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Second-order ordinary differential equations

Special functions, Sturm-Liouville theory and transforms

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ISBN 978-87-7681-972-9

Contents

Preface to these three texts

The three texts in this one cover, entitled 'The series solution of second order, ordinary differential equations and special functions' (Part I), 'An introduction to Sturm-Liouville theory' (Part II) and 'Integral transforms' (Part III), are three of the 'Notebook' series available as additional and background reading to students at Newcastle University (UK). These three together present a basic introduction to standard methods typically met in modern courses on ordinary differential equations (although the topic in Part III is relevant, additionally, to studies in partial differential equations and integral equations). The material in Part I is the most familiar topic encountered in this branch of university applied mathematical methods, and that in Part II would be included in a slightly more sophisticated approach to any work on second order, linear ODEs. The transform methods developed in Part III are likely to be included, at some point, in most advanced studies; here, we cover most of the standard transforms, their properties and a number of applications.

Each text is designed to be equivalent to a traditional text, or part of a text, which covers the relevant material, with many worked examples and a few set exercises (with answers provided). The appropriate background for each is mentioned in the preface to each Notebook, and each has its own comprehensive index.

Part I

The series solution of second order, ordinary differential equations and special functions

List of Equations

This is a list of the types of equation, and specific examples, whose solutions are discussed. (Throughout, we write $y = y(x)$ and a prime denotes the derivative; the power series are about $x = 0$, unless stated otherwise.)

Preface

This text is intended to provide an introduction to the methods for solving second order, ordinary differential equations (ODEs) by the *method of Frobenius*. The topics covered include all those that are typically discussed in modern mathematics degree programmes. The material has been written to provide a general introduction to the relevant ideas, rather than as a text linked to a specific course of study. Indeed, the intention is to present the material in a way that enhances the understanding of the topic, and so can be used as an adjunct to a number of different modules – or simply to help the reader gain a broader experience of mathematics. The aim is to go beyond the methods and techniques that are presented in a conventional module, but all the standard ideas are included (and can be accessed through the comprehensive index).

It is assumed that the reader has a basic knowledge of, and practical experience in, various aspects of the differential and the integral calculus. In particular, familiarity with the basic calculus and elementary differential-equation work, typically encountered in a first year of study, is assumed. This brief notebook does not attempt to include any applications of the differential equations; this is properly left to a specific module that might be offered in a conventional applied mathematics or engineering mathematics or physics programme. However, the techniques are applied to some specific equations whose solutions include important 'special' functions that are met in most branches of mathematical methods; thus we will discuss: Bessel functions, and the polynomials of Legendre and Hermite.

The approach adopted here is to present some general ideas, which might involve a notation, or a definition, or a theorem, or a classification, but most particularly methods of solution, explained with a number of carefully worked examples (there are 11 in total). A small number of exercises, with answers, are also offered to aid the understanding of the material.

1 Power-series solution of ODEs

In this chapter we will describe the fundamental ideas and method that underpin this approach to the solution of (second order, linear) ordinary differential equations. This will be presented with the help of a simple example, which will provide much of the motivation for the more general methods that follow later. However, as we explain in §1.2, this first analysis is very restrictive. We extend its applicability by first classifying the equations for which we can employ this technique, and then we carefully formulate (Chapter 2) the general method (due to G. Frobenius). All the possible cases will be described, with examples, and then we apply the procedure to a number of important equations.

1.1 Series solution: essential ideas

The methods for finding exact solutions of ordinary differential equations (ODEs) are familiar; see, for example, the volume 'The integration of ordinary differential equations' in *The Notebook Series*. So, for example, the equation

$$y'' + 3y' + 2y = 3 + 2x$$

(where the prime denotes the derivative with respect to x) has the complete general solution (complementary function + particular integral)

$$y(x) = A\mathrm{e}^{-x} + B\mathrm{e}^{-2x} + x\,,$$

where A and B are the arbitrary constants. A solution expressed like this is usually referred to as being 'in closed form'; on the other hand, if we wrote the solution (of some problem) in the form

$$y(x) = A\left(\sum_{n=1}^{\infty}\frac{x^n}{n^2}\right) + x^2$$

then this is not in closed form. (It would become closed form if we were able to sum the series in terms of elementary functions.) A solution is, of course, best written in closed form, but we may not always be able to do this; it is, nevertheless, sufficient for most purposes to represent the solution as a power series (provided that this series is convergent for some x, so that the solution exists somewhere). We should note that the solution of

$$y'' + 3y' + 2y = 3 + 2x$$

could be written

$$y(x) = A\left(\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right) + B\left(\sum_{n=0}^{\infty}\frac{(-2x)^n}{n!}\right) + x$$

(with the usual identification: $0! = 1$). In other words, we could always seek a power-series solution, and this will be particularly significant if we cannot solve the equation any other way. Indeed, it is evident that this approach provides a more general technique for tackling the problem of solving differential equations, even if the downside is the construction of a more complicated-looking form of the solution.

Thus the procedure is to set $y(x) = \sum_{n=0}^{\infty} a_n x^n$, and then aim to determine the coefficients, a_n, of the series, in order to ensure that this series is the solution of the given equation.

Example 1

Seek a solution of $y'' + 3y' + 2y = 3 + 2x$ in the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Given the power series, we find

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \text{ and } y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and so the equation becomes

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 3\sum_{n=0}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 3 + 2x .$$

We have assumed that *y*, y' and y'', all expressed via the given series, exist for some *x* i.e. all three series are convergent for some *x*s common to all three series. With this in mind, we require the equation, expressed in terms of the series, to be valid for *all x* in some domain – so we do not generate an equation for *x*! For this to be the case, *x* must vanish identically from the equation. Now our equation, written out in more detail, becomes

$$2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \ldots + 3\left(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots\right)$$
$$+2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\right) = 3 + 2x$$

and so, to be an identity in *x*, we require

$2a_2 + 3a_1 + 2a_0 = 3$; $6a_3 + 6a_2 + 2a_1 = 2$;

$12a_4 + 9a_3 + 2a_2 = 0$; $20a_5 + 12a_4 + 2a_3 = 0$, and so on.

We choose to solve these equations in the form

$$a_2 = \tfrac{3}{2} - \tfrac{3}{2}a_1 - a_0\,;\; a_3 = \tfrac{1}{3}(1-a_1) - a_2 = -\tfrac{7}{6}(1-a_1) + a_0\,;$$

$$a_4 = -\tfrac{1}{6}a_2 - \tfrac{3}{4}a_3 = \tfrac{5}{8}(1-a_1) - \tfrac{7}{12}a_0\,;\; a_5 = -\tfrac{1}{10}a_3 - \tfrac{3}{5}a_4 = -\tfrac{31}{120}(1-a_1) + \tfrac{1}{4}a_0, \text{ etc.}$$

Further, see that, in general, for every term x^n in the equation, for $n \geq 2$, we may write

$$(n+1)(n+2)a_{n+2} + 3(n+1)a_{n+1} + 2a_n = 0$$

i.e. $$a_{n+2} = -\frac{2a_n}{(n+1)(n+2)} - \frac{3a_{n+1}}{n+2}\,;$$

this is called a *recurrence relation*.

Because we have the combination ($1-a_1$) appearing here, it is convenient to write $a_1 = 1+b_1$, then all the coefficients a_2, a_3, etc., depend on only two constants: a_0 and b_1. These are undetermined in this system, so they are arbitrary: the two arbitrary constants expected in the general solution of a second order ODE. Our solution therefore takes the form

$$y(x) = a_0 + (1+b_1)x - \left(a_0 + \tfrac{3}{2}b_1\right)x^2 + \left(a_0 + \tfrac{7}{6}b_1\right)x^3 - \left(\tfrac{7}{12}a_0 + \tfrac{5}{8}b_1\right)x^4 + \left(\tfrac{1}{4}a_0 + \tfrac{31}{120}b_1\right)x^5 + \ldots.$$

$$= x + a_0\left(1 - x^2 + x^3 - \tfrac{7}{12}x^4 + \tfrac{1}{4}x^5 + \ldots\right) + b_1\left(x - \tfrac{3}{2}x^2 + \tfrac{7}{6}x^3 - \tfrac{5}{8}x^4 + \tfrac{31}{120}x^5 + \ldots\right).$$

This is more conveniently written by relabelling the arbitrary constants as

$$a_0 = A + B\,;\ b_1 = -(A + 2B)$$

although this is certainly not a necessary manoeuvre. (We choose to do this here to show directly the connection with the general solution quoted earlier.) This gives

$$y(x) = A\left(1 - x + \tfrac{1}{2!}x^2 - \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 - \tfrac{1}{5!}x^5 + \ldots\right) + B\left(1 - 2x + \tfrac{1}{2!}(2x)^2 - \tfrac{1}{3!}(2x)^3 + \tfrac{1}{4!}(2x)^4 - \tfrac{1}{5!}(2x)^5 + \ldots\right) + x\,,$$

which recovers our original (closed form) solution, because the infinite series (on using the formula that defines all the coefficients) can be summed and are convergent for all finite *x*. This approach, we observe, has automatically generated both the complementary function and the particular integral.

In summary, direct substitution of a suitable series, followed by the identical elimination of *x*, enables the complete general solution to be found. This approach, we note – perhaps with some relief – does not invoke any integration processes; any technical difficulties are now associated with the issue of convergence of the resulting series.

1.2 ODEs with regular singular points

The essential idea described in §1.1 needs to be developed further because it cannot accommodate, as it stands, functions such as $y(x) = \sqrt{x(1+x)}$. This function, represented as a power series, becomes

$$y(x) = \sqrt{x}\left(1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \ldots\right),\ |x| < 1\,,$$

obtained by using the familiar binomial theorem; this contains non-integral powers of *x*. Thus a solution assumed to be of the form used in §1.1:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n ,$$

can never find a solution that contains a term such as $\sqrt{x}$. The method – a simple extension of our basic approach – will enable a wide class of important equations to be solved. However, we must be clear about this class, and this then leads to an important classification of linear, second order, ordinary differential equations.

Let us suppose that the homogeneous ODE

$$a(x)y'' + b(x)y' + c(x)y = 0,$$

where the coefficients *a*, *b* and *c* are given functions, possesses a solution, near some point $x = x_0$, of the form

$$y(x) \approx A(x - x_0)^{\lambda} \quad (\text{as } x \to x_0)$$

for some constant λ (where *A* is an arbitrary constant). Then we see that

$$y' \approx A\lambda(x - x_0)^{\lambda - 1}; \; y'' \approx A\lambda(\lambda - 1)(x - x_0)^{\lambda - 2},$$

and so y'', $y'/(x - x_0)$ and $y/(x - x_0)^2$ are each proportional to $(x - x_0)^{\lambda - 2}$. Thus a linear combination of these will be zero, and this must recover (approximately) the original ODE (if such a solution exists) in the form

$$y'' + \frac{b(x)}{a(x)} y' + \frac{c(x)}{a(x)} y = 0$$

i.e. $b(x)/a(x) \propto (x - x_0)^{-1}$ and $c(x)/a(x) \propto (x - x_0)^{-2}$ (as $x \to x_0$).

This observation is the basis for the classification of the equations that we are able to solve by this method. Note that we are not concerned, at this stage, with the complete general solution which contains a particular integral.

Consider the second-order, homogeneous, linear ODE, written in the form

$$y'' + p(x)y' + q(x)y = 0;$$

if $p(x)$ or $q(x)$ (or both) are undefined – in this context, this means infinite – at $x = x_0$, then we have a *singular point* at $x = x_0$. Then, in the light of the observations just made, we form

$$P(x) = (x - x_0)p(x) \text{ and } Q(x) = (x - x_0)^2 q(x);$$

if both $P(x)$ and $Q(x)$ remain finite as $x \to x_0$, then we have a *regular singular point* at $x = x_0$.

If either $P(x)$ or $Q(x)$ (or both) are undefined (infinite) at $x = x_0$, then we have an *essential singular point* at $x = x_0$. On the other hand, if both $p(x)$ and $q(x)$ are defined at $x = x_0$, then we have an *ordinary point* at $x = x_0$.

The methods that we shall develop here are applicable to equations with regular singular (or ordinary) points; we cannot, in general, use this same approach to find a solution in the neighbourhood of an essential singular point.

Example 2

Identify and classify the singular points of these equations.

(a) $y'' + 3y' + 2y = 0$; (b) $x^2 y'' + 3xy' + 2y = 0$;

(c) $x(1-x)y'' + xy' + y = 0$; (d) $x^3 y'' + x^2 y' + (1-x)y = 0$.

In each, first we identify the functions $p(x)$ and $q(x)$, and then proceed.

(a) $p(x) = 3$, $q(x) = 2$: defined for all *x*, so all points are ordinary points.

(b) $p(x) = 3/x$, $q(x) = 2/x^2$: there is a singular point at $x = 0$; we form $xp(x) = 3$ and $x^2 q(x) = 2$, both of which are defined at $x = 0$, so we have a regular singular point at $x = 0$.

(c) $p(x) = 1/(1-x)$, $q(x) = 1/x(1-x)$: there are singular points at $x = 0$ and $x = 1$; for $x = 0$ we form $xp(x) = x/(1-x)$ and $x^2 q(x) = x/(1-x)$, both of which are defined at $x = 0$, so $x = 0$ is a regular singular point; for $x = 1$ we form $(x-1)p(x) = -1$ and $(x-1)^2 q(x) = (1-x)/x$, both of which are defined at $x = 1$, so $x = 1$ is another regular singular point.

(d) $p(x) = 1/x$, $q(x) = (1-x)/x^3$: there is a singular point at $x = 0$; we form $xp(x) = 1$ and $x^2 q(x) = (1-x)/x$, and the second of these is not defined at $x = 0$, so we have an essential singular point at $x = 0$.

Comment: This classification can also be applied where $|x| \to \infty$. To accomplish this, we transform the equation so that $y(x) = Y(x^{-1})$ which ensures that $|x| \to \infty$ now corresponds to $x^{-1} = u \to 0$; $u = 0$ is called the *point at infinity*. The point at infinity will, according to our classification, be an ordinary point or a regular singular point or an essential singular point.

Example 3

Classify the point at infinity for the equation $x^2 y'' + 3xy' + 2y = 0$ (cf. Ex. 2(b)).

We write $y(x) = Y(x^{-1})$ then

$$y'(x) = -\frac{1}{x^2} Y'(x^{-1}) \text{ and } y''(x) = \frac{2}{x^3} Y'(x^{-1}) + \frac{1}{x^4} Y''(x^{-1}),$$

so we have $y = Y(u)$, $y' = -u^2 Y'(u)$, $y'' = 2u^3 Y'(u) + u^4 Y''(u)$ (where $u = x^{-1}$). Thus the equation becomes

$$\frac{1}{u^2}\left(2u^3 Y' + u^4 Y''\right) + \frac{3}{u}\left(-u^2 Y'\right) + 2Y = 0$$

$$\text{i.e. } u^2 Y'' - uY' + 2Y = 0.$$

We see that $p(u) = -1/u$, $q(u) = 2/u^2$: the point at infinity ($u = 0$) is singular point. Now consider $up(u) = -1$ and $u^2 q(u) = 2$, both of which are defined at $u = 0$, so the point at infinity is a regular singular point.

Exercises 1

1. Identify and classify the singular points (excluding the point at infinity) of these equations.

(a) $x^2(1+x)y'' + x(1-x)y' + (1-x)y = 0$; (b) $\left(1-x^2\right)y'' + \left(\frac{x}{1+x}\right)y' + x^2 y = 0$;;

(c) $y'' + x^2 y = 0$, and in this case, also classify the point at infinity.

2. Show that the equation (the *hypergeometric equation*)

$$x(1-x)y'' + \{c-(a+b+1)x\}y' - aby = 0,$$,

has three regular singular points: at $x = 0$, $x = 1$ and at infinity (where *a*, *b* and *c* are given constants, although *c* will not be zero, nor will $|c|$ be an integer).

2 The method of Frobenius

We consider the second order, linear ordinary differential equation

$$a(x)y'' + b(x)y' + c(x)y = r(x)$$

for which we know that the solution can be expressed as

$$y(x) = y_{CF}(x) + y_{PI}(x),$$

where the particular integral, $y = y_{PI}(x)$, is any function that will generate $r(x)$. It is known that, if we can find one or both solutions that contribute to the complementary function, $y_{CF}(x)$, then the complete general solution can be found (by the method of variation of parameters, for example). Thus it is sufficient here to work only with the homogeneous equation i.e. $r(x) \equiv 0$. (For the relevant background ideas and techniques, see volume 'The integration of ordinary differential equations' in *The Notebook Series*, or any other suitable text.) We shall consider a power-series solution, in powers of $(x - x_0)$ (and quite often we will have $x_0 = 0$) and if $x = x_0$ happens to be an ordinary point of the equation, then the simple method described in the previous chapter is applicable. We allow the equation to have, at most, regular singular points, and then the most useful results are obtained by expanding about a regular singular point i.e. $x = x_0$ may be a regular singular point. The method cannot be applied, in general, if $x = x_0$ is an essential singular point. If there are, for example, two regular singular points, then it is often necessary to construct two power series: one about each of the regular singular points; we will comment on this in more detail later.

