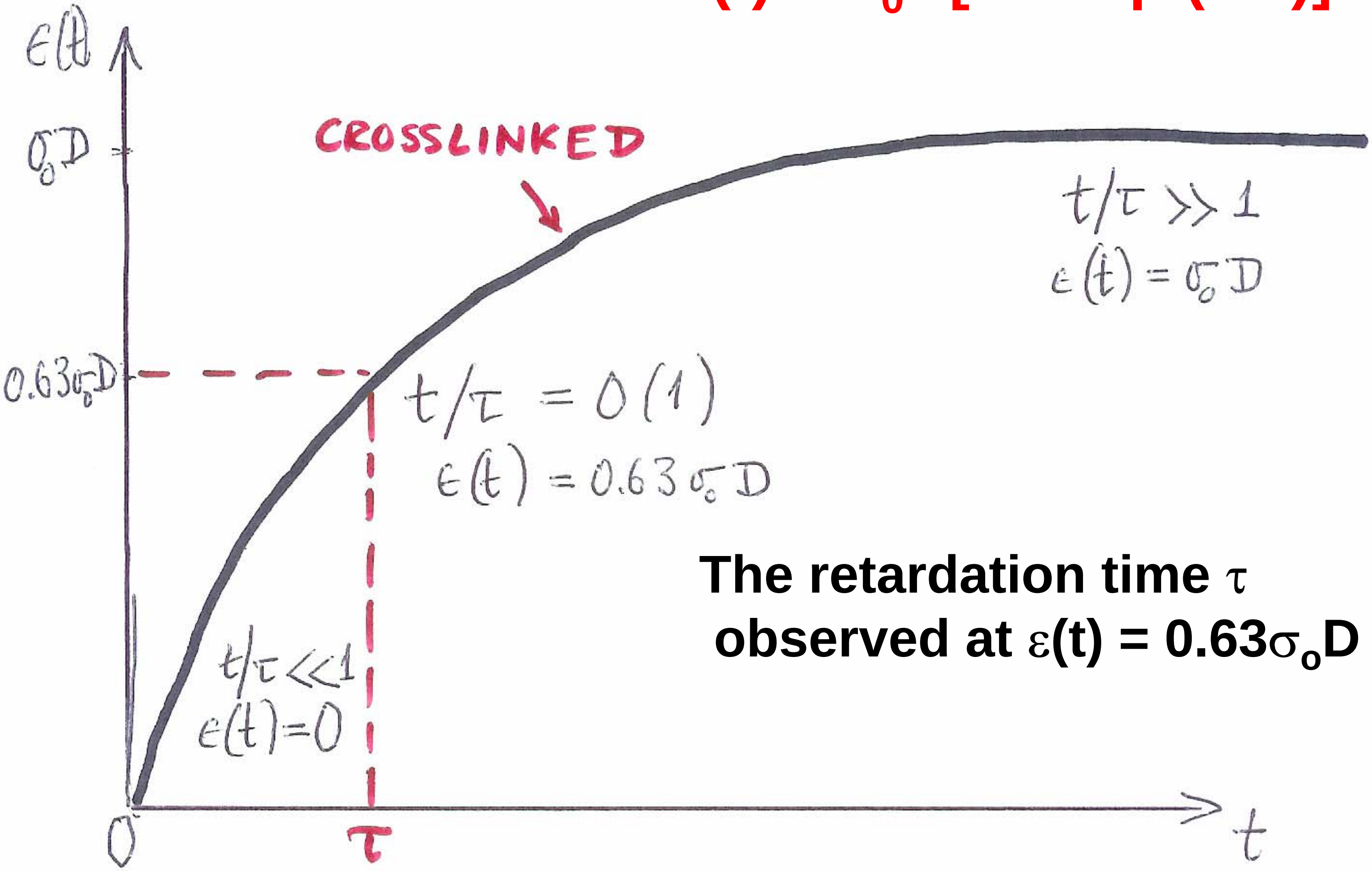
$$-\frac{\varepsilon(t)}{\eta D}=-\frac{\sigma_o}{\eta}+\frac{\sigma_o}{\eta}\exp\left(-t/\eta D\right)$$

$$\varepsilon(t)=\sigma_o D-\sigma_o D\exp\left(-t/\eta D\right)$$

$$\varepsilon(t)=\sigma_o D\left[1-\exp\left(-t/\eta D\right)\right]$$

$$\varepsilon(t)=\sigma_o D\left[1-\exp\left(-t/\tau\right)\right]$$

$$\varepsilon(t) = \sigma_0 D[1 - \exp(-t/\tau)]$$



The retardation time τ observed at $\varepsilon(t) = 0.63\sigma_o D$

4. Mechanical memory.

Mechanical memory of tissues

- Do tissues recover their original shape when released from a load? Find out by developing a general criterion for mechanical memory.
- The constitutive equation for a linear viscoelastic body is written as an nth order, linear differential equation with constant coefficients.
- The related homogeneous equation is obtained by setting the stress and all its derivatives equal to zero, corresponding to sudden release of the load. (Roots of the characteristic equation. Poles.)
- A linear viscoelastic body recovers completely following removal of the load if the general solution of the homogeneous equation approaches zero as time becomes very long.