2.1 The basic method

Given the equation

$$a(x)y'' + b(x)y' + c(x)y = 0,$$

which has a regular singular point at $x = x_0$, we seek a solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{\lambda+n}, \; a_0 \neq 0,$$

where λ is a parameter (called the *index*) which is to be determined; this is usually called a 'power series about $x = x_0$'. This type of series solution is at the heart of the *method of Frobenius*. [Georg F. Frobenius (1849-1917), German mathematician; developed the concept of the abstract group; published his method for finding power-series solution of second order ODEs early in his career (in 1874).] Apart from the appearance of λ, the method follows precisely that described in Chapter 1. Note that λ gives the power of the first term ($n = 0$) in the series, whatever that may be. Thus, provided that a power series of this form exists, there must be a first term and this has been designated $a_0(x - x_0)^{\lambda}$, and so we require $a_0 \neq 0$. As we shall see, our previous method can be simplified and made more systematic, and this is the approach we will incorporate here.

Example 4

Use the method of Frobenius to find the general solution of the equation

$$4xy'' + 2y' - y = 0,$$

as a power series about $x = 0$.

We can note that the equation possesses a regular singular point at $x = 0$, so the method is applicable. We set $y(x) = \sum_{n=0}^{\infty} a_n x^{\lambda+n}$ ($a_0 \neq 0$), then we obtain

$$y'(x) = \sum_{n=0}^{\infty} a_n(\lambda+n)x^{\lambda+n-1}\ ;\ y''(x) = \sum_{n=0}^{\infty} a_n(\lambda+n)(\lambda+n-1)x^{\lambda+n-2},$$

and so the equation becomes

$$4x\sum_{n=0}^{\infty} a_n(\lambda+n)(\lambda+n-1)x^{\lambda+n-2} + 2\sum_{n=0}^{\infty} a_n(\lambda+n)x^{\lambda+n-1} - \sum_{n=0}^{\infty} a_n x^{\lambda+n} = 0.$$

It is convenient to introduce new dummy counters into each summation, so that each term now takes the form $x^{\lambda+m}$ – this is the essential manoeuvre in simplifying and organising the calculation. Thus in the first and second series we write $n-1=m$, and in the third we simply set $n=m$; we then obtain

$$4x\sum_{n=0}^{\infty} a_n(\lambda+n)(\lambda+n-1)x^{\lambda+n-2} + 2\sum_{n=0}^{\infty} a_n(\lambda+n)x^{\lambda+n-1} - \sum_{n=0}^{\infty} a_n x^{\lambda+n} = 0.$$

This equation can be rearranged to give

$$4a_0\lambda(\lambda-1)x^{\lambda-1} + 2a_0\lambda x^{\lambda-1}$$

$$+\sum_{m=0}^{\infty}\left\{4a_{m+1}(\lambda+m+1)(\lambda+m) + 2a_{m+1}(\lambda+m+1) - a_m\right\}x^{\lambda+m} = 0$$

which is an identity, for all *x*, if

$$2a_0(2\lambda^2 - \lambda) = 0;$$

$$2(\lambda+m+1)(2\lambda+2m+1)a_{m+1} - a_m = 0,\ m = 0, 1, 2, \ldots .$$

But, by definition, $a_0 \neq 0$ so we require

$$2\lambda^2 - \lambda = 0,$$

and the equation for λ is called the *indicial equation* (being the equation for the index λ); here we see that $\lambda = 0$ or $1/2$.

We also have

$$a_{m+1} = \frac{a_m}{2(\lambda + m + 1)(2\lambda + 2m + 1)} \quad (m = 0, 1, 2, \dots)$$

which is called the *recurrence relation*. (Note that we are not dividing by zero here, for the given λs and *m*s – but this is something that we must be wary of later.)

Now for each value of λ, we can find the coefficients a_m, ultimately in terms of a_0. However, because the equation is linear, we may perform the two calculations (one each for $\lambda = 0$ and $\lambda = 1/2$) and then construct a general linear combination of the two in order to generate the complementary function. In particular, for $\lambda = 0$, we obtain

$$a_{m+1} = \frac{a_m}{2(m+1)(2m+1)}: \ a_1 = \tfrac{1}{2}a_0; \ a_2 = \tfrac{1}{2.2.3}a_1 = \frac{a_0}{(2.2)!}, \text{ etc.,}$$

and for $\lambda = 1/2$:

$$a_{m+1} = \frac{a_m}{2(2m+3)(m+1)}: \ a_1 = \tfrac{1}{2.3}a_0; \ a_2 = \tfrac{1}{2.2.5}a_1 = \frac{a_0}{(2.2+1)!}, \text{ etc.}$$

The general solution can then be written as

$$y(x) = A\sum_{n=0}^{\infty}\frac{x^n}{(2n)!} + Bx^{1/2}\sum_{n=0}^{\infty}\frac{x^n}{(2n+1)!}.$$

Comment: These two series converge for all finite *x*, as can be seen by applying the ratio test.

In this example, we found that the indicial equation was quadratic – it always will be – and that its roots were not equal (nor did they differ by an integer, which will be an important observation). The upshot is that we have two special cases that must be addressed.

2.2 The two special cases

First, we write the general homogeneous equation in the form

$$y'' + p(x)y' + q(x)y = 0$$

and seek a solution
$$y(x) = \sum_{n=0}^{\infty} a_n x^{\lambda+n};$$

this will, for general λ, take the form

$$y(x) = a_0 x^{\lambda}\left\{1 + \alpha_1(\lambda)x + \alpha_2(\lambda)x^2 + \ldots\right\}$$

where the $\alpha_i(\lambda)$ are obtained from the recurrence relation. Let us write

$$Y(x,\lambda) = x^{\lambda}\left\{1 + \alpha_1(\lambda)x + \alpha_2(\lambda)x^2 + \ldots\right\},$$

then the equation, with $y = Y$, gives (after a little calculation, which is left as an exercise)

$$Y'' + p(x)Y' + q(x)Y = x^{\lambda-2}(\lambda-\lambda_1)(\lambda-\lambda_2),$$

where $\lambda = \lambda_1, \lambda_2$ are the roots of the indicial equation. This result immediately confirms what happened in Example 4: $y = Y(x,\lambda)$ is a solution for $\lambda = \lambda_1$ and for $\lambda = \lambda_2$. However, if $\lambda_1 = \lambda_2$ (repeated root), we clearly obtain only one solution by this method (and we know that there must be two linearly independent solutions of a second order ODE). To see how to continue in this case, we choose to differentiate with respect to λ:

$$Y_\lambda'' + p(x)Y_\lambda' + q(x)Y_\lambda = \frac{\partial}{\partial\lambda}\left\{x^{\lambda-2}(\lambda-\lambda_1)^2\right\}$$

where we have set $\lambda_2 = \lambda_1$. But the right-hand side of this equation is still zero for $\lambda = \lambda_1$, so a second solution is $\frac{\partial}{\partial\lambda}\left[Y(x,\lambda)\right]$ evaluated on $\lambda = \lambda_1$. This solution is

$$\frac{\partial}{\partial\lambda}\left[x^\lambda\left\{1+\alpha_1(\lambda)x+\alpha_2(\lambda)x^2+\ldots.\right\}\right]_{\text{on } \lambda=\lambda_1}$$

$$= \frac{\partial}{\partial\lambda}\left[\mathrm{e}^{\lambda\ln|x|}\left\{1+\alpha_1 x+\alpha_2 x^2+\ldots.\right\}\right]_{\lambda=\lambda_1}$$

$$= x^{\lambda_1}\ln|x|\left\{1+\alpha_1 x+\alpha_2 x^2+\ldots.\right\}+x^{\lambda_1}\left\{\alpha_1'(\lambda_1)x+\alpha_2'(\lambda_1)x^2+\ldots\right\}$$

– this second solution contains a logarithmic term, which is therefore certainly linearly independent of our first solution (Y). As we have demonstrated, this second solution can be obtained directly from the first solution or, as is quite often done, it is computed by seeking an appropriate solution of the differential equation.

Example 5

Use the method of Frobenius to find the general solution of the equation

$$xy'' + y' - xy = 0$$

as a suitable power series about $x = 0$.

We note that the equation has a regular singular point at $x = 0$, so we may indeed write $y(x) = \sum_{n=0}^{\infty} a_n x^{\lambda+n}$ ($a_0 \neq 0$), to give

$$\sum_{n=0}^{\infty} a_n(\lambda+n)(\lambda+n-1)x^{\lambda+n-1} + \sum_{n=0}^{\infty} a_n(\lambda+n)x^{\lambda+n-1} - \sum_{n=0}^{\infty} a_n x^{\lambda+n+1} = 0 .$$

In the first two summations, we choose to write $m = n-1$, and $m = n+1$ in the third; this gives

$$\sum_{m=-1}^{\infty} a_{m+1}(\lambda+m+1)^2 x^{\lambda+m} - \sum_{m=1}^{\infty} a_{m-1}x^{\lambda+m} = 0 .$$

This can now be expressed as

$$a_0\lambda^2 x^{\lambda-1} + a_1(\lambda+1)^2 x^{\lambda} + \sum_{m=1}^{\infty} \left\{ a_{m+1}(\lambda+m+1)^2 - a_{m-1} \right\} x^{\lambda+m} = 0$$

and this is an identity for all x if

$$a_0\lambda^2 = 0; \; a_1(\lambda+1)^2 = 0; \; a_{m+1}(\lambda+m+1)^2 - a_{m-1} = 0 \; (m = 1, 2, \ldots).$$

But $a_0 \neq 0$, so $\lambda = 0$ (repeated) and then $a_1 = 0$; this leaves

$$a_{m+1} = \frac{a_{m-1}}{(m+1)^2} \; (m = 1, 2, \ldots),$$

with implies that $0 = a_1 = a_3 = a_5 = \ldots$. Otherwise we obtain

$$a_2 = \frac{a_0}{2^2}; \; a_4 = \frac{a_2}{4^2} = \frac{a_0}{(2.4)^2} = \frac{a_0}{2^4(2!)^2}, \text{ etc.,}$$

which provides one solution:

$$y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{(m!)^2} ,$$

where a_0 is the arbitrary constant.

To proceed, let us write $y_1(x) = a_0 Y(x)$ (so, of course, $xY'' + Y' - xY = 0$), then a second solution must necessarily take the form

$$y_2(x) = Y(x)\ln|x| + \sum_{n=0}^{\infty} b_n x^n ;$$

the equation therefore gives

$$x\left[Y''\ln|x|+2Y'.\frac{1}{x}+Y.\left(-\frac{1}{x^2}\right)+\sum_{n=0}^{\infty}b_n n(n-1)x^{n-2}\right]$$
$$+Y'\ln|x|+Y.\frac{1}{x}+\sum_{n=0}^{\infty}b_n n x^{n-1}-x\left[Y\ln|x|+\sum_{n=0}^{\infty}b_n x^n\right]=0\,.$$

Thus, when we use the fact that $Y(x)$ is a solution of the original equation, we obtain

$$\sum_{n=1}^{\infty}b_n n^2 x^{n-1}-\sum_{n=0}^{\infty}b_n x^{n+1}+2\sum_{n=0}^{\infty}\frac{2nx^{2n-1}}{2^{2n}(n!)^2}=0$$

so $0=b_1=b_3=b_5=\dots.$ and

$$(2m+2)^2 b_{2m+2}-b_{2m}=-\frac{(m+1)}{2^{2m}[(m+1)!]^2}\,,\quad m=0,1,2,\dots,$$

(where we have written $2m$ for n – the even values). We may select $b_0 = 0$ (because this will simply generate y_1 again), then

$$b_2 = -\frac{1}{4};\ b_4 = -\frac{3}{8.16},\ \text{etc.}$$

The complete, general solution can therefore be written as

$$y(x) = A\sum_{n=0}^{\infty}\frac{(x/2)^{2n}}{(n!)^2} + B\left\{\left(\sum_{n=0}^{\infty}\frac{(x/2)^{2n}}{(n!)^2}\right)\ln|x| - \frac{1}{4}x^2 - \frac{3}{8.16}x^4 + \ldots\right\},$$

where the arbitrary constants are now A and B.

The second special case arises when the roots for λ differ by an integer; let us write $\lambda_1 = \lambda_2 + n$, where n is a positive integer i.e. $\lambda_1 > \lambda_2$. A solution for general λ will take the same form as before, namely

$$y(x) = a_0 x^{\lambda}\left\{1 + \alpha_1(\lambda)x + \alpha_2(\lambda)x^2 + \ldots.\right\},$$

but this time each $\alpha_i(\lambda)$, starting at $\alpha_n(\lambda)$ (with n as just defined), will have a factor $(\lambda - \lambda_2)$ in the denominator. (In exceptional cases, this factor is cancelled by one in the numerator and then the standard procedure (§2.1) can be employed; we discuss this special situation in §3.2.)

Because of the appearance of this factor, it is convenient to write

$$\hat{Y}(x,\lambda) = (\lambda - \lambda_2)x^{\lambda}\left\{1 + \alpha_1(\lambda)x + \alpha_2(\lambda)x^2 + \ldots.\right\}$$

then the original equation, with $y = \hat{Y}$, gives

$$\hat{Y}'' + p(x)\hat{Y}' + q(x)\hat{Y} = x^{\lambda-2}(\lambda - \lambda_1)(\lambda - \lambda_2)^2,$$

so that possible solutions are $\hat{Y}(x,\lambda_1)$, $\hat{Y}(x,\lambda_2)$ and $\hat{Y}_{\lambda}(x,\lambda_2)$. But the second of these, it turns out, is a multiple of the first (where there is a zero in the denominator of the α_is, $i > n$); thus the second solution we require is $\hat{Y}_{\lambda}(x,\lambda_2)$ which follows the same pattern as for repeated roots. We have avoided giving the details of this general argument; we have simply aimed to indicate what we should expect. The precise details, and any particular difficulties, will become evident when we tackle specific problems of this type, as we shall now see.

Example 6

Use the method of Frobenius to find the general solution of the equation

$$xy'' + y = 0,$$

as a suitable power series about $x = 0$.

We see that there is a regular singular point at $x = 0$, so we may proceed by writing

$$y(x) = \sum_{n=0}^{\infty} a_n x^{\lambda+n} \quad (a_0 \neq 0),$$

and then we obtain

$$\sum_{n=0}^{\infty} a_n(\lambda+n)(\lambda+n-1)x^{\lambda+n-1} + \sum_{n=0}^{\infty} a_n x^{\lambda+n} = 0.$$

In the first series, we set $n-1=m$, and in the second simply put $n=m$, to give

$$\sum_{m=-1}^{\infty} a_{m+1}(\lambda+m+1)(\lambda+m)x^{\lambda+m} + \sum_{m=0}^{\infty} a_m x^{\lambda+m} = 0$$

i.e. $$a_0\lambda(\lambda-1)x^{\lambda-1} + \sum_{m=0}^{\infty}\{a_{m+1}(\lambda+m+1)(\lambda+m) + a_m\}x^{\lambda+m} = 0.$$

This is an identity for all *x* if

$$\lambda(\lambda-1)=0 \text{ and } a_{m+1}(\lambda+m+1)(\lambda+m)+a_m = 0 \quad (m=0,1,2,\ldots).$$

So $\lambda = 0, 1$ – they differ by an integer. If we select $\lambda = 1$ (the larger one), then the recurrence relation becomes

$$(m+1)(m+2)a_{m+1} + a_m = 0 \text{ or } a_{m+1} = -\frac{a_m}{(m+1)(m+2)} \quad (m=0,1,2,\ldots),$$

which is defined for all the given *m*. Thus

$$a_1 = -\frac{a_0}{1.2};\ a_2 = -\frac{a_1}{2.3} = \frac{a_0}{1.2.2.3}, \text{ etc.}$$

to give $a_m = \dfrac{(-1)^m}{(1+m)(m!)^2}a_0$, $m = 1, 2, 3, \ldots$, and hence one solution is

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(1+n)(n!)^2}.$$

Now if we select $\lambda = 0$ (the lower value), we obtain

$$m(m+1)a_{m+1} = -a_m \quad (m = 0, 1, 2, \ldots),$$

so that, for $m = 0$, we get $a_0 = 0$, which is impossible; equivalently, we have division by zero in the definition of the coefficient a_1 (exactly as we described in our commentary earlier). The second solution, we believe, therefore contains a logarithmic term, just as we encountered in Example 5. Thus we set

$$y(x) = Y(x)\ln|x| + \sum_{n=0}^{\infty} b_n x^n$$

where our first solution is written as $y(x) = a_0 Y(x)$; thus we obtain

$$x\left[Y''\ln|x| + Y'.\frac{2}{x} + Y.\left(-\frac{1}{x^2}\right) + \sum_{n=0}^{\infty} b_n n(n-1)x^{n-2}\right] + Y\ln|x| + \sum_{n=0}^{\infty} b_n x^n = 0.$$

This is simplifies to give

$$\sum_{m=0}^{\infty}\{m(m+1)b_{m+1}+b_m\}x^m=\frac{1}{x}Y-2Y'$$

$$=\sum_{m=0}^{\infty}a_m x^m-2\sum_{m=0}^{\infty}(1+m)a_m x^m=-\sum_{m=0}^{\infty}(1+2m)a_m x^m,$$

where $a_m=(-1)^m\Big/\left[(1+m)(m!)^2\right]$.