The general constitutive equation.

$$a_r \frac{d^n \varepsilon}{dt^n} + \ldots + a_2 \frac{d^2 \varepsilon}{dt^2} + a_1 \frac{d\varepsilon}{dt} + a_o \varepsilon =$$

$$= b_m \frac{d^m \sigma}{dt^m} + \ldots + b_2 \frac{d^2 \sigma}{dt^2} + b_1 \frac{d\sigma}{dt} + b_o \sigma$$

HOMOGENEOUS EQUATION

$$a_n \frac{d^n \varepsilon}{dt^n} + \ldots + a_2 \frac{d^2 \varepsilon}{dt^2} + a_1 \frac{d\varepsilon}{dt} + a_o \varepsilon = 0$$

COMPLETE MEMORY

$$\ell im \varepsilon_h(t) = 0$$
$$t \to \infty$$

INCOMPLETE MEMORY

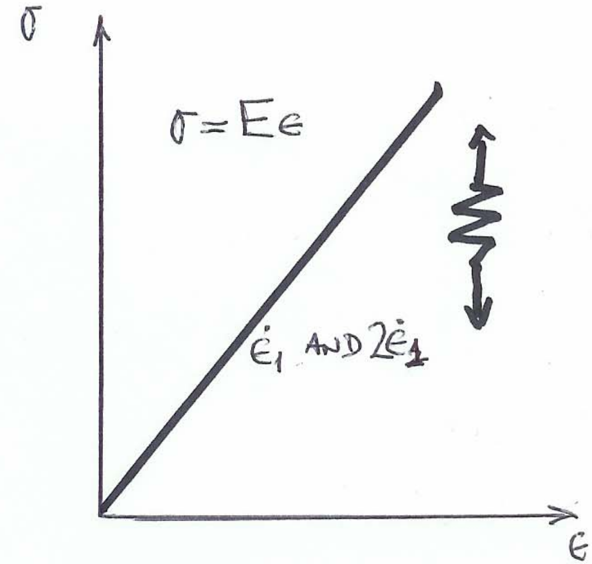
$$\ell im \varepsilon_h(t) = cons\tan t$$
$$t \to \infty$$

Diagnostic tests for rheological behavior give information about memory

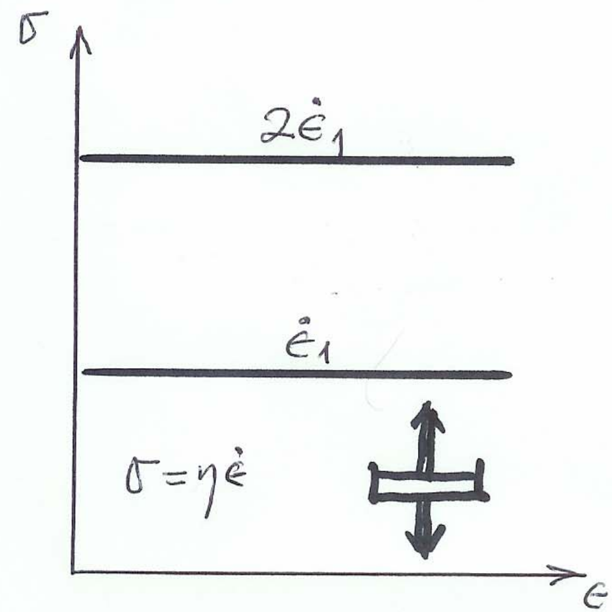
- **Use an imaginary or real Instron machine to identify the rheological nature of an unknown body. Is it a solid? a liquid? How rapidly does it relax?**
- **Stretch the body at constant extension rate, $d\varepsilon/dt = \dot{\varepsilon}$ = constant. For example, the specimen of a rigid plastic may be stretched at 12 inches (1 foot) per sec., about the rate at which we often stretch specimens in our hands.**
- **The shape of the resulting stress-strain curve gives out the identity of the unknown.**