Thus the coefficients b_m satisfy

$$m(m+1)b_{m+1}+b_m=-(1+2m)a_m \text{ (for } m=0,1,2,\ldots\text{)},$$

so $b_0=-1$, $2b_2+b_1=-3a_1=3/2$ (and we may choose $b_1=0$), etc.; a second solution is therefore

$$y(x)=Y(x)\ln|x|+\left(-1+\tfrac{3}{4}x^2-\tfrac{3}{76}x^4+\ldots.\right).$$

In conclusion, we have the complete, general solution as

$$y(x)=A\sum_{n=0}^{\infty}\frac{(-1)^n x^{1+n}}{(1+n)(n!)^2}+B\left\{\ln|x|\sum_{n=0}^{\infty}\frac{(-1)^n x^{1+n}}{(1+n)(n!)^2}-1+\frac{3}{4}x^2-\frac{7}{36}x^4+\ldots\right\}.$$

The exceptional case, where the roots for λ differ by an integer but no logarithmic terms appear, will be discussed in detail when we consider the Bessel equation (Chap. 3).

Comment: All our examples thus far have involved seeking power series about $x=0$, but this is not always what we require. Let us consider the equation

$$(1-x)y''+xy'+2y=0;$$

this has a regular singular point at $x=1$, so our conventional application of the method of Frobenius is to seek a solution

$$y(x)=\sum_{n=0}^{\infty}a_n(x-1)^{\lambda+n}.$$

However, it is often more convenient to transform the original equation so that the 'standard' power series, i.e. equivalently about $x = 0$, can be invoked. To do this, we write

$$y(x) = Y(x-1) = Y(z) \quad \text{where } z = x-1,$$

and then we obtain

$$-zY'' + (1+z)Y' + 2Y = 0.$$

This equation has a regular singular point at $z = 0$ (i.e. $x = 1$), and so we may seek a solution

$$Y(z) = \sum_{n=0}^{\infty} a_n z^{\lambda+n} \text{ – the standard form.}$$

We conclude with one more example which contains two regular singular points, and show what happens when we expand about each in turn. In this case we will write a little more about the convergence of the series that we generate (and we will take the opportunity to mention one of the special cases that can arise, but we will write more of this later).

Example 7

Use the method of Frobenius to find a power-series solution, about $x = 0$, of the equation

$$x^2(x-1)y'' + xy' - \tfrac{1}{4}(3-x)y = 0.$$

Examine the convergence of this solution, and then construct a corresponding solution about the other singular point.

It is clear that we have a regular singular point at $x = 0$ (and also at $x = 1$), so we write

$$y(x) = \sum_{n=0}^{\infty} a_n x^{\lambda+n},$$

to obtain

$$(x-1)\sum_{n=0}^{\infty} a_n(\lambda+n)(\lambda+n-1)x^{\lambda+n} + \sum_{n=0}^{\infty} a_n(\lambda+n)x^{\lambda+n} - \frac{1}{4}(3-x)\sum_{n=0}^{\infty} a_n x^{\lambda+n} = 0$$

which is rewritten as

$$\sum_{m=1}^{\infty} a_{m-1}\left\{(\lambda+m-1)(\lambda+m-2)+\tfrac{1}{4}\right\}x^{\lambda+m} + \sum_{m=0}^{\infty} a_m\left\{-(\lambda+m)(\lambda+m-1)+(\lambda+m)-\tfrac{3}{4}\right\}x^{\lambda+m} = 0 .$$

Thus we have

$$a_0\left\{-\lambda(\lambda-1)+\lambda-\tfrac{3}{4}\right\} + \sum_{m=1}^{\infty}\left\{a_{m-1}\left[(\lambda+m-1)(\lambda+m-2)+\tfrac{1}{4}\right]-a_m\left[(\lambda+m)(\lambda+m-2)+\tfrac{3}{4}\right]\right\}x^{\lambda+m} = 0 ,$$

which is an identity for all x if

$$\lambda^2 - 2\lambda + \tfrac{3}{4} = 0 \ \ (a_0 \neq 0)$$

and $a_m\left[(\lambda+m)(\lambda+m-2)+\tfrac{3}{4}\right] = a_{m-1}\left[(\lambda+m-1)(\lambda+m-2)+\tfrac{1}{4}\right]$ $(m = 1, 2, \ldots)$

i.e. $(\lambda-1)^2 - \tfrac{1}{4} = 0$; $a_m\left(\lambda+m-\tfrac{1}{2}\right)\left(\lambda+m-\tfrac{3}{2}\right) = a_{m-1}\left(\lambda+m-\tfrac{3}{2}\right)^2$ $(m = 1, 2, \ldots)$.

So we have $\lambda = 1/2$ and $\lambda = 3/2$ (which differ by an integer); for $\lambda = 3/2$ we obtain

$$a_m m(m+1) = a_{m-1}m^2 \ \text{ or } \ a_m = \left(\frac{m}{m+1}\right)a_{m-1} \ (m = 1, 2, \ldots).$$

This gives one solution:

$$y(x) = a_0 x^{3/2}\left(1 + \tfrac{1}{2}x + \tfrac{1}{3}x^2 + \ldots. + \tfrac{1}{r+1}x^r + \ldots.\right).$$

On the other hand, for $\lambda = 1/2$, the recurrence relation becomes

$$a_m m(m-1) = a_{m-1}(m-1)^2 \ \text{ for } \ m = 1, 2, \ldots,$$

which shows that, on $m = 1$, we have both a_0 and a_1 arbitrary. (Comment: This special case of values of λ differing by an integer, but for which the recurrence is valid even for the lower value of λ, is one that we shall discuss more fully later. The procedure always involves the testing of the recurrence relation, and we use it if it does not produce an inconsistent solution – and, as we have seen, it is valid here.) Thus we now have

$$a_m = \left(\frac{m-1}{m}\right)a_{m-1} \ \text{ for } \ m = 2, 3, \ldots,$$

which produces the solution

$$y(x) = x^{1/2}\left(a_0 + a_1 x + \tfrac{1}{2}a_1 x^2 + \ldots.\right)$$

$$= a_0 x^{1/2} + a_1 x^{3/2}\left(1 + \tfrac{1}{2}x + \tfrac{1}{3}x^2 + \ldots + \tfrac{1}{r+1}x^r + \ldots\right),$$

the complete, general solution. (Comment: We see that the solution obtained for $\lambda = 3/2$ merely recovers one contribution to the general solution; it is always the case that, if the recurrence gives a solution for both values of λ,when they differ by an integer, then the smaller value will always generate the complete, general solution.)

Now the infinite series

$$1+\tfrac{1}{2}x+\tfrac{1}{3}x^2+\dots\frac{1}{r+1}x^r+\dots$$

can be investigated, for example, by applying the *ratio test*; this shows us that the series is certainly convergent if

$$\left|\frac{x^r/(r+1)}{x^{r-1}/r}\right|<1 \text{ as } r\to\infty \text{ i.e. if } |x|<1,$$

and it is divergent for $|x|>1$. For $x=\pm 1$, we must examine the series directly: this gives

$$1\pm\tfrac{1}{2}+\tfrac{1}{3}\pm\tfrac{1}{4}+\dots.$$

and for the lower sign ($x=-1$) this series converges, but it does not for the upper sign ($x=+1$). Thus we have constructed a power-series representation of the solution which is valid for $-1\le x<1$ – and the inadmissible choice $x=+1$ corresponds precisely to the other singular point of the equation. In order to find the form of the solution valid in the neighbourhood of $x=1$, we now seek a power-series solution about $x=1$.

First, therefore, we set $x-1=z$ and write $y(x)=Y(x-1)=Y(z)$, to give

$$(1+z)^2 zY''+(1+z)Y'-\tfrac{1}{4}(2-z)Y=0;$$

we seek a solution $Y(z)=\sum_{n=0}^{\infty}a_n z^{\lambda+n}$ (where we have elected to retain the standard notation although, presumably, λ and a_n will take different forms here). We obtain

$$(1+z)^2\sum_{n=0}^{\infty}a_n(\lambda+n)(\lambda+n-1)z^{\lambda+n-1}+(1+z)\sum_{n=0}^{\infty}a_n(\lambda+n)z^{\lambda+n-1}$$

$$-\tfrac{1}{4}(2-z)\sum_{n=0}^{\infty}a_n z^{\lambda+n}=0,$$

which becomes

$$\sum_{m=-1}^{\infty} a_{m+1}\{(\lambda+m+1)(\lambda+m)+(\lambda+m+1)\}z^{\lambda+m}$$
$$+\sum_{m=0}^{\infty} a_m\left\{2(\lambda+m)(\lambda+m-1)+(\lambda+m)-\tfrac{1}{2}\right\}z^{\lambda+m}$$
$$+\sum_{m=1}^{\infty} a_{m-1}\left\{(\lambda+m-1)(\lambda+m-2)+\tfrac{1}{4}\right\}z^{\lambda+m}=0\,.$$

With $m=-1$, we obtain $\lambda^2=0$, so $\lambda=0$ is a repeated root (implying that logarithmic terms will definitely arise); with $m=0$, we get $a_1-\frac{1}{2}a_0=0$, leaving (with $\lambda=0$)

$$a_{m+1}(m+1)^2+a_m\left(2m^2-m-\tfrac{1}{2}\right)+a_{m-1}\left[(m-1)(m-2)+\tfrac{1}{4}\right]=0,\ m=1,2,\dots\ .$$

This generates the coefficients $a_2=-a_0/8,\ a_3=a_0/16$, etc., so we have one solution

$$Y(z)=a_0\left(1+\tfrac{1}{2}z-\tfrac{1}{8}z^2+\tfrac{1}{16}z^3+\dots\right).$$

A second solution necessarily takes the form

$$Y(z) = \hat{Y}(z)\ln|z| + \sum_{n=0}^{\infty} b_n z^n \,,$$

where $\hat{Y}(z) = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \ldots$; then we obtain

$$z(1+z)^2\left[\hat{Y}''\ln|z| + \frac{2}{z}.\hat{Y}' - \frac{1}{z^2}.\hat{Y} + \sum_{n=2}^{\infty} b_n n(n-1)z^{n-2}\right]$$
$$+(1+z)\left[\hat{Y}'\ln|z| + \frac{1}{z}.\hat{Y} + \sum_{n=1}^{\infty} b_n n z^{n-1}\right] - \frac{1}{4}(2-z)\left[\hat{Y}\ln|z| + \sum_{n=0}^{\infty} b_n z^{\lambda+n}\right] = 0\,.$$

But $\hat{Y}$ is a solution of the original equation, so we are left with

$$(1+z)^2 \sum_{n=2}^{\infty} b_n n(n-1)z^{n-1} + (1+z)\sum_{n=1}^{\infty} b_n n z^{n-1} - \tfrac{1}{4}(2-z)\sum_{n=0}^{\infty} b_n z^{\lambda+n}$$

$$= (1+z)\hat{Y} - 2(1+z)^2\hat{Y}'$$

$$= (1+z)\left(1 + \tfrac{1}{2}z - \tfrac{1}{8}z^2 + \ldots\right) - 2(1+z)^2\left(\tfrac{1}{2} - \tfrac{1}{4}z + \tfrac{3}{16}z^2 + \ldots\right).$$

Then terms z^0 give $b_1 - \frac{1}{2}b_0 = 0$, and we may select $b_0 = 0$, so that $b_1 = 0$. Terms z^1 give $4b_2 = 0$, and terms z^2 then give $9b_3 = 0$, and so on. There is a solution for which $b_n = 0$ for all *n*, so the complete, general solution can be written as

$$Y(z) = A\left(1 + \tfrac{1}{2}z - \tfrac{1}{8}z^2 + \ldots\right) + B\left(1 + \tfrac{1}{2}z - \tfrac{1}{8}z^2 + \ldots\right)\ln|z|$$

or $$y(x) = \left[1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \tfrac{1}{16}(x-1)^3 + \ldots\right]\left(A + B\ln|x-1|\right).$$

We conclude, therefore, that the singular point at $x = 1$ corresponds to a logarithmic singularity in the solution here. Thus if we want to use the solution not too near $x = 1$, we should employ the power-series representation about $x = 0$. However, if we wished to capture more accurately the behaviour near $x = 1$ then we should use the power series valid about $x = 1$ (i.e. about $z = 0$). (Note that the solution does not exist on $x = 1$ unless $B = 0$.)

Comment: You might recognise $1 + \frac{1}{2}x + \frac{1}{3}x^2 + \ldots + \frac{1}{r+1}x^r + \ldots$ as the Maclaurin expansion of $\ln(1-x)$, valid for $-1 \le x < 1$, and also that $1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 + \ldots$ is the binomial expansion of $(1+z)^{1/2}$. Indeed, our original equation has the *exact* general solution

$$y(x)=\sqrt{x}\left(A+B\ln|x-1|\right).$$

Exercises 2

1. Use the method of Frobenius to find the general solutions of these equations as power series about $x=0$. In each case, find explicitly the first three terms of each series and also record the recurrence relation that will generate the rest.

(a) $4xy''+2y'+y=0$; (b) $xy''+(x+\frac{1}{2})y'+y=0$;

(c) $2x^2y''+7x(x+1)y'-3y=0$; (d) $x^2y''+x(x-2)y'+2y=0$;

(e) $xy''+y'+xy=0$; (f) $x^2y''+x(3+x)y'+(1+x+x^2)y=0$.

2. Repeat Q.1, but now find the power series about $x=1$ for the equation

$$2(x-1)\left(3-4x+2x^2\right)y''-y'-24(x-1)y=0.$$

3 The Bessel equation and Bessel functions

An important equation in applied mathematics, engineering mathematics and most branches of physics is the second order, linear, ordinary differential equation

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right)y = 0,$$

where $\nu \geq 0$ is a real number; this is *Bessel's equation of order* ν. [Friedrich W. Bessel (1784-1846), German astronomer and mathematician; first to use the method of parallax to determine the distance away of a star; developed his equation, and its solution, in his study of the perturbations of planetary orbits.] We shall use the method of Frobenius to construct solutions of this equation, for various ν; we note that this is permitted because the equation possesses a regular singular point at $x = 0$.

3.1 First solution

We set

$$y(x) = \sum_{n=0}^{\infty} a_n x^{\lambda+n} \quad (a_0 \neq 0)$$

to give

$$\sum_{n=0}^{\infty} a_n(\lambda+n)(\lambda+n-1)x^{\lambda+n} + \sum_{n=0}^{\infty} a_n(\lambda+n)x^{\lambda+n} + \left(x^2 - \nu^2\right)\sum_{n=0}^{\infty} a_n x^{\lambda+n} = 0;$$

in the first, second and the constant multiple of the third series we set $n = m$, and in the other series we write $n + 2 = m$. Thus we obtain

$$a_0\left(\lambda^2 - \nu^2\right)x^{\lambda} + a_1\left[(\lambda+1)^2 - \nu^2\right]x^{\lambda+1} + \sum_{m=2}^{\infty}\left\{a_m\left[(\lambda+m)^2 - \nu^2\right] + a_{m-2}\right\}x^{\lambda+m} = 0,$$

which is an identity for all *x* if

$$\lambda = \pm\nu \ (a_0 \neq 0);\ a_1 = 0 \ (\lambda + 1 \neq \pm\nu) \text{ and } a_m\left[(\lambda+m)^2 - \nu^2\right] = -a_{m-2} \ (m = 2, 3, \dots).$$

For $\lambda = +\nu$, we certainly have a solution with

$$a_m = -\frac{a_{m-1}}{m(m+2\nu)}, \; m = 2, 3, \ldots,$$

and $0 = a_1 = a_3 = a_5 = \ldots.$. Thus we obtain

$$a_2 = -\frac{a_0}{2.2(1+\nu)}, \; a_4 = -\frac{a_2}{4.2(2+\nu)} = \frac{a_0}{4.2^3(1+\nu)(2+\nu)}, \text{ etc.,}$$

giving $$a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m!)(\nu+m)(\nu+m-1)\ldots(\nu+1)},$$

which can be expressed in terms of the gamma function:

$$\Gamma(\nu) = \int_0^\infty t^{\nu-1}\mathrm{e}^{-t}\mathrm{d}t \;\; (\nu > 0)$$

as $$a_{2m} = \frac{(-1)^m \Gamma(\nu+1)}{2^{2m}(m!)\Gamma(\nu+m+1)} a_0 \cdot.$$

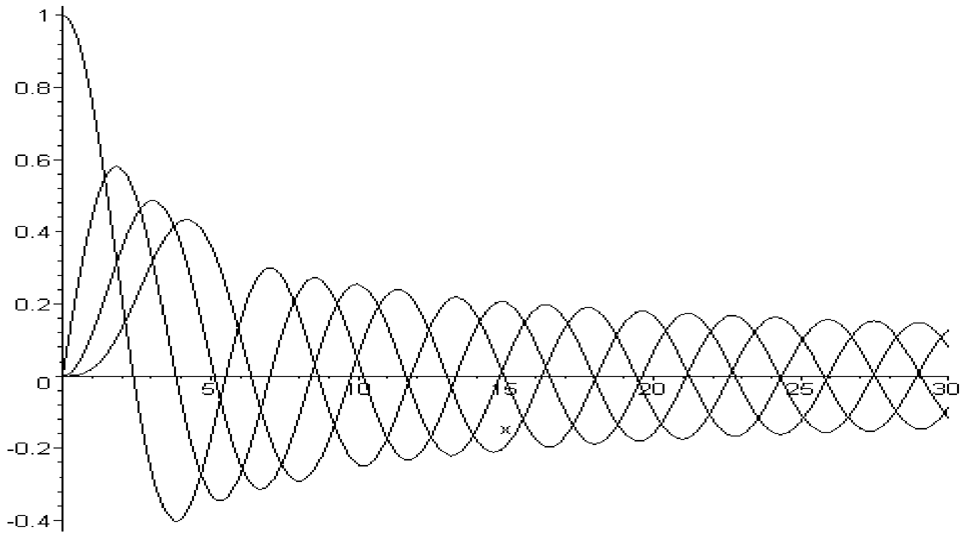
Thus we have a solution

$$y(x) = a_0 2^\nu \Gamma(\nu+1) \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)\Gamma(\nu+m+1)} \left(\frac{x}{2}\right)^{2m+\nu}$$

which is usually written as the *Bessel function of the first kind*, *of order* ν:

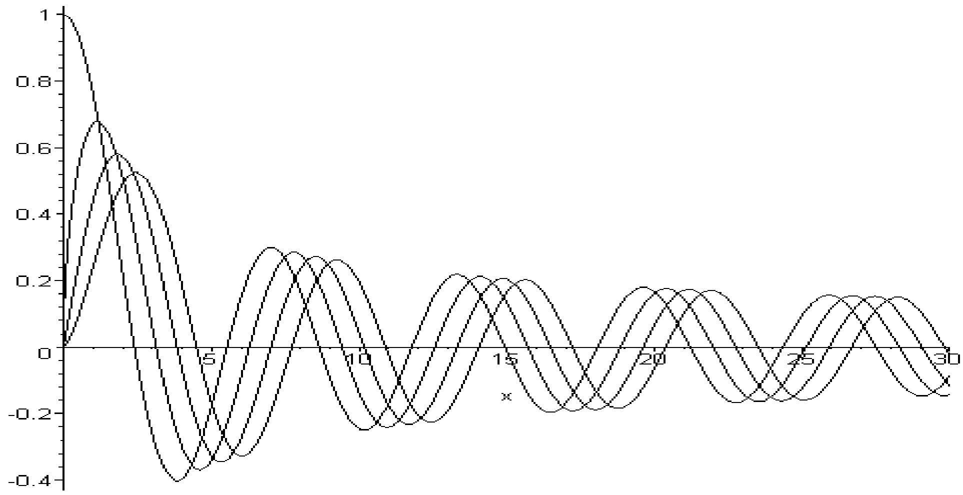
$$\mathrm{J}_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)\Gamma(\nu+m+1)} \left(\frac{x}{2}\right)^{2m}.$$

This series converges for all $x \geq 0$. Examples of these Bessel functions are given in the two figures below.



The Bessel functions J_0, J_1, J_2, J_3, identified in this order by the ordering, from left to right, of the first peak on each curve.

The Bessel functions J_0, J_1, J_2, J_3, identified in this order by the ordering, from left to right, of the first peak on each curve.



The Bessel functions $J_0, J_{1/2}, J_1, J_{3/2}$, identified in this order by the ordering, from left to right, of the first peak on each curve.

The Bessel functions $J_0, J_{1/2}, J_1, J_{3/2}$, identified in this order by the ordering, from left to right, of the first peak on each curve.

3.2 The second solution

From our work with the method of Frobenius, in the previous chapter, we know that the second solution of these equations depends critically on the relationship between the two roots for λ. First we note that only for $\nu = 0$ are the roots repeated, and this case is essentially the same as $\nu = n$ (n integer), for then $\pm\nu$ will differ by an integer; in these cases, we must expect that logarithmic terms will arise (unless something exceptional occurs). In addition, there are non-integral values of ν which also give rise to λs that differ by an integer: all the half-integers e.g. $\nu = 1/2$, for then $\lambda = \pm 1/2$ (which differ by 1). For all other ν, the solution that we have just described is appropriate when ν is replaced by $-\nu$, the solutions then being $\mathrm{J}_{-\nu}(x)$; this is the second solution.

In the case $\nu = n$ ($n = 1, 2, \ldots$), with $\lambda = -\nu = -n$, the recurrence relation becomes $a_m m(m-2n) = -a_{m-2}$ ($m = 2, 3, \ldots$)

which is not defined when $m = 2n$, for a given n. Thus this case, and that of a repeated root, will certainly require the inclusion of a logarithmic term. The calculation follows that described in Chapter 2 and, although it is rather cumbersome and lengthy, it is altogether routine. So, for example, for $n = 0$, we find that the second solution can be written

$$y(x) = \mathrm{J}_0(x)\ln x + \left(\frac{x}{2}\right)^2 - \frac{3}{2}\left(\frac{1}{2!}\right)^2\left(\frac{x}{2}\right)^4 + \ldots .$$

However, this solution is usually combined with $\mathrm{J}_0(x)$ in the form

$$\mathrm{Y}_0(x)=\frac{2}{\pi}\left\{\left(\gamma+\ln\left(\frac{x}{2}\right)\right)\mathrm{J}_0(x)+\left(\frac{x}{2}\right)^2-\left(1+\frac{1}{2}\right)\left(\frac{1}{2!}\right)^2\left(\frac{x}{2}\right)^4 +\left(1+\frac{1}{2}+\frac{1}{3}\right)\left(\frac{1}{3!}\right)^2\left(\frac{x}{2}\right)^6-\ldots\right\},$$

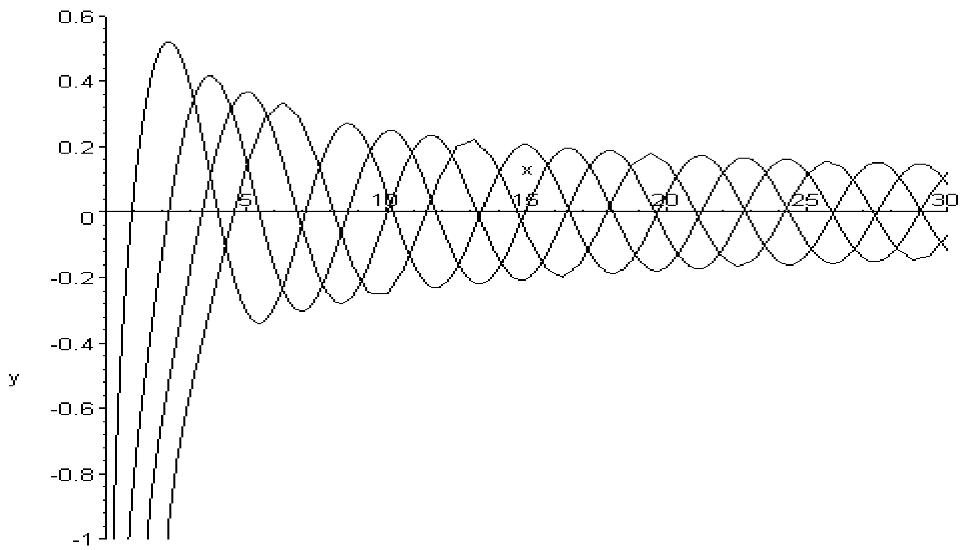
where $\gamma=-\int_0^\infty \mathrm{e}^{-t}\ln t\,\mathrm{d}t\approx 0\cdot 5772$ is Euler's constant; the resulting solution, Y_0, is the *Bessel function of the second kind, of order zero*. Correspondingly, with $\nu=n$ (>0), we find

$$\mathrm{Y}_n(x)=\frac{2}{\pi}\left\{\left(\gamma+\ln\left(\frac{x}{2}\right)\right)\mathrm{J}_n(x)-\frac{1}{2}\sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}\left(\frac{x}{2}\right)^{2r-n} -\frac{1}{2}\frac{(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n})}{n!}\left(\frac{x}{2}\right)^n+\frac{1}{2}\frac{(1)+(1+\frac{1}{2}+\ldots+\frac{1}{1+n})}{(1!)[(n+1)!]}\left(\frac{x}{2}\right)^{n+2}+\ldots\right\}.$$

Finally, we consider the case where ν is half-integer. The recurrence relation now becomes, with $\lambda=-\nu$ and $\nu=-(1+2n)/2$ $(n=0,1,2,\ldots)$,

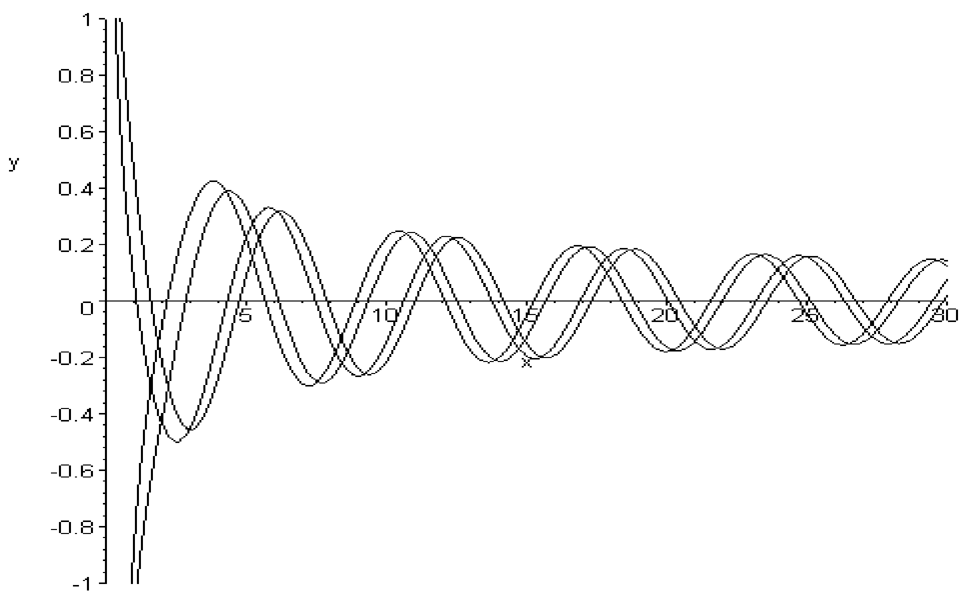
$$a_m m\left(m-n-\tfrac{1}{2}\right)=-a_{m-2}\quad (m=2,3,\ldots),$$

and we see that $m-n-\frac{1}{2}\neq 0$ for every valid choice of *m* and *n*; thus the recurrence is defined and the second solution exists without a logarithmic term – and it simply takes the form $\mathrm{J}_{-\nu}(x)$ for the appropriate ν. Some examples of these Bessel functions are shown in the next two figures; note that all these solutions diverge as $x\to 0$.



The Bessel functions Y_0, Y_1, Y_2, Y_3, identified in this order by the ordering,

from left to right, of the first peak on each curve.



The Bessel functions $J_{-7/4}, J_{-3/2}, J_{-3/4}, J_{-1/2}$, identified in this order by the ordering, from left to right, of the first peak (ignoring the asymptote along the *y*-axis) on each curve.

Example 8

Write down the general solution of the equation $xy'' + y' + a^2xy = 0$, where $a > 0$ is a constant.

When the equation is multiplied by *x* it becomes – almost – a zero-order Bessel equation; it can be recast precisely in this form if we write $y(x) = u(\alpha x)$, where α is a constant to be determined. The equation becomes

$$x^2\alpha^2 u'' + x\alpha u' + a^2x^2u = 0,$$

$$\text{or } (\alpha x)^2 u'' + (\alpha x)u' + \left(\frac{a}{\alpha}\right)^2 (\alpha x)^2 u = 0,$$

so we choose $\alpha = a$: $z^2u'' + zu' + z^2u = 0$ ($u = u(z)$, $z = \alpha x$). This is the zero-order Bessel equation, with general solution $u(z) = A\mathrm{J}_0(z) + B\mathrm{Y}_0(z)$ i.e.

$$y(x) = A\mathrm{J}_0(ax) + B\mathrm{Y}_0(ax).$$

3.3 The modified Bessel equation

We comment briefly on an equation that is intimately related to the conventional Bessel equation, namely $x^2y''+xy'-\left(x^2+\nu^2\right)y=0$, called the *modified Bessel equation of order* ν. We observe that the only change occurs in the term x^2y – it is negative here. This immediately suggests that we should consider, formally at least, a change of variable $x=\mathrm{i}z$ in the standard Bessel equation:

$$x^2y''+xy'-\left(x^2+\nu^2\right)y=0.$$

This yields $z^2Y''+zY'-\left(z^2+\nu^2\right)Y=0$, $\quad y(\mathrm{i}z)=Y(z)$,

for which we have a solution. Thus, for example, we define

$$\mathrm{I}_n(z)=\mathrm{i}^{-n}\mathrm{J}_n(\mathrm{i}z)=\sum_{r=0}^{\infty}\frac{(z/2)^{2r+n}}{r!(r+n)!},$$

which is the *modified Bessel function of the first kind, of order n*. The function that corresponds to Y_n is usually defined by

$$\mathrm{K}_n(z)=\frac{\pi}{2}\mathrm{i}^{n+1}\left[\mathrm{J}_n(\mathrm{i}z)+\mathrm{i}\mathrm{Y}_n(\mathrm{i}z)\right],$$

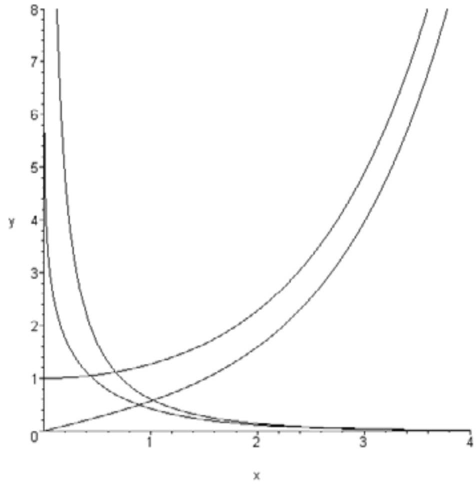
which is the *modified Bessel function of the second kind, of order n*. Written explicitly, for $n=0$ (and reverting to the more conventional *x*), we obtain

$$\mathrm{I}_0(x)=1+\frac{(x/2)^2}{(0!)^2}+\frac{(x/2)^4}{(2!)^2}+\frac{(x/2)^6}{(3!)^2}+\ldots.$$

$$\mathrm{K}_0(x)=-(\gamma+\ln(x/2))\mathrm{I}_0(x)+(1)\frac{(x/2)^2}{(1!)^2}+\left(1+\tfrac{1}{2}\right)\frac{(x/2)^4}{(2!)^2}+\left(1+\tfrac{1}{2}+\tfrac{1}{3}\right)\frac{(x/2)^6}{(3!)^2}+\ldots;$$

;

some examples of modified Bessel functions are shown in the figure below.