Diagnostic tests for rheological behavior

An elastic solid is a Hookean spring with $\sigma = E\varepsilon$



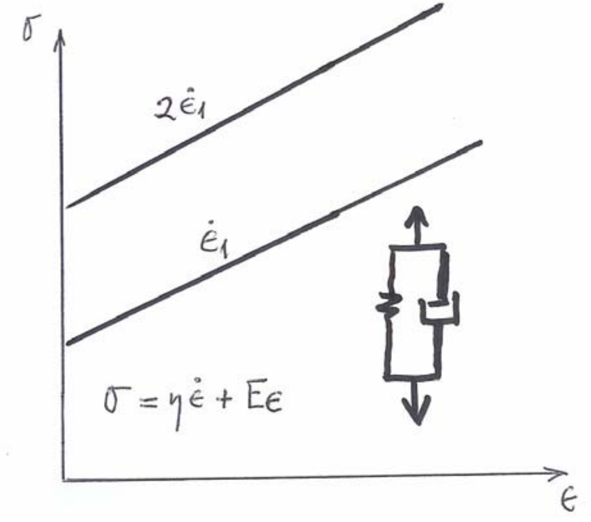
A Newtonian viscous liquid with $\sigma = E\dot{\varepsilon}$



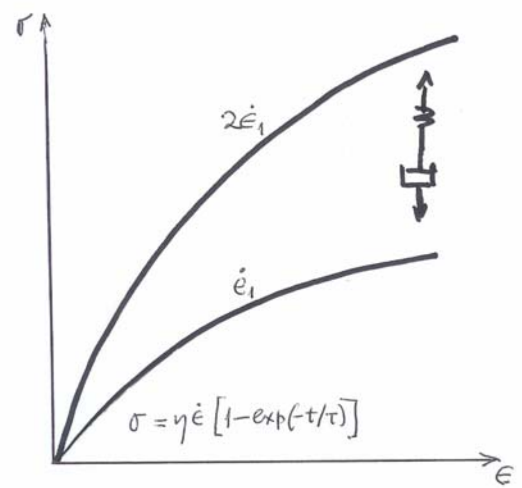
NB. Liquid supports only <u>shear</u> stress, τ. Symbol σ used here for uniformity of presentation.

(cont.) <u>Diagnostic tests for rheological behavior.</u>

A Kelvin-Voigt body has a yield stress and will be classified as a "viscoelastic solid".



A Maxwell body has no yield stress. Classified as a "viscoelastic liquid".



Which of these rheological bodies shows recovery from deformation

when the load has been removed?

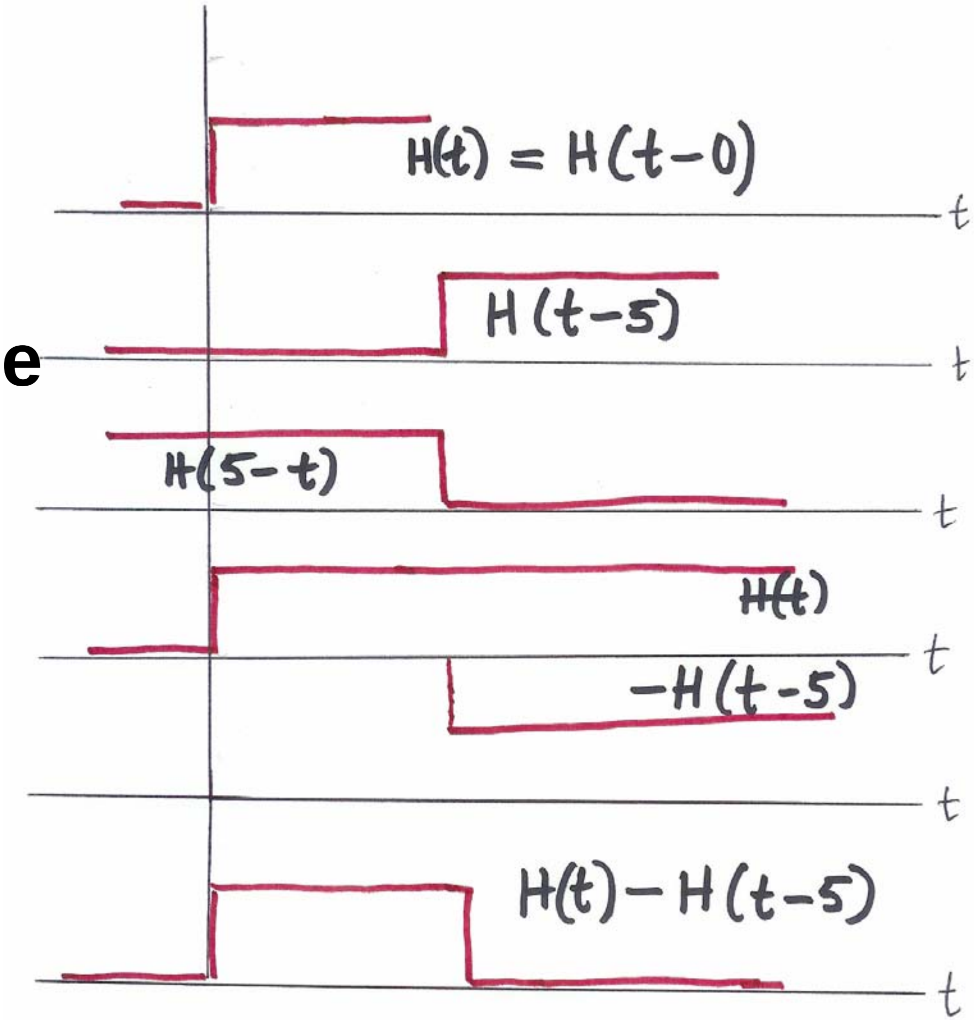
5. The Boltzmann equation (integral representation of the constitutive equation)