The modified Bessel functions $\mathrm{K}_0, \mathrm{K}_1, \mathrm{I}_0, \mathrm{I}_1$, identified in this order by the ordering,

from left to right, of the upper part of the curves.

Exercises 3

1. Write the general solutions of these equations in terms of Bessel functions:

   (a) $x^2 y'' + xy' + \left(x^2 - 1\right)y = 0$; ; (b) $9x^2 y'' + 9xy' + \left(9x^2 - 4\right)y = 0$. .

2. Seek a solution of the equation $xy'' + (1-2\nu)y' + xy = 0$ in the form $y(x) = x^{\nu} Y(x)$; use your result to find the general solution of the equation

   $$xy'' - 2y' + xy = 0.$$

3. Show that the differential equation

   $$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^a y'\right) + bx^c y = 0 \qquad (c - a + 2 \neq 0,\ b > 0)$$

   can be transformed into a Bessel equation for $Y(X)$:

   $$X^2 Y'' + XY' + \left(X^2 - \nu^2\right)Y = 0$$

by $X = \frac{\sqrt{b}}{\lambda} x^{\lambda}$, $Y = x^{-\lambda\nu} y$, for suitable λ and ν. Find λ and ν and hence obtain the general solution of the equation

$$xy'' + 3y' + y = 0.$$

4. In the definition of the Bessel function of the first kind (J_{ν}), select $\nu = \pm 1/2$ and hence show that

$$J_{1/2} = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{-1/2} = \sqrt{\frac{2}{\pi x}} \cos x.$$

5. The Bessel functions of the first kind, with integer orders, can be represented by

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x \sin\theta) \, dx.$$

Use this expression to show that

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x); \quad \frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x),$$

and hence obtain the recursion formula

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x).$$

6. Use the recursion formula in Q.5, and the results in Q.4, to obtain closed-form representations of $J_{3/2}(x)$ and $J_{-3/2}(x)$.

4 The Legendre polynomials

Another equation that plays a significant rôle in many branches of the applications of mathematics (particularly in problems that involve spherical symmetry) is the *Legendre equation*

$$\left(1-x^2\right)y'' - 2xy' + \lambda y = 0 \, ,$$

where λ is a parameter. [A.-M. Legendre (1752-1833), French mathematician who made contributions to number theory, the theory of elliptic functions and to celestial mechanics, which is where his equation first arose.] This equation has regular singular points at $x = \pm 1$, and so the domain of interest is normally $-1 < x < 1$ (for general λ); however, it is usual to seek a power-series solution about $x = 0$ (which is, of course, an ordinary point). Thus we write

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

to give

$$\left(1-x^2\right)\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - 2\sum_{n=0}^{\infty} a_n n x^{n-1} + \lambda \sum_{n=0}^{\infty} a_n x^n = 0$$

which becomes

$$\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m - \sum_{m=0}^{\infty} a_m\{m(m-1)+2m-\lambda\}x^m = 0\,.$$

This equation is an identity for all x if

$$a_{m+2}(m+1)(m+2) = a_m[m(m+1)-\lambda],\quad m = 0,1,2,\ \ldots,$$

with both a_0 and a_1 arbitrary. Thus, for general λ, we have a general solution:

$$y(x) = a_0\left[1 - \tfrac{1}{2}\lambda x^2 + \tfrac{1}{24}\lambda(\lambda-6)x^4 - \tfrac{1}{720}\lambda(\lambda-6)(\lambda-20)x^6 + \ldots\right]$$

$$+a_1\left[x - \tfrac{1}{6}(\lambda-2)x^3 + \tfrac{1}{120}(\lambda-2)(\lambda-12)x^5 + \ldots\right].$$

The ratio test then gives

$$\left|\frac{a_{m+2}x^{m+2}}{a_m x^m}\right| < 1 \text{ as } m \to \infty \text{ i.e. } \left|\frac{m(m+1)-\lambda}{(m+1)(m+2)}x^2\right| < 1 \text{ as } m \to \infty,$$

so that each series that contributes to the general solution converges for $-1 < x < 1$.

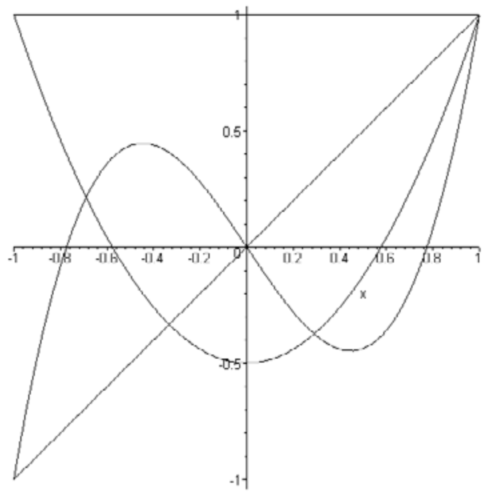
In many applications we need bounded solutions for $-1 \le x \le 1$ and, for general λ, this is impossible: the series are not convergent at $x = \pm 1$. (Indeed, a Frobenius approach about $x = +1$ or $x = -1$ shows that the solution diverges like $\ln(x \mp 1)$ as $x \to \pm 1$.) However, if the series possesses a finite number of terms i.e. it is polynomial, then it certainly will remain bounded for all $x \in [-1,1]$; we then say that the series *terminates*. This situation will occur here for any $\lambda = n(n+1)$, $n = 0,1,2,\ \ldots$; the first few such solutions are then

$$n = 0,\ y = 1;\ n = 1,\ y = x;\ n = 2,\ y = 1 - 3x^2;\ n = 3,\ y = x - \tfrac{5}{3}x^3,$$

and so on. Of course, these are still solutions when multiplied by constants, and when these are chosen to ensure that $y(1) = 1$, for each n, they are known as the *Legendre polynomials* ($\mathrm{P}_n(x)$), so we have

$$\mathrm{P}_0(x) = 1,\ \mathrm{P}_1(x) = x,\ \mathrm{P}_2(x) = \tfrac{1}{2}\left(3x^2 - 1\right),\ P_3(x) = \tfrac{1}{2}\left(5x^3 - 3x\right), \text{ etc.}$$

These first few polynomials are shown in the figure below.



The Legendre polynomials, P_0, P_1, P_2, P_3, identified in this order by

the number of times (n) that each crosses the x-axis.

Exercises 4

1. Obtain the Legendre polynomials $P_4(x)$ and $P_5(x)$.
2. Confirm that $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right]$ for $n = 0,1,2,3,4$. (This is called *Rodrigues' formula*.)
3. Legendre's equation in the form $\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0$ has one solution $y(x) = P_n(x)$

   ; use the method of variation of parameters to find a second solution (usually written $Q_n(x)$) in the cases

   $n = 0$ and $n = 1$.

4. Use Rodrigues' formula (Q.2) to obtain the *recursion formula*

   $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$.

5 The Hermite polynomials

Another standard second order ODE is

$$y'' - 2xy' + \lambda y = 0\,,\ -\infty < x < \infty\,,$$

which appears, most typically, in elementary solutions of Schrödinger's equation; λ is a parameter. This is *Hermite's equation*, where special choices of λ give rise to the *Hermite polynomials*. [Charles Hermite (1822-1901), French mathematician who made important contributions to the theory of differential equations and also worked on the theory of matrices; in 1873 he proved that e is a transcendental number.] The equation has no singular points – all (finite) points are ordinary points – so we seek a solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

which gives

$$\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - 2\sum_{n=0}^{\infty} a_n n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0 \,.$$

This is written as

$$\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m - \sum_{m=0}^{\infty} a_m(2m-\lambda)x^m = 0$$

which is an identity for all values of x if

$$a_{m+2}(m+2)(m+1) = a_m(2m-\lambda),\quad m = 0,1,2,\ \dots,$$

with both a_0 and a_1 arbitrary. This recurrence relation gives directly the general solution

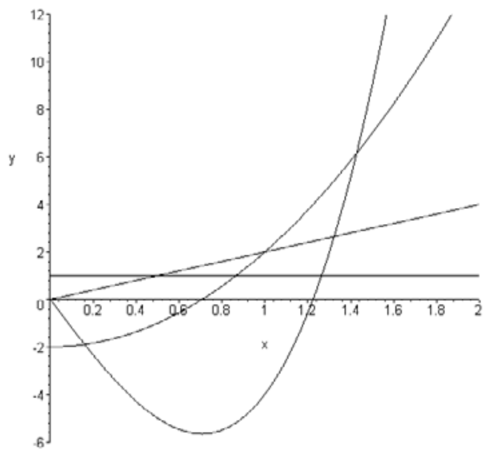
$$y(x) = a_0\left[1 - \tfrac{1}{2!}\lambda x^2 + \tfrac{1}{4!}\lambda(\lambda-4)x^4 - \dots\right]$$

$$+a_1\left[x - \tfrac{1}{3!}(\lambda-2)x^3 + \tfrac{1}{5!}(\lambda-2)(\lambda-6)x^5 + \dots\right].$$

It is immediately clear that there exists a polynomial solution of the original equation whenever $\lambda = 2n$, $n = 0, 1, 2, \dots$. With the choice $\lambda = 2n$, and the arbitrary multiplicative constant chosen so that the coefficient of the term x^n is 2^n, the resulting solution is the *Hermite polynomial*, $\mathrm{H}_n(x)$. Thus we have

$\mathrm{H}_0(x) = 1$, $\mathrm{H}_1(x) = 2x$, $\mathrm{H}_2(x) = 4x^2 - 2$, $\mathrm{H}_3(x) = 8x^3 - 12x$, $\mathrm{H}_4(x) = 16x^4 - 48x^2 + 12$, etc.;

the first four of these are shown in the figure below.



The Hermite polynomials, $\mathrm{H}_0, \mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3$, identified in this order by

counting up on the far right of the figure.

Exercises 5

1. Write down $\mathrm{H}_5(x)$.
2. Confirm that $\mathrm{H}_n(x) = (-1)^n \mathrm{e}^{x^2} \dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{-x^2}\right)$ for $n = 0, 1, 2, 3, 4$. (This is Rodrigues' formula for the Hermite polynomials.)
3. The equation $y'' - 2xy' + 2ny = 0$ has one solution $y = \mathrm{H}_n(x)$; use the method of variation of parameters to find a second solution in the cases $n = 0$ and $n = 1$.
4. Use Rodrigues' formula (Q.2) to obtain the *recursion formula*

   $$\mathrm{H}_{n+1}(x) = 2x\mathrm{H}_n(x) - 2n\mathrm{H}_{n-1}(x).$$

6 Generating functions

We conclude with an additional and intriguing property of the functions that we have discussed in this Notebook (and similar properties arise for other functions and polynomials). The existence of recursion formulae, and also the Rodrigues' formulae [O. Rodrigues (1794-1851), French mathematician] for the polynomials, is related to another important idea: the *generating function*.

Suppose that we are given a set of functions, $F_n(x)$ say; further, suppose that these can be obtained as the coefficients of some power series (in *z*, say) i.e. we write

$$\sum_{n=0}^{\infty} F_n(x) z^n .$$

If we can find a function, $f(z;x)$, whose Maclaurin expansion in *z* recovers the series, so that we have

$$f(z;x) = \sum_{n=0}^{\infty} F_n(x) z^n ,$$

we call $f(z;x)$ a *generating function*. (It obviously *generates*, in a purely algebraic way, the set of functions that are of interest.) An elementary example of this idea is the choice

$$f(z;x) = \sqrt{z+x}$$

which has the Maclaurin – actually binomial in this case – expansion

$$f(z;x) = \sqrt{x}\left\{1 + \frac{1}{2}\frac{z}{x} - \frac{1}{8}\left(\frac{z}{x}\right)^2 + \frac{1}{16}\left(\frac{z}{x}\right)^3 - \dots\right\};$$

thus we have generated the set of functions

$$\sqrt{x}, \frac{1}{2}\frac{1}{\sqrt{x}}, -\frac{1}{8}\frac{1}{x\sqrt{x}}, \frac{1}{16}\frac{1}{x^2\sqrt{x}}, \dots$$

(and the coefficients that appear here may, or may not, be included). We may therefore regard $f(z;x) = \sqrt{z+x}$ as a generating function for $F_n(x) = x^{\frac{1}{2}-n}$.

For each of our three examples (Bessel, Legendre, Hermite), simple generating functions exist; we will present them – detailed proofs of their correctness are beyond this text – and make a few observations about each.

6.1 Legendre polynomials

First we consider the function $1\big/\sqrt{1-2xz+z^2}$, which has the expansion (in z)

$$\left(1-2xz+z^2\right)^{-1/2} = 1-\tfrac{1}{2}\left(-2xz+z^2\right)+\tfrac{3}{8}\left(-2xz+z^2\right)^2-\dots.$$

$$= 1+xz+\left(\tfrac{3}{2}x^2-\tfrac{1}{2}\right)z^2+\dots.$$

and we recognise the coefficients of z^0, z^1, z^2: 1, x, $\tfrac{1}{2}\left(3x^2-1\right)$, respectively. These are the first three Legendre polynomials; see Chapter 4. In general, we find that

$$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} \mathrm{P}_n(x)\,z^n$$

and this property can be analysed in a number of ways; the most powerful is to differentiate each side with respect to z and, separately, with respect to x. First, the z-derivative gives

$$-\tfrac{1}{2}\left(1-2xz+z^2\right)^{-3/2}\left(-2x+2z\right) = \sum_{n=0}^{\infty} \mathrm{P}_n(x)\,nz^{n-1}$$

which can be written (upon using the original definition of the generating function) as

$$(x-z)\sum_{n=0}^{\infty} \mathrm{P}_n(x)\,z^n = \left(1-2xz+z^2\right)\sum_{n=0}^{\infty} n\mathrm{P}_n(x)\,z^{n-1}$$

or $$-\sum_{m=1}^{\infty} \mathrm{P}_{m-1}\,z^m + x\sum_{m=0}^{\infty} \mathrm{P}_m\,z^m$$

$$= \sum_{m=-1}^{\infty}(1+m)\mathrm{P}_{m+1}\,z^m - 2x\sum_{m=0}^{\infty} m\mathrm{P}_m\,z^m + \sum_{m=1}^{\infty}(m-1)\mathrm{P}_{m-1}\,z^m .$$

This is an identity in z if

$$x\mathrm{P}_0 = \mathrm{P}_1 \text{ and } x\mathrm{P}_m - \mathrm{P}_{m-1} = (m+1)\mathrm{P}_{m+1} - 2xm\mathrm{P}_m + (m-1)\mathrm{P}_{m-1} \;\; (m = 1, 2, \dots)$$

and the first of these is automatically satisfied; the second can be written as

$$(n+1)\mathrm{P}_{n+1}(x) = x(2n+1)\mathrm{P}_n(x) - n\mathrm{P}_{n-1}(x) ,$$

which is the *recursion formula* for Legendre polynomials (see Exercises 4).

On the other hand, if we differentiate with respect to x, we obtain

$$-\tfrac{1}{2}\left(1-2xz+z^2\right)^{-3/2}(-2z) = \sum_{n=0}^{\infty}\mathrm{P}_n'(x)\,z^n$$

$$\text{or} \quad z\sum_{n=0}^{\infty}\mathrm{P}_n(x)\,z^n = \left(1-2xz+z^2\right)\sum_{n=0}^{\infty}\mathrm{P}_n'(x)\,z^n \ ,$$

which can be simplified when we use the previous derivative:

$$z\sum_{n=0}^{\infty}n\mathrm{P}_n(x)\,z^{n-1} = (x-z)\sum_{n=0}^{\infty}\mathrm{P}_n'(x)\,z^n .$$

This is conveniently written as

$$\sum_{m=0}^{\infty}m\mathrm{P}_m(x)\,z^m = x\sum_{m=0}^{\infty}\mathrm{P}_m'(x)\,z^m - \sum_{m=1}^{\infty}\mathrm{P}_{m-1}'(x)\,z^m$$

which is an identity in z if

$$\mathrm{P}_0' = 0 \text{ and } m\mathrm{P}_m = x\mathrm{P}_m' - \mathrm{P}_{m-1}' \ (m = 1, 2, \ldots);$$

we know that $\mathrm{P}_0' = 0$, so we are left with

$$\mathrm{P}_{n-1}'(x) = x\mathrm{P}_n'(x) - n\mathrm{P}_n(x)$$

– another type of recurrence relation.