Integral representation of constitutive equation

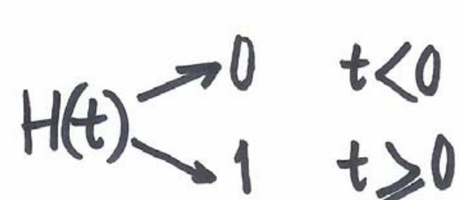
The linear differential equation (LDE) representation lacks a ”switching” function (that can turn stress or strain on and off rapidly) which is useful in description of stress and strain histories as well as many testing modes.

To incorporate a switching function use the integral representation (IE) of the constitutive equation. Derivation of the Boltzmann integral follows in 4 steps.

<u>Step 1</u>. Various mathematical switches that depend on use of the Heaviside function. These switches can be used to "cut-off" or start a function suddenly at a desired time.

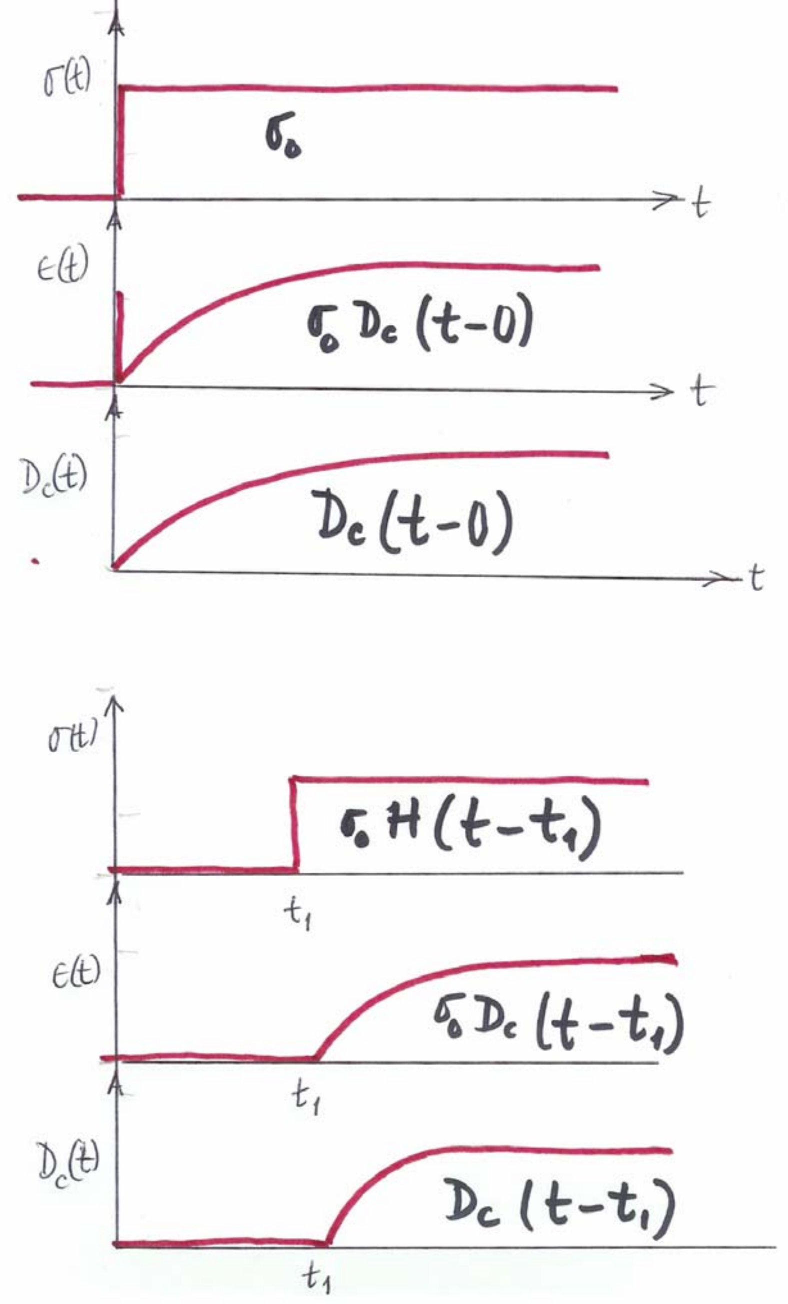


The Heaviside function

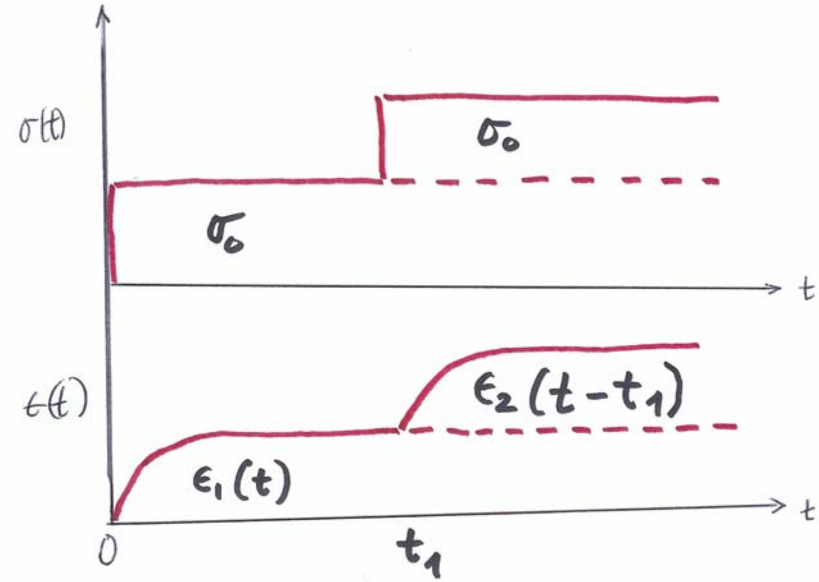


<u>Step 2</u>. Time invariance.

$D_c(t)$ is a function of material structure. It does not vary with time of loading.

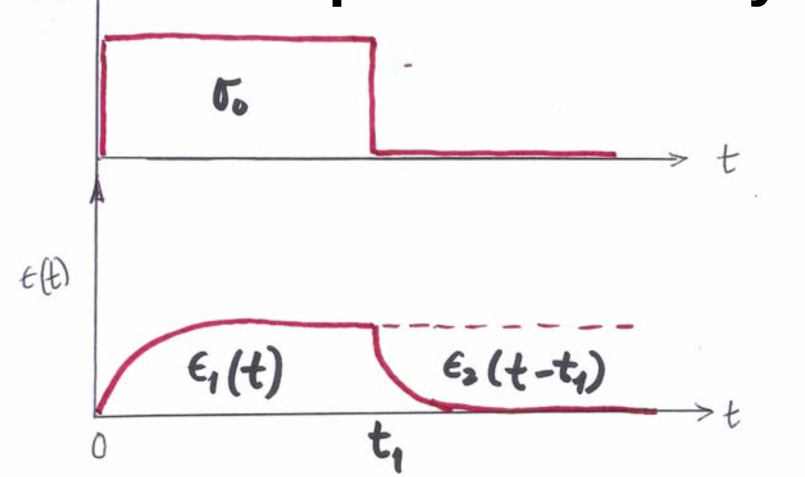


Step 3. Additivity.
The total strain following addition of a second load is the sum of the strains due to each load.



$$\varepsilon(t) = \varepsilon_1(t) \pm \varepsilon_2(t - t_1) =$$
$$= \sigma_o D_c(t) \pm \sigma_o D_c(t - t_1) =$$
$$= \sigma_o D \pm \sigma_o D_c(t - t_1)$$

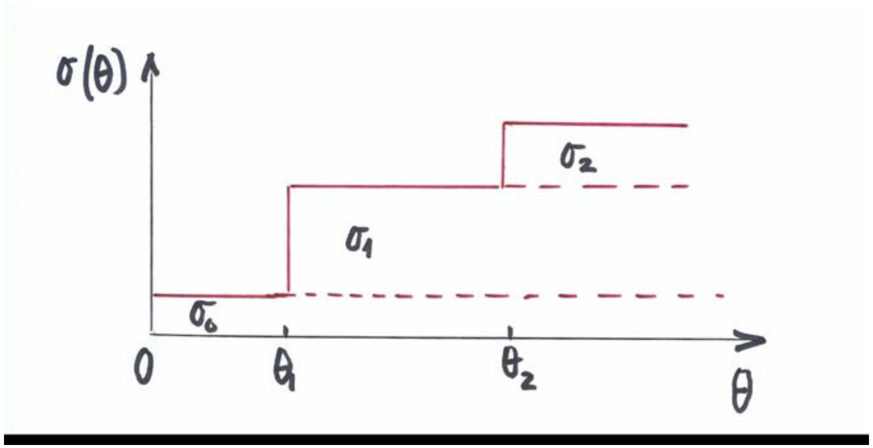
“creep and recovery”



Taken together, time invariance and additivity make up the “linearity” of viscoelastic behavior.