6.2 Hermite polynomials

The function that we consider here is

$$\exp\left(2xz - z^2\right)$$

which possesses the expansion

$$\exp\left(2xz - z^2\right) = \left(1 + 2xz + \frac{(2xz)^2}{2!} + \frac{(2xz)^3}{3!} + \frac{(2xz)^4}{4!} + \ldots\right)\left(1 - z^2 + \frac{z^4}{2!} + \ldots\right)$$
$$= 1 + 2xz + \left(2x^2 - 1\right)z^2 + \left(\tfrac{4}{3}x^3 - 2x\right)z^3 + \ldots$$

and again we recognise (see Chapter 5) the first four H_ns, in the form

$$\mathrm{H}_0, \mathrm{H}_1, \tfrac{1}{2}\mathrm{H}_2, \tfrac{1}{6}\mathrm{H}_3.$$

Thus we have a generating function for the Hermite polynomials, written as

$$\exp\left(2xz - z^2\right) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\mathrm{H}_n(x)\, z^n$$

(with the usual identification: $0! = 1$). This generating function can be used (as described in the preceding section) to derive various recurrence relations.

The z-derivative yields

$$2(x-z)\exp\left(2xz-z^2\right)=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\mathrm{H}_n(x)\,z^{n-1}$$

or
$$2(x-z)\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{H}_n(x)\,z^n=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\mathrm{H}_n(x)\,z^{n-1}$$

which can be written as

$$2x\sum_{m=0}^{\infty}\frac{1}{m!}\mathrm{H}_m(x)\,z^m-2\sum_{m=1}^{\infty}\frac{1}{(m-1)!}\mathrm{H}_{m-1}(x)\,z^m=\sum_{m=0}^{\infty}\frac{1}{m!}\mathrm{H}_{m+1}(x)\,z^m \; .$$

This is an identity in z if

$$2x\mathrm{H}_0=\mathrm{H}_1 \text{ and } \frac{2x}{m!}\mathrm{H}_m-\frac{2}{(m-1)!}\mathrm{H}_{m-1}=\frac{1}{m!}\mathrm{H}_{m+1} \;(m=1,2,\,\dots);$$

the first of these is automatically satisfied, and the second becomes

$$\mathrm{H}_{n+1}(x)=2x\mathrm{H}_n(x)-2n\mathrm{H}_{n-1}(x) \;(n=1,2,\,\dots),$$

which appears in Exercises 5.

6.3 Bessel functions

Finally, we return to the functions which possess a far more complicated structure: the Bessel functions (discussed in Chapter 3). We shall present the relevant details for the Bessel functions $\mathrm{J}_{\pm n}(x)$ (for $n=0,1,2,\,\dots$) by considering the function

$$\exp\left\{\frac{1}{2}\left(z-\frac{1}{z}\right)x\right\} \quad (z\neq 0)$$

which, we observe, will necessarily generate a more complicated powers series (in z) than has arisen in our earlier examples. This time we must expand in a Laurent series (i.e. terms z^n and z^{-n} will appear), producing a power series that is convergent for $0<|z|<\infty$. The expansion is

$$\mathrm{e}^{xz/2}.\mathrm{e}^{-x/2z} = \left(1+\frac{(xz/2)}{1!}+\frac{(xz/2)^2}{2!}+\frac{(xz/2)^3}{3!}+\ldots.\right)$$

$$\times\left(1-\frac{(x/2z)}{1!}+\frac{(x/2z)^2}{2!}-\frac{(x/2z)^3}{3!}+\ldots.\right)$$

$$=\left\{1-\left(\frac{x}{2}\right)^2+\frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4-\ldots\right\}+z\frac{x}{2}\left\{1-\frac{1}{2}\left(\frac{x}{2}\right)^2+\frac{1}{2!3!}\left(\frac{x}{2}\right)^4-\ldots\right\}$$

$$-\frac{1}{z}\frac{x}{2}\left\{1-\frac{1}{2}\left(\frac{x}{2}\right)^2+\frac{1}{2!3!}\left(\frac{x}{2}\right)^4-\ldots.\right\}+\ldots..$$

$$=\sum_{n=-\infty}^{\infty} z^n \mathrm{J}_n(x) \quad .$$

Thus we have demonstrated (in outline, at least) that

$$\exp\left\{\tfrac{1}{2}\left(z-z^{-1}\right)x\right\}=\sum_{n=-\infty}^{\infty} z^n \mathrm{J}_n(x)$$

which, as developed earlier, can be used to obtain various recurrence formulae. The z-derivative gives

$$\tfrac{1}{2}\left(1+z^{-2}\right)x\exp\left\{\tfrac{1}{2}\left(z-z^{-1}\right)x\right\}=\sum_{n=-\infty}^{\infty} nz^{n-1}\mathrm{J}_n(x)$$

$$\text{or } \tfrac{1}{2}\left(z+z^{-1}\right)x\sum_{n=-\infty}^{\infty} z^n \mathrm{J}_n(x)=\sum_{n=-\infty}^{\infty} nz^n \mathrm{J}_n(x)$$

which leads directly to

$$\mathrm{J}_{n+1}(x)=\frac{2n}{x}\mathrm{J}_n(x)-\mathrm{J}_{n-1}(x) \text{ (see Exercises 3).}$$

Correspondingly, the x-derivative produces

$$\tfrac{1}{2}\left(z-z^{-1}\right)\exp\left\{\tfrac{1}{2}\left(z-z^{-1}\right)x\right\}=\sum_{n=-\infty}^{\infty} z^n \mathrm{J}'_n(x)$$

$$\text{or } \tfrac{1}{2}\left(z+z^{-1}\right)x\sum_{n=-\infty}^{\infty} z^n \mathrm{J}_n(x)=\sum_{n=-\infty}^{\infty} nz^n \mathrm{J}_n(x)$$

which produces

$$J_n'(x) = \tfrac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right].$$

Exercises 6

1. Use the appropriate generating functions to find $P_3(x)$ and $H_4(x)$.

2. Use the appropriate recursion formulae (and any results already available) to find $P_6(x)$ and $H_6(x)$.

3. Use the recursion formulae (obtained from the generating functions; see §6.1) to show that $\left(1-x^2\right)P_n'' - 2xP_n' + n(n+1)P_n = 0$

   (thereby confirming that the generating function does produce functions P_n).

4. Repeat Q.3 for $H_n(x)$ (§6.2) and the equation $H_n'' - 2xH_n' + 2nH_n = 0$.

5. Repeat Q.3 for $J_n(x)$ (§6.3) and the equation $J_n'' + x^{-1}J_n' + \left(1-n^2x^{-2}\right)J_n = 0$..

6. Use the generating function for the Bessel functions, J_n, to show that

   $J_n(-x) = (-1)^n J_n(x)$ and $J_{-n}(x) = (-1)^n J_n(x)$,

   and hence deduce that $J_0'(x) = -J_1(x)$ and then that

   $$J_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x).$$

Answers

Exercises 1

1. (a) Regular singular points at $x=0$, $x=-1$; (b) regular singular point at $x=1$ and an essential singular point at $x=-1$; (c) no singular points – all (finite) points are ordinary points, but the point at infinity is an essential singular point.

Exercises 2

Where the result is easily obtained, the full series is presented.

1. (a) $y(x)=A\sum_{n=0}^{\infty}\frac{(-x)^n}{(2n)!}+B\sqrt{x}\sum_{n=0}^{\infty}\frac{(-x)^n}{(2n+1)!}$ (with $\lambda=0,\ 1/2$);

   (b) $y(x)=A\sqrt{x}\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}+Bx\sum_{n=0}^{\infty}\frac{(2n+1)!}{n!}(-4x)^n$ (with $\lambda=1/2, 1$) and note that the first solution

   here can be written $A\sqrt{x}\,\mathrm{e}^{-x}$;

(c) $y(x) = A\sqrt{x}\left(1 - \frac{7.1}{2.9}x + \frac{1.3.7.7}{2.4.9.11}x^2 + \ldots\right) + Bx^{-3}\left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3\right)$ (with $\lambda = -3, 1/2$);

(d) $y(x) = Ax^2 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} + B\left\{y_1(x)\ln|x| - x + x^2 - \frac{3}{4}x^4 + \ldots\right\}$ (with $\lambda = 1, 2$) and note that the first solution here can be written $Ax^2 e^{-x}$;

(e) $y(x) = A\sum_{n=0}^{\infty} \frac{\left(-x^2/4\right)^n}{(n!)^2} + B\left\{y_1(x)\ln|x| + \frac{1}{4}x^2 - \frac{3}{128}x^4 + \ldots\right\}$ (with $\lambda = 0, 0$);

(f) $y(x) = Ax^{-1}\left(1 - \frac{1}{4}x^2 + \ldots\right) + B\left\{y_1(x)\ln|x| - 1 + \frac{1}{2}x + \ldots\right\}$ (with $\lambda = -1, 1$).

2. Set $x - 1 = z$: $4zY'' + 2Y' - Y = 0$, then
$y(x) = A\sum_{n=0}^{\infty} \frac{(x-1)^n}{(2n)!} + B\sqrt{x-1}\sum_{n=0}^{\infty} \frac{(x-1)^n}{(2n)!}$ (with $\lambda = 0, 1/2$) which can be written
$y(x) = \left(A + B\sqrt{x-1}\right)\cosh(\sqrt{x-1})$.

Exercises 3

1. (a) $y(x) = A\mathrm{J}_1(x) + B\mathrm{Y}_1(x)$; (b) $y(x) = A\mathrm{J}_{2/3}(x) + B\mathrm{J}_{-2/3}(x)$.

2. $x^2Y'' + xY + \left(x^2 - \nu^2\right)Y = 0$; $y(x) = x^{3/2}\left(A\mathrm{J}_{3/2}(x) + B\mathrm{J}_{-3/2}(x)\right)$.

3. $\lambda = \frac{1}{2}(c - a + 2)$, $\nu = (1-a)/(c-a+2)$; $y(x) = \frac{1}{x}\left(A\mathrm{J}_2(x) + B\mathrm{Y}_2(x)\right)$.

4. $\mathrm{J}_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{1}{x}\sin x - \cos x\right)$; $\mathrm{J}_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\sin x + \frac{1}{x}\cos x\right)$.

Exercises 4

1. $\mathrm{P}_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$; $\mathrm{P}_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$.

2. $\mathrm{Q}_0(x) = A + B\ln\left|\frac{1+x}{1-x}\right|$; $\mathrm{Q}_1(x) = Ax + B\left(x\ln\left|\frac{1+x}{1-x}\right| - 2\right)$.

Exercises 5

1. $\mathrm{H}_5(x) = 32x^5 - 160x^3 + 120x$.

2. $n = 0$: $y(x) = \int^x e^{x^2}\,dx$; $n = 1$: $y(x) = x\int^x \frac{e^{x^2}}{x^2}\,dx$.

Exercises 6

1. $\mathrm{P}_3(x)=\frac{1}{2}\left(5x^3-3x\right)$; $\mathrm{H}_4(x)=16x^4-48x^2+12$.
2. $\mathrm{P}_6(x)=\frac{1}{16}\left(231x^6-315x^4+105x^2-5\right)$; $\mathrm{H}_6(x)=64x^6-480x^4+720x^2-120$.

Part II

An introduction to Sturm-Liouville theory

Preface

This text is intended to provide an introduction to some of the methods and ideas that come under the heading of 'Sturm-Liouville theory'. This topic is likely to be included in any modern, fairly comprehensive mathematical degree programme, probably linked to the method of series solution (Part I). This volume in the 'Notebook Series' has been written to extend, and explain in more detail, these ideas, but it has not been designed to replicate any typical module that includes this material. Indeed, the intention is to present the material so that it can be used as an adjunct to a number of different modules – or simply to provide a different and broader experience of these mathematical ideas. The aim is to go beyond the methods and techniques that are presented in the relevant course of study, but the standard ideas are discussed here, and can be accessed through the comprehensive index.

It is assumed that the reader has a basic knowledge of, and practical experience in, the elementary methods for solving ordinary differential equations (as encountered in any standard course in the first year of study). Some knowledge of the material taught beyond the first year (e.g. series solution: Part I) would be an advantage, but this is not essential. Further, this brief notebook does not attempt to include any applications of these equations; this is properly left to a specific module that might be offered in a conventional applied mathematics or engineering mathematics or physics programme.

The approach adopted here is to present some general ideas, which might involve a notation, or a definition, or a theorem, or a classification, but most particularly methods of solution, explained with a number of carefully worked examples – we present 11. A small number of exercises, with answers, are also offered in order to aid the learning process.

List of Equations

These are the equations, and associated problems, that are discussed in the examples.

1 Introduction and Background

The thrust of this volume is to present and describe some of the essential features of the properties of solutions of second order, linear, ordinary differential equations that contain a parameter. Most significantly, these ideas are developed within a general framework which does not require explicit solutions to have been obtained. Indeed, it will become apparent that it is possible to derive some important (and often quite detailed) properties of the solution of these equations, even when we cannot write down a solution. First, we shall provide a general description of the type of equations, and problems, that we shall discuss here.

1.1 The second-order equations

We start with the homogeneous equation for $y(x)$:

$$L(y) \equiv p_0(x)y'' + p_1(x)y' + p_2(x)y = 0 \tag{1}$$

where the prime denotes the derivative with respect to x and L represents the differential operator

$$p_0(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + p_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + p_2(x);$$

the functions p_0, p_1 and p_2 are continuous throughout the domain of the solution. Because this is a second-order equation, the general solution will be constructed from two linearly independent solutions ($y_1(x)$, $y_2(x)$, say) i.e.

$$y(x) = Ay_1(x) + By_2(x),$$

where A and B are arbitrary constants.

As an introduction to our more detailed work, it is convenient to write down the two equations that, separately, define $y_1(x)$ and $y_2(x)$:

$$p_0y_1'' + p_1y_1' + p_2y_1 = 0 \text{ and } p_0y_2'' + p_1y_2' + p_2y_2 = 0,$$

and then we form $y_2 \times \text{first} - y_1 \times \text{second}$, to give

$$p_0\left(y_2y_1'' - y_1y_2''\right) + p_1\left(y_2y_1' - y_1y_2'\right) = 0.$$

Now we introduce

$$W(y_1, y_2) = y_1y_2' - y_2y_1' \quad \left(= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \right), \tag{2}$$

then $\dfrac{\mathrm{d}W}{\mathrm{d}x} = y_1 y_2'' - y_2 y_1''$ and so our new equation can be written as

$$p_0 \frac{\mathrm{d}W}{\mathrm{d}x} + p_1 W = 0 \text{ or } W = \exp\left\{-\int \frac{p_1(x)}{p_0(x)}\mathrm{d}x\right\},$$

and if we have data given somewhere, at $x = x_0$, say, then

$$W = \exp\left\{-\int \frac{p_1(x)}{p_0(x)}\mathrm{d}x\right\},$$

where $W_0 = W\big(y_1(x_0), y_2(x_0)\big) = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0)$. Consequently, either $W = 0$ everywhere (because $W_0 = 0$), or *W* is never zero (because $W_0 \neq 0$ and the exponential term can never be zero). Further, *W* (which is called the *Wronskian* of the two functions) possesses an additional and fundamental property: $W = 0$ if and only if y_1 and y_2 are linearly dependent functions.

Proof

Suppose that y_1 and y_2 *are* linearly dependent, then $y_2(x) = ky_1(x)$ for some (non-zero) constant *k*; then $W = ky_1y_1' - ky_1y_1' = 0$. On the other hand, if $W = 0$ then

$$y_1y_2' - y_2y_1' = 0 \text{ so } \frac{y_2'}{y_2} = \frac{y_1'}{y_1} \text{ or } \ln|y_2| = \ln|y_1| + \text{constant i.e. } y_2 = \text{constant} \times y_1 :$$

the functions $y_1(x)$ and $y_2(x)$ are linearly dependent. ■

Thus we deduce, for the solution of a linear, second-order ordinary differential equation, comprising two linearly independent solutions, that $W \neq 0$. However, if the two solutions are linearly dependent (so that we do not have the general solution), then $W = 0$ everywhere; indeed, this property of *W* can be used to test if the solution we have constructed is the general solution. When $W \neq 0$, we say that the functions form a *fundamental set*.

Example 1

Find the general solution of the equation $y'' + \omega^2 y = 0$ (where $\omega > 0$ is a real constant), and then find the Wronskian corresponding to the two independent solutions.

We have $y'' + \omega^2 y = 0$, a constant coefficient, second-order equation, so the two solutions are $y_1 = \sin(\omega x)$ and $y_2 = \cos(\omega x)$; the general solution is therefore

$$y(x) = A\sin(\omega x) + B\cos(\omega x).$$

Now $W = y_1y_2' - y_2y_1' = \sin(\omega x)[-\omega\cos(\omega x)] - \cos(\omega x)[\omega\cos(\omega x)]$

$$= -\omega \quad (\neq 0).$$

[Josef Maria Hoëné-Wronski (1778-1853) was a Polish mathematician with great manipulative skills, but not much concerned with rigor (which he regarded as a 'pendantry which prefers means to ends'). He spent most of his career in France – he became a French citizen in 1800 – where he often quarrelled with other mathematicians and the academic institutions: he was a troubled man. For many years his work was dismissed as having no merit. However, in recent times, it has been realised that he produced some important results – even if a lot of what he did is erroneous.]