Step 4. Generalize and integrate.

Add several stresses, σ_o, σ_1, σ_2, etc., one after another, at respective times θ_o, θ_1, θ_2. Sum up strains assuming linearity of viscoelastic behavior (time invariance and additivity). Convert to incremental time intervals between each load addition, then integrate to get Boltzmann integral. This is the new constitutive equation we looked for.



$$\epsilon(t) = \sigma_0 D_c(t) + \sigma_1 D_1(t-\tau_1) + \sigma_2 D_c(t-\tau_2) + \cdots$$

$$\epsilon(t) = \sum_{\tau_i=0}^{\tau=t} D_c(t-\tau_1)\frac{\Delta\sigma(\tau_i)}{\Delta\tau}\Delta\tau$$

$$\epsilon(t) = \int_{\tau=-\infty}^{\tau=t} D_c(t-\tau)\frac{d\sigma(\tau)}{d\tau}\,d\tau$$

$$\varepsilon(t) = \varepsilon_1(t) + \varepsilon_2(t) =$$

$$= \sigma_o D_c(t) + \sigma_o D_c(t - t_i)$$

$$\varepsilon(t) = \varepsilon_1(t) - \varepsilon_2(t) =$$

$$= \sigma_o D_c(t) - \sigma_o D_c(t - t_1)$$

$$\varepsilon(t) = \sigma_o D_c(t) + \sigma_1 D_1(t - \tau_1) + \sigma_2 D_c(t - \tau_2) +$$

$$\varepsilon(t) = \sum_{\tau_i = o}^{\tau = t} D_c(t - \tau_i) \frac{\Delta\sigma(\tau_1)}{\Delta\tau} \Delta\tau$$

$$\varepsilon(t) = \int_{\tau = -\infty}^{\tau = t} D_c(t - \tau) \frac{d_\sigma(\tau)}{d\tau} d\tau$$

$$w(t) = \sigma_o W_H - \sigma_o W(t - \tau_i)$$

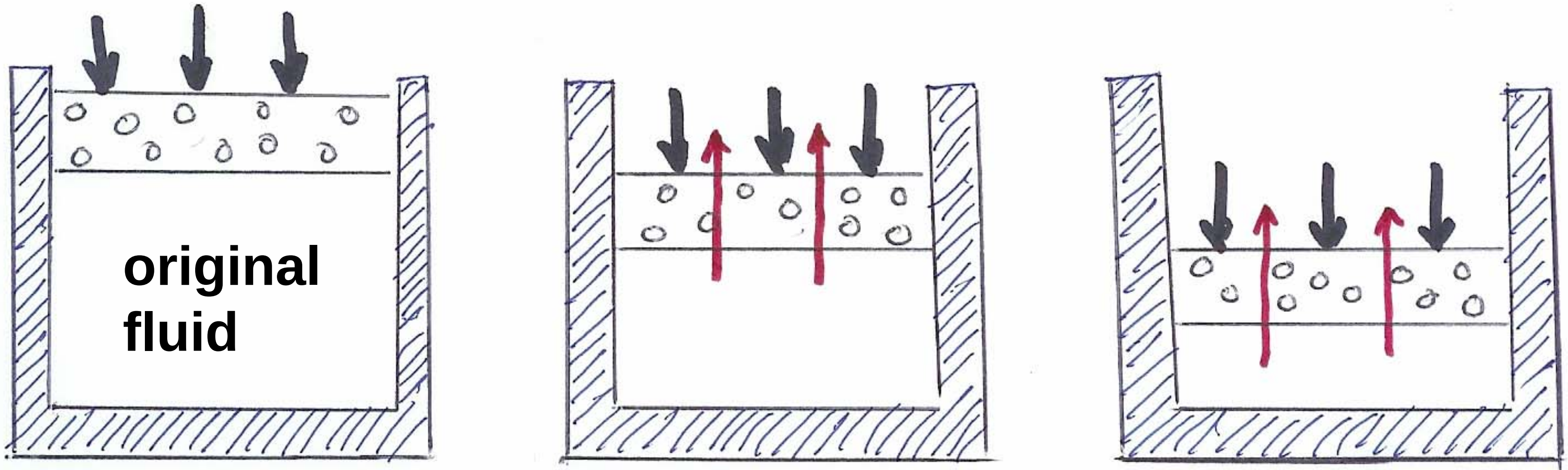
Compare the LDE to the IE representation

- **Even though differing in its descriptive power from the LDE approach, the new constitutive equation, in the form of an integral equation, provides the same information about the material described as the LDE approach.**
- **In particular, the material functions are represented in LDE as n constants while being represented in IE as the “kernel” function of the integral, Dc(t).**

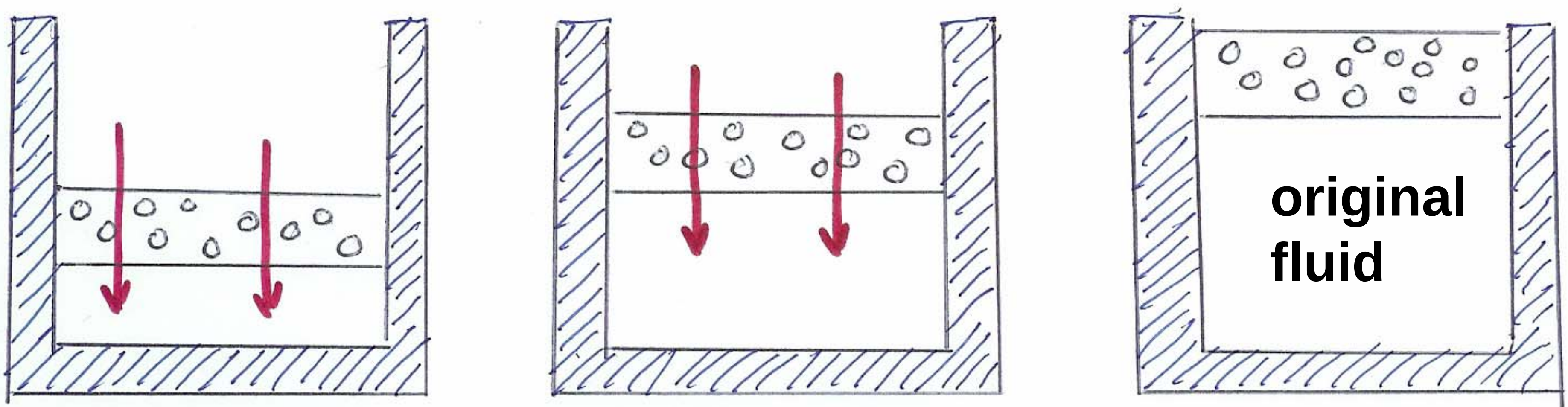
6. The “weeping memory” of cartilage.

Application: Weeping of joint cartilage modeled in terms of linear viscoelasticity

- Joint (articular) cartilage “weeps” fluid and contracts in volume when loaded in uniaxial compression (in the form of a specimen cut out of the intact cow tissue).
- Following removal of the compressive load, the specimen reswells in the fluid that had been expressed and expands to its original volume.
- Analyze data by Edwards and Maroudas et al. using a model of weeping cartilage as if the time-dependent deformation of a viscoelastic body and the mass of fluid expressed by weeping were mathematically identical variables.



Load. Fluid comes out and cartilage contracts.



Unload. Fluid re-enters and cartilage expands.

Analyze cartilage weeping and re-expansion

- The experimental data were obtained in the form of mass of fluid that came out following loading, or re-entered in following unloading, the cartilage specimen.
- Since loading of cartilage was accompanied by loss of mass a simple analysis of cartilage deformation as a linear viscoelastic “body” is not possible.
- Treat the two-step experiment as if it were a creep and recovery cycle. Use the mathematical symbolism of linear visco-elasticity to predict the kinetics of reswelling step from knowledge of the weeping step.

Graph removed for copyright reasons.
“Figure 7. Indentation against time curves for four areas of cartilage”

Kempson et al.,1971

Graph removed for copyright reasons.
“Figure 4.8. Effect of ionic (Donnan) contribution to osmotic pressure of cartilage...”

Maroudas, 1972

Graph removed for copyright reasons.
“Curves of weight of fluid expressed versus time at different loads.”

Linn and Sokoloff, 1965

Analysis of cartilage weeping data (cont.)

- **Define a new “weeping” function, W(t), as the ratio of time-dependent mass of fluid transferring in or out, m(t), and the constant stress, σ_o, acting on the specimen.**