We conclude this section by commenting on two other variants of the second-order equations that we shall discuss. The first is at the heart of Sturm-Liouville theory; write equation (1) in the form

$$y'' + \frac{p_1}{p_0}y' + \frac{p_2}{p_0}y = 0 \text{ or } \frac{\mathrm{d}}{\mathrm{d}x}\left[y' \exp\left(\int \frac{p_1}{p_0}\mathrm{d}x\right)\right] + \frac{p_2}{p_0} y \exp\left(\int \frac{p_1}{p_0}\mathrm{d}x\right) = 0$$

(provided that $p_0 \neq 0$). This makes use of the integrating factor associated with the first two terms in the equation. We write this equation more compactly as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[P(x)y'\right] + Q(x)y = 0, \tag{3}$$

for given $P(x)$ and $Q(x)$; see §2.1 for a development of this form. A further generalisation is to allow the equation to have a non-zero right-hand side i.e. it has a forcing term; then we write (cf. equation (3))

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[P(x)y'\right]+Q(x)y=R(x) \quad (4)$$

This last equation is usually referred to as Sturm's equation.

Example 2

Write $xy''+2x^2y'+(2+x)y=x^4$ in the form of Sturm's equation.

We need to find, first, the integrating factor associated with the first two terms i.e. $y''+2xy'$ (after division by $x\ (\neq 0)$), so we use the integrating factor $\exp\left(\int 2x\,\mathrm{d}x\right)=\exp(x^2)$. Thus we now write

$$\mathrm{e}^{x^2}\left[y''+2xy'+\left(1+\frac{2}{x}\right)y\right]=x^3\mathrm{e}^{x^2}$$

$$\text{or}\quad \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{x^2}y'\right]+\left(1+\frac{2}{x}\right)\mathrm{e}^{x^2}y=x^3\mathrm{e}^{x^2},$$

which is Sturm's version of the equation.

1.2 The boundary-value problem

The equation, and the relevant boundary conditions, which will be central to the ideas that we present here are described by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)y'\right]+\left[q(x)+\lambda r(x)\right]y=s(x),\ a\le x\le b, \quad (5)$$

where λ is a constant and, with $s(x)\equiv 0$, we use the boundary conditions

$$\alpha y(a)+\beta y'(a)=0;\ \gamma y(b)+\delta y'(b)=0. \quad (6)$$

We record that *a* and/or *b* could be infinite, allowing us to have infinite or semi-infinite domains. The functions $p(x)$, $q(x)$ and $r(x)$ are all real, with the additional requirements that

$$p(x)>0,\ p'(x),\ q(x),\ r(x)>0,$$

are all continuous on the domain of the solution; the constants α, β, γ and δ are also real (and not both of the pair (α,β), nor both the pair (γ,δ), are zero). These (two-point) boundary conditions are *homogeneous* (because the transformation $y \to \text{constant} \times y$ leaves them unchanged); the constant λ is the *eigenvalue* and is to be determined. The problem posed by (5) (with $s(x) \equiv 0$) and (6) is usually called the *Sturm-Liouville problem*; this was first discussed in a number of papers that were published by these authors in 1836 and 1837. [Charles-François Sturm (1803-1855), Professor of Mechanics at the Sorbonne, had been interested, since about 1833, in the problem of heat flow in bars, so was well aware of eigenvalue-type problems. He worked closely with his friend Joseph Liouville (1809-1882), Professor of Mathematics at the Collège de France, on the general properties of second order differential equations. Liouville also made many contributions to the general field of analysis.]

1.3 Self-adjoint equations

One final, but general aspect of second-order equations, closely related to the Sturm-Liouville form, needs to be mentioned before we proceed with the more detailed discussion. We return to equation (1):

$$L(y) \equiv p_0(x)y'' + p_1(x)y' + p_2(x)y = 0,$$

and ask if there is a function, $v(x)$, that makes $vL(y)$ an exact differential i.e. does an integrating factor, for the whole equation, exist? To examine this possibility, we write

$$\begin{aligned} vL(y) &= \frac{\mathrm{d}}{\mathrm{d}x}(p_0vy') - (p_0v)'\,y' + \frac{\mathrm{d}}{\mathrm{d}x}(p_1vy) - (p_1v)'\,y + p_2vy \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\Big[p_0vy' - (p_0v)'\,y\Big] + (p_0v)''\,y + \frac{\mathrm{d}}{\mathrm{d}x}(p_1vy) - (p_1v)'\,y + p_2vy \end{aligned}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\Big[p_0vy' - (p_0v)'\,y + p_1vy\Big] + yM(v)$$

where M is the differential operator

$$M \equiv p_0\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (2p_0' - p_1)\frac{\mathrm{d}}{\mathrm{d}x} + (p_0'' - p_1' + p_2).$$

Thus, for $vL(y)$ to be an exact differential, we require $M(v) = 0$ – another second-order, ordinary differential equation (so probably not that helpful).

We call the operator M the *adjoint* to L, and $M(v) = 0$ is then the *adjoint equation*. In general, the differential operators M and L are not identical, but if $M \equiv L$ then the equation $L(y) = 0$ is said to be *self-adjoint*. It is immediately apparent that the operator *is* self-adjoint if $p_0' \equiv p_1$, for then

$$M \equiv p_0\frac{\mathrm{d}^2}{\mathrm{d}x^2} + p_0'\frac{\mathrm{d}}{\mathrm{d}x} + p_2 \text{ and so } M \equiv L \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left(p_0\frac{\mathrm{d}}{\mathrm{d}x}\right) + p_1,$$

which is the Sturm-Liouville form. However, we have already demonstrated that any linear, second-order ordinary differential equation can be recast in this form; thus every such equation can be written in self-adjoint form.

Exercises 1

1. Find the general solutions, and associated Wronskians, of these equations:
   (a) $y'' + 2y' + 5y = 0$; (b) $2x^2 y'' + 3xy' - y = 0$; (c) $y'' + 2y' + y = 0$.

2. Which of these functions could be Wronskians ?
   (a) e^{-2x}; (b) x^2; (c) $1/(2-x)$; (d) 72.

3. Write these equations in the form of Sturm's equation:
   (a) $y'' + 2y' + xy = 0$; (b) $2y'' + 4xy' + 3\mathrm{e}^x y = x$;

   (c) $y'' + 4y = \sin x$; (d) $(1+x)y'' + 5xy = x$.

4. Which of these are Sturm-Liouville problems?
   (a) $y'' + \lambda y = 0$, $0 \le x \le 1$, $y(0) = y(1) = 0$;

   (b) $(xy')' + (1 + \lambda x)y = 0$, $-1 \le x \le 1$, $y(-1) = 0$, $y'(1) = 0$;

   (c) $[(1+x)y']' + \left(x + \lambda \mathrm{e}^x\right)y = 0$, $0 \le x \le 1$, $y'(0) = 0$, $y(1) + y'(1) = 0$;

   (d) $y'' + y' + \mathrm{e}^x y = 0$, $-1 \le x \le 1$, $y(-1) = 0$, $y(1) - 2y'(1) = 0$.

5. Find the adjoint equation associated with the equation

$$xy'' + 2y' + (3x-1)y = 0.$$

6. Write this general equation

$$p_0 y'' + p_1 y' + p_2 y = 0$$

in self-adjoint (Sturm-Liouville) form, with $p_1 = p_0'$ (see §1.3); hence write down

the equation for the adjoint function, $v(x)$, and then obtain (a version of)

Lagrange's identity:

$$vL(y) - yL(v) = \frac{\mathrm{d}}{\mathrm{d}x}\left[p_0(x)W(v,y)\right].$$

2 The Sturm-Liouville problem: the eigenvalues

The strength, and beauty, of the approach developed by Sturm and Liouville is that considerable general information, and some specific detail, can be obtained without ever finding the solution to the problem. Thus we are able to find, for example, integrals involving these functions – orthogonality integrals are the prime example – without knowing the solution. Such integral results are likely to be useful in their own right, and they are essential in the context of Fourier series. We shall describe, first, the nature of the eigenvalues and then, in the following chapter, turn to the associated solutions (the eigenfunctions).

2.1 Real eigenvalues

We return to the classical Sturm-Liouville problem (equations (5) and (6)):

$$\left.\begin{array}{l} \left[p(x)y'\right]' + \left[q(x) + \lambda r(x)\right]y = 0, \quad a \le x \le b, \\ \text{with} \qquad \alpha y(a) + \beta y'(a) = 0; \quad \gamma y(b) + \delta y'(b) = 0 \end{array}\right\} \tag{7}$$

where $p(x) > 0$, $r(x) > 0$ and $q(x)$ are real functions, and α, β, γ and δ are real constants. Suppose that the two (independent) solutions of the equation are $y_1(x;\lambda)$ and $y_2(x;\lambda)$, where the dependence on the parameter λ has been included. Thus the general solution becomes

$$y = Ay_1(x;\lambda) + By_2(x;\lambda).$$

The boundary conditions then require

$$\alpha\left[Ay_1(a;\lambda) + By_2(a;\lambda)\right] + \beta\left[Ay_1'(a;\lambda) + By_2'(a;\lambda)\right] = 0$$

and $$\gamma\left[Ay_1(b;\lambda) + By_2(b;\lambda)\right] + \delta\left[Ay_1'(b;\lambda) + By_2'(b;\lambda)\right] = 0,$$

where the prime denotes the derivative with respect to *x*. The elimination of A/B between these two equations then yields

$$\frac{\alpha y_1(a;\lambda) + \beta y_1'(a;\lambda)}{\alpha y_2(a;\lambda) + \beta y_2'(a;\lambda)} = \frac{\gamma y_1(b;\lambda) + \delta y_1'(b;\lambda)}{\gamma y_2(b;\lambda) + \delta y_2'(b;\lambda)} \tag{8}$$

which, in general, can be an extremely complicated equation for λ. (We may note that this is a statement of the condition $|M| = 0$, necessary for the existence of non-zero solutions of the matrix equation $MX = 0$, where $MX = 0$, where $X = \begin{pmatrix} A \\ B \end{pmatrix}$. This demonstrates a direct connection with the familiar definition of the eigenvalues of a matrix *A*: $|A - \lambda I| = 0$, associated with the matrix equation $(A - \lambda I)X = 0$.).)

Now equation (8) may have, in general, complex roots; indeed, this must be expected as a possibility. However, we do not have y_1 and y_2 available to use in (8), which would allow us – in principle, at least – to determine the λ s by solving (8). It turns out that we can deduce the nature of the roots directly from the differential equation itself. Although *p*, *q*, *r*, α, β, γ and δ are all real (as is the domain), λ might be complex and then, in turn, so will be the associated solutions *y*; we shall prove, however, that all the eigenvalues of a Sturm-Liouville problem are real.

Proof

We proceed by assuming that both λ and *y* <u>are</u> complex valued, and then we have the pair of equations

$$\left[p(x)y'\right]' + \left[q(x) + \lambda r(x)\right]y = 0 \text{ and } \left[p(x)\bar{y}'\right]' + \left[q(x) + \bar{\lambda} r(x)\right]\bar{y} = 0,$$

where the over bar denotes the complex conjugate. Correspondingly we have

$$\alpha y + \beta y' = 0 \text{ and } \alpha\bar{y} + \beta\bar{y}' = 0 \text{ (both on } x = a\text{)},$$

and

$\gamma y + \delta y' = 0$ and $\gamma\bar{y} + \delta\bar{y}' = 0$ (both on $x = b$).

We form

$$\bar{y}\left[(py')' + (q + \lambda r)y\right] - y\left[(p\bar{y}')' + (q + \bar{\lambda} r)\bar{y}\right] = 0,$$

which simplifies to give

$$\bar{y}(py')' - y(p\bar{y}')' + (\lambda - \bar{\lambda})ry\bar{y} = 0;$$

this equation is integrated over the given domain:

$$\int_a^b \left[\bar{y}(py')' - y(p\bar{y}')'\right]\mathrm{d}x + (\lambda - \bar{\lambda})\int_a^b ry\bar{y}\,\mathrm{d}x = 0.$$

Integration by parts (on the first integral) then produces

$$\left[\bar{y}py' - yp\bar{y}'\right]_a^b - \int_a^b py'\bar{y}'\,\mathrm{d}x + \int_a^b p\bar{y}'y'\,\mathrm{d}x + (\lambda - \bar{\lambda})\int_a^b ry\bar{y}\,\mathrm{d}x = 0,$$

in which two of the integrals cancel identically. The evaluation at $x = b$, for example, generates the term

$$p(b)\left[\bar{y}y' - y\bar{y}'\right]_{x=b} \text{ where } p(b) > 0;$$

if we have the case $\delta = 0$, then $y(b) = \bar{y}(b) = 0$ and so this evaluation gives zero; if $\gamma = 0$, then $y'(b) = \bar{y}'(b) = 0$ and again the value is zero. If, however, both $\gamma \neq 0$ and $\delta \neq 0$, then $y(b) = -(\delta/\gamma)y'(b)$, so that we now obtain

$$\left[\bar{y}y' - y\bar{y}'\right]_{x=b} = \bar{y}(b)y'(b) + \frac{\delta}{\gamma}y'(b)\bar{y}'(b)$$

$$= \frac{1}{\gamma}y'(b)\left[\gamma\bar{y}(b) + \delta\bar{y}'(b)\right] = 0,$$

so again zero. Hence the evaluation at $x = b$ is zero; the same argument obtains at $x = a$, leaving

$$(\lambda - \bar{\lambda})\int_a^b ry\bar{y}\,\mathrm{d}x = 0.$$

But $r(x) > 0$ and $y\bar{y} = |y|^2 > 0$ for any non-identically-zero solution; thus if there exists a non-trivial solution (i.e. not identically zero), then necessarily

$$\int_a^b r(x)y(x;\lambda)\bar{y}(x;\bar{\lambda})\,\mathrm{d}x > 0$$

requiring that $\lambda - \bar{\lambda} = 0$, which implies that λ is real. ■

We have demonstrated that the eigenvlues of the general Sturm-Liouville problem are necessarily real, and then so are all the associated solutions – the eigenfunctions.

Comment: This proof requires $r(x) > 0$, and we also impose $p(x) > 0$. If a particular equation has $p(x) < 0$, we simply multiply through by -1; if the resulting equation then has $r(x) < 0$, we redefine λ to absorb the minus sign.

The word 'eigen' is German, meaning something like 'proper' or 'characteristic' or 'own' (as in 'belonging to').

2.2 Simple eigenvalues

Although we have written little so far about the eigenvalues (and eigenfunctions) – other than to prove that the eigenvalues must be real – it is probable that the reader assumes that to each eigenvalue there corresponds one (linearly independent) solution. Is this the case? If there is only one solution to each choice of the eigenvalue, we describe this as 'simple' i.e. the eigenvalues are said to be *simple* if there is one solution to each eigenvalue. We will provide a proof of this property for the Sturm-Liouville problem; this will require the introduction of a suitable Wronskian (§1.1).

Proof

For a given λ, let us suppose that there are two linearly independent solutions, $y_1(x)$

and $y_2(x)$, say. (Note that we may certainly allow a constant multiple of any particular solution, because a constant $\times y$ is necessarily a solution of the Sturm-Liouville problem, if $y(x)$ is.) The Wronskian of these two solutions is then

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1',$$

and we may compute *W* at any point in the domain; let us do this at one end e.g. $x = a$ (and we may equally evaluate at $x = b$, if we wished). The boundary condition (in (5) and (7)), at $x = a$ applies to each solution, y_1 and y_2, because each is a solution of the same Sturm-Liouville problem. Hence, if $\beta = 0$, then $y_1(a) = y_2(a) = 0$; if $\alpha = 0$, then $y_1'(a) = y_2'(a) = 0$; if $\alpha \neq 0$ and $\beta \neq 0$, then

$$y_1 y_2' - y_2 y_1' = -\frac{\beta}{\alpha} y_1' y_2' + \frac{\beta}{\alpha} y_2' y_1' = 0 \text{ at } x = a.$$

Hence in all cases, $W = 0$. But (see §1.1) the functions y_1 and y_2 are linearly independent, if and only if $W = 0$; thus y_1 and y_2 are linearly dependent: to each λ there is one and only one linearly independent solution. Thus the eigenvalues of the Sturm-Liouville problem are simple. ■

Example 3

Use the Sturm-Liouville problem: $y'' + \lambda y = 0$, $0 \leq x \leq 1$, with $y(0) = y(1) = 0$, to demonstrate explicitly (in this case) that the eigenvalues are simple.