$$W(t) = m(t)/\sigma_o$$

- **Make the following substitutions :**

$D_c(t) \rightarrow W(t)$

$\varepsilon(t) \rightarrow m(t)$

- **Assuming time invariance and additivity hold:**

$$m(t) = \sigma_o W(t) - \sigma_o W(t - t_1)$$

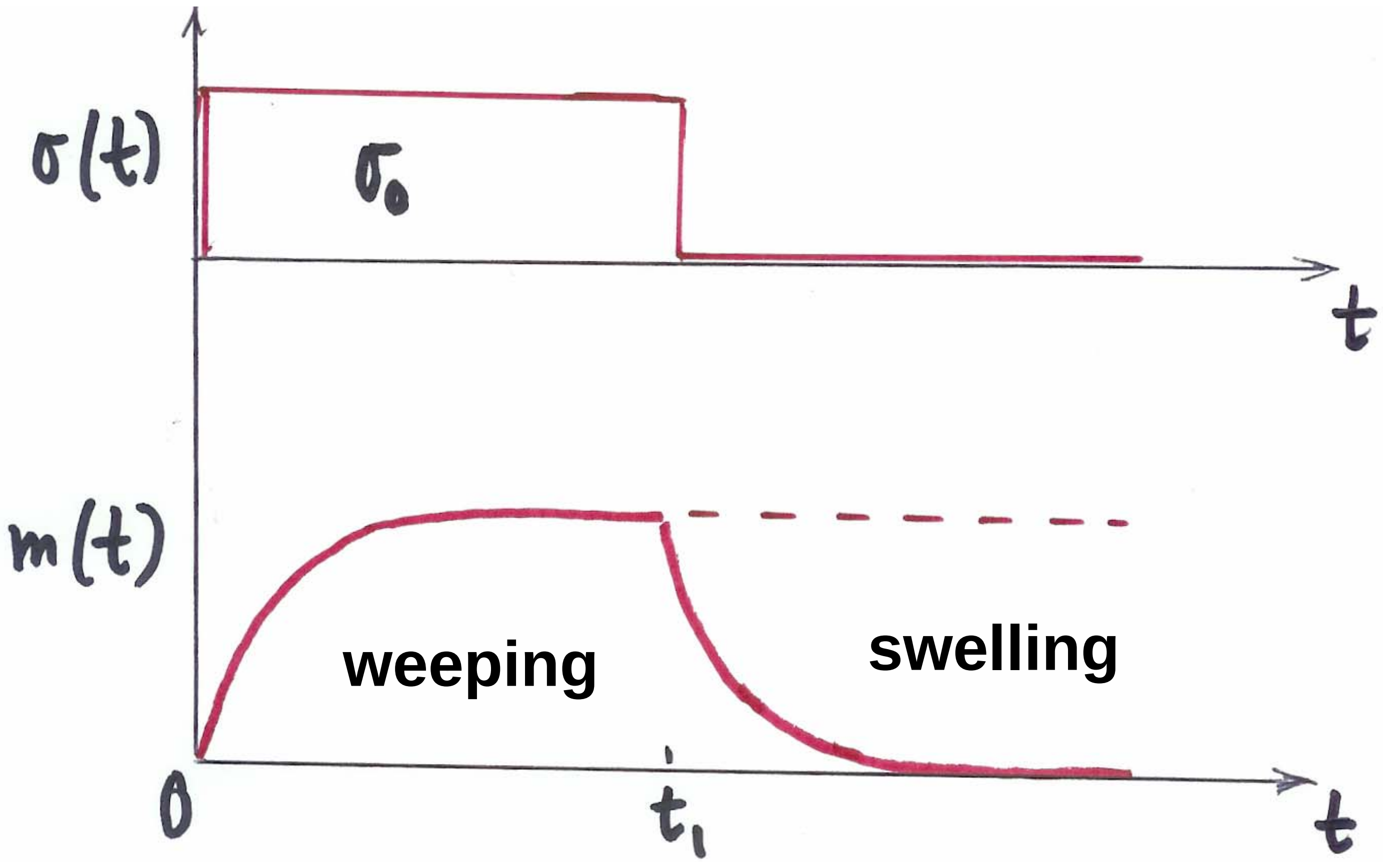
Analysis of cartilage weeping data (cont.)

- The data show that, by $t = t_1$, the mass of expressed fluid had reached a constant value, $\sigma_o W$; no more fluid was weeping out. However, fluid started going in when the load was removed at $t = t_1$.
- It follows that the mass of fluid going in during the swelling step can be calculated as follows:

$$m(t) = \sigma_o W - \sigma_o W(t - t_1)$$

- The analysis agrees well with the data. We conclude that the weeping step can be used to predict the data from the swelling step.

Analyze cartilage weeping data (cont.)



Calculate swelling step data from:

$$m(t) = \sigma_o W - \sigma_o W(t - t_1)$$

Summary of linear viscoelastic theory

1. Time dependence is very common during deformation of many tissues. Linear elasticity theory is not useful for lengthy loading experiences. Theory of linear viscoelasticity focuses on the history of strain or stress.
2. Time dependent behavior simulated either using a differential equation or integral equation representation.
3. Viscoelastic behavior is simply modeled using differential equations describing spring-dashpot combinations. Stress relaxation and creep behavior are modeled. Mechanical memory is also modeled in this way.

Summary (cont.)

4. Diagnostic tests for failure of elasticity are developed.
5. Integral equation representation of the constitutive equation (Boltzmann equation) is useful when the stress or strain are being switched on or off suddenly.
6. The weeping behavior of cartilage can be well represented using an integral equation representation.