This is an elementary problem with a familiar solution: set $\lambda = \omega^2$ (> 0) then

$$y(x) = A\sin(\omega x) + B\cos(\omega x);$$

see Example 1. (In Exercises 2, Q.1, you are invited to show that this is the only choice of λ which allows the application of the boundary conditions, to obtain a non-zero solution.) The boundary conditions then require

$$0 = y(0) = B \text{ and } 0 = y(1) = A\sin\omega + B\cos\omega,$$

so $B=0$, A is arbitrary and $\sin\omega=0$ i.e. $\omega=n\pi$ for $n=0,\pm1,\pm2,\ldots$. Thus the eigenvalues are $\lambda=\omega^2=n^2\pi^2$ (and we may now use just $n=1,2,\ldots$; $n=0$ is of no interest since it generates the zero solution), and the solution corresponding to each n is then $y(x)=A\sin(n\pi x)$. But $\sin(n\pi x)$ and $\sin(m\pi x)$, for $n\neq m$, are linearly independent functions, so the eigenvalues are simple.

2.3 Ordered eigenvalues

In the example above, we see that the eigenvalues are

$$\pi^2, 4\pi^2, 9\pi^2, \text{etc.}$$

i.e. they are discrete, ordered and extend to infinity. This is a general property of eigenvalues of the Sturm-Liouville problem; the proof of this follows from the *oscillation theorems*, which are described in Chapter 3. It is natural, however, to note this important property here, in the context of a discussion of the eigenvalues. However, it is worthy of comment that if we relax the conditions that underpin the formulation of the Sturm-Liouville problem, then we can produce a very different picture.

Example 4

Find the eigenvalues of the problem: $x^2y''+\lambda y=0$, $0\le x\le 1$, with $y(0)=y(1)=0$.

In this case, the general solution is obtained by setting $y=x^m$:

$$m(m-1)+\lambda=0 \text{ so } \left(m-\tfrac{1}{2}\right)^2=\tfrac{1}{4}-\lambda \text{ or } m=\tfrac{1}{2}\pm\sqrt{\tfrac{1}{4}-\lambda}.$$

Let us write these two roots as m_1 and m_2, then the general solution is

$$y(x)=Ax^{m_1}+Bx^{m_2} \quad (m_1\neq m_2).$$

Now a choice that allows the two boundary conditions to be satisfied (in order to generate a non-zero solution) is: $m_1>0$ and $m_2>0$, both real and unequal; it is clear that these cannot be real and negative. (The demonstration that other choices lead to only the zero solution is left as an exercise; there is, however, one exceptional solution: repeated roots; see the comment that follows this example.) Here, we require $\frac{1}{4}-\lambda=\mu^2$ with $\frac{1}{2}>\mu>0$ i.e.

$$m_1=\tfrac{1}{2}+\mu,\ m_2=\tfrac{1}{2}-\mu \text{ (so that } \tfrac{1}{2}<m_1<1 \text{ and } 0<m_2<\tfrac{1}{2}).$$

Then we have $0=y(0)=A\times0+B\times0$; $0=y(1)=A+B$, which gives the solution

$$y(x) = A\left(x^{\mu+1/2} - x^{-\mu+1/2}\right),\ 0 < \mu < \tfrac{1}{2};$$

the eigenvalues are $\lambda = \frac{1}{4} - \mu^2$, so that $0 < \lambda < \frac{1}{4}$.

In this example, we do not have a discrete, ordered set of eigenvalues: the eigenvalues are continuous on the open interval $\left(0, \frac{1}{4}\right)$. How can this have happened? To answer this, note that when we write the differential equation in Sturm-Liouville form we obtain

$$y'' + \frac{\lambda}{x^2} y = 0,$$

and so we identify $p(x) = 1$, $q(x) = 0$ and $r(x) = 1/x^2$. Thus $r(x)$ is not continuous on the domain of the solution: $x \in [0,1]$. Thus not all the conditions of the Sturm-Liouville problem are met; nevertheless, as we have demonstrated, the problem has a solution that is readily accessible. Indeed, if we include the exceptional solution that corresponds to repeated roots ($\lambda = 1/4$), namely, $y(x) = A\sqrt{x}\ln x$ (with $y \to 0$ as $x \to 0^+$), then we have a solution for all $\lambda \in (0, 1/4]$.

Exercises 2

1. For the simplest Sturm-Liouville problem:
   $y'' + \lambda y = 0$, $0 \le x \le 1$, with $y(0) = y(1) = 0$, show that the choices $\lambda < 0$ and $\lambda = 0$ produce only the zero solution.

2. For the equation $y'' + \lambda y = 0$, $0 \le x \le 1$, find the eigenvalues and the corresponding solutions to the following problems; confirm, in each case, that the eigenvalues are real, ordered and simple.
   (a) $y(0) = y'(1) = 0$; (b) $y'(0) = y(1) = 0$; (c) $y'(0) = y'(1) = 0$.

3. Find the eigenvalues, and corresponding solutions, of this Sturm-Liouville problem:
   $x^2 y'' + 3xy' + \lambda y = 0$, $1 \le x \le \mathrm{e}$, with $y(1) = y(\mathrm{e}) = 0$.

4. Find the eigenvalues, and corresponding solutions, of this problem:
   $x^2 y'' - xy' + \lambda y = 0$, $0 \le x \le 1$, with $y(0) = y(1) = 0$.

   (Note that this is not a Sturm-Liouville problem.)

3 The Sturm-Liouville problem: the eigenfunctions

We now turn to a consideration of the solutions of the Sturm-Liouville problem, that is, we analyse the *eigensolutions* (more often referred to as *eigenfunctions*; each is associated with an eigenvalue, as described in §2.2). The main tool for this analysis is a set of theorems, usually called *oscillation theorems* (but some authors prefer *comparison theorems*). The equation that we consider is (see (5) and (7))

$$\left[p(x)y'\right]' + \left[q(x) + \lambda r(x)\right]y = 0\,,\ a \le x \le b\,;$$

to proceed, we introduce two equations, each of this form, namely

$$\left[p(x)u'\right]' - k(x)u = 0 \text{ and } \left[p(x)v'\right]' - \ell(x)v = 0\,,\ a \le x \le b\,. \tag{9}$$

These two equations define, respectively, the functions $u(x)$ and $v(x)$; note that, at this stage at least, the dependence on the parameter λ is irrelevant (and the negative sign is simply a convenience). Consistent with our standard Sturm-Liouville problem, we require that $p(x) > 0$ in the domain of the solution.

3.1 The fundamental oscillation theorem

We examine equations (9), with the auxiliary condition that $k(x) \geq \ell(x)$ for $\forall x \in [a,b]$, the equality occurring (if at all) at only a finite number of points. We form $v \times \text{first} - u \times \text{second}$ to give

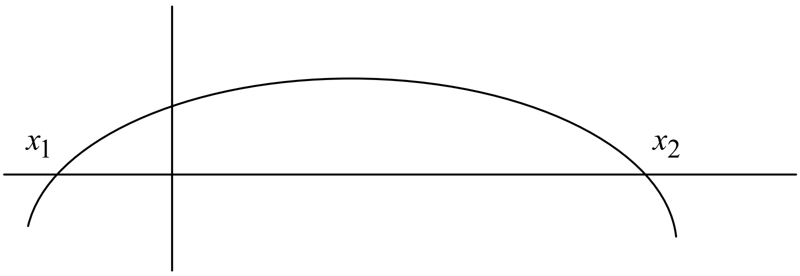
$$v\left[(pu')' - ku\right] - u\left[(pv')' - \ell v\right] = 0$$

so that $$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(u'v - uv')\right] = (k - \ell)uv\,. \tag{10}$$

We now suppose that $u(x)$ oscillates for $x \in [a,b]$, that is, $u(x) = 0$ occurs with at least two zeros on $[a,b]$. These zeros are assumed to be simple i.e. the function $y = u(x)$ crosses the *y*-axis with $u'(x)$ non-zero and finite at these points. Let two consecutive zeros of $u(x)$ be at $x = x_1, x_2$, then we integrate equation (10) from x_1 to x_2:

$$\left[p(x)(u'v - uv')\right]_{x_1}^{x_2} = \int_{x_1}^{x_2} (k - \ell)uv\,\mathrm{d}x\,. \tag{11}$$

Suppose that $v(x)$ is <u>not</u> zero for $x_1 \leq x \leq x_2$, and then take $u(x) > 0$ and $v(x) > 0$ in this same interval (which we may always do, by adjusting the arbitrary, multiplicative constants accordingly). These conditions ensure that the integral on the right in (11) is positive. However, on the left, we have $u(x_1) = u(x_2) = 0$ with $u'(x_1) > 0$ and $u'(x_2) < 0$; see the figure below, which represents this situation.



Sketch of $y = u(x)$, showing the two zeros: at x_1 and x_2.

The left-hand side therefore becomes

$$p(x_2)v(x_2)u'(x_2) - p(x_1)v(x_1)u'(x_1) < 0\,,$$

which presents us with a contradiction: right-hand side > 0 and left-hand side < 0. Since the conditions describing $u(x)$ are given, we conclude that $v(x)$ must change sign between $x = x_1$ and $x = x_2$. Thus $v(x) = 0$ (at least once) between the zeros of $u(x)$, and if it happens that $v(x_1) = 0$ (which could be at, for example, $x = x_1 = a$) then the next zero of $v(x)$ appears before that of $u(x)$. Hence $v(x)$ oscillates more rapidly than $u(x)$; this is our fundamental result.

Example 5

Demonstrate the correctness of the above theorem, in the case $u'' + n^2 u = 0$, $v'' + m^2 v = 0$, for $m > n$.

The general solutions are

$$u(x) = A\sin(nx) + B\cos(nx);\ v(x) = C\sin(mx) + D\cos(mx),$$

where *A*, *B*, *C* and *D* are the arbitrary constants. The periods of these oscillations are, respectively, $2\pi/n$ and $2\pi/m$, and so $v(x)$ oscillates more rapidly than $u(x)$, given that $m > n$. We observe that, in this example, $p = 1$ and $k - \ell = -n^2 + m^2 > 0$.

An important extension of this fundamental theorem is to the pair of equations

$$[p(x)u']' - k(x)u = 0 \text{ and } [P(x)v']' - \ell(x)v = 0$$

where $p(x) \geq P(x) > 0$ and $k(x) \geq \ell(x)$ (and equality occurs, at most, at only a finite set of points). The result of the theorem just proved, namely that $v(x)$ oscillates more rapidly than $u(x)$, applies in this case also, although the proof is less straightforward. The extension (usually associated with the name M. Picone) will not be presented here: it produces exactly the same result, even if the technicalities are more involved. An example should be sufficient to describe this new situation.

Example 6

Demonstrate the correctness of this latter theorem, in the case $4u'' + n^2 u = 0$, $v'' + m^2 v = 0$, for $m > n$.

The general solutions are

$$u(x) = A\sin(nx) + B\cos(nx);\ v(x) = C\sin(mx) + D\cos(mx),$$

where *A*, *B*, *C* and *D* are the arbitrary constants. The periods of these oscillations are, respectively, $2\pi/n$ and $2\pi/m$ and $2\pi/m$ i.e. $v(x)$ oscillates more rapidly than $u(x)$, since $m > n/2$ (and obviously so, since $m > n$). (In this example, we have $p - P = 4 - 1 = 3 > 0$ and $k - \ell = -n^2 + m^2 > 0$.)

3.2 Using the fundamental oscillation theorem

The really useful results are obtained when we suppose that we have upper and lower bounds on $p(x)$ and $k(x)$ for $x \in [a,b]$. Let us be given that

$$0 < P \le p(x) \le Q \text{ and } N \le k(x) \le M \quad (a \le x \le b)$$

and then we compare the equation

$$[p(x)y']' - k(x)y = 0 \tag{12}$$

with $u'' - \frac{N}{P}u = 0$ and $v'' - \frac{M}{Q}v = 0$, (13a,b)

which are constant-coefficient equations. From the extension to the main theorem (in part proved and otherwise described), it follows that $y(x)$ oscillates more rapidly than $v(x)$, but less rapidly than $u(x)$. Indeed, if $N \ge 0$, then the solution for $u(x)$ is non-oscillatory – it is either a linear function ($N = 0$) or an exponential function ($N > 0$) – and so $y(x)$ is also non-oscillatory. Correspondingly, if $N < 0$, then $u(x)$ has a distance of at least $\pi\sqrt{-P/N}$ between consecutive zeros, and so the solution will again be non-oscillatory if $\pi\sqrt{-P/N} > b - a$.

When we consider the other associated function, $v(x)$, with $M<0$, we observe that consecutive zeros are a distance $\pi\sqrt{-Q/M}$ apart, and so $y(x)$ has at least *m* zeros on $[a,b]$ if

$$(m-1)\pi\sqrt{-Q/M} \le b-a \cdot$$

(Note that *m* zeros for $x\in[a,b]$ require $(m-1)$ spacings between them.) Thus a sufficient condition that $y(x)$ oscillates on $[a,b]$ is that

$$-\frac{M}{Q} \ge \frac{\pi^2}{(b-a)^2},$$

corresponding to the choice $m=2$.

These ideas can now be applied directly to the Sturm-Liouville equation:

$$\left(py'\right)' + (q+\lambda r)y = 0,\ a \le x \le b,$$

with $p(x)>0$ and $r(x)>0$, which corresponds to the general problem that we have just described (and with $k(x) = -q(x) - \lambda r(x)$. Then, as above, we consider the situation where we have $0 < P \le p(x) \le Q$, where the bounds *P* and *Q* are independent of λ; similarly, we write

$$N - \lambda T \le -q(x) - \lambda r(x) \le M - \lambda S$$

where $N \le -q(x) \le M$, $0 < S \le r(x) \le T$ and we have assumed $\lambda \ge 0$ (see below and also Example 7); and all these bounds are independent of λ. Thus $y(x)$ has at least *m* zeros in $[a,b]$ if

$$m-1 \le \left(\frac{b-a}{\pi}\right)\sqrt{\frac{\lambda S - M}{Q}};$$

thus, starting from $\lambda = 0$, $\lambda S - M$ increases as λ increases: the number of zeros increases with the eigenvalue, λ.

Example 7

Apply the relevant oscillation theorems to the equation $\left(y'/x\right)' + \lambda xy = 0$, $1 \le x \le 2$, and then solve the associated Sturm-Liouville problem with $y(1) = y(2) = 0$.

We identify $p(x) = 1/x$ (so $1/2 \le p(x) \le 1$) and $k(x) = -\lambda x$ (so $-2\lambda \le k(x) \le -\lambda$ for $\lambda > 0$). Thus we may compare the given equation with the pair

$$u'' + 4\lambda u = 0 \text{ and } v'' + \lambda v = 0;$$

these two have solutions, for $\lambda > 0$,

$$u(x) = A\sin\left(2x\sqrt{\lambda}\right) + B\cos\left(2x\sqrt{\lambda}\right);\ v(x) = C\sin\left(x\sqrt{\lambda}\right) + D\cos\left(x\sqrt{\lambda}\right).$$

Thus the number of intervals between consecutive zeros, for $x \in [1,2]$, lies between $2\sqrt{\lambda}/\pi$ and $\sqrt{\lambda}/\pi$ (because we require $2(2-1)\sqrt{\lambda} = m\pi$ and $(2-1)\sqrt{\lambda} = m\pi$, respectively). This number, and so the number of zeros, increases as λ increases. We may note that, for $\lambda < 0$, there are no oscillatory solutions.

Now let us introduce $y(x) = Y(x^2)$, then we obtain

$$\frac{d}{dx}\left[2Y'(x^2)\right] + \lambda xY = 0 \text{ and so } 4Y'' + \lambda Y = 0 \ (x \neq 0).$$

We write $\lambda = 4\omega^2 > 0$ and then the general solution is

$$y(x) = A\sin\left(\omega x^2\right) + B\cos\left(\omega x^2\right)$$

with $$0 = y(1) = A\sin\omega + B\cos\omega;\ 0 = y(2) = A\sin(4\omega) + B\cos(4\omega).$$

Thus we must have

$$\tan\omega = \tan 4\omega = 4(\tan\omega - \tan^3\omega)/(1 - 6\tan^2\omega + \tan^4\omega)$$

which has solutions $\tan\omega = 0$ and $\tan\omega = \pm\sqrt{3}$ i.e. $\omega = n\pi, \pm(\pi/3) + n\pi$ (for $n = 0, \pm 1, \pm 2, \ldots$). The solution then takes the form

$$y(x) = A\left[\sin\left(\omega x^2\right) - \tan\omega\cos\left(\omega x^2\right)\right]$$

which reduces to either

$$y(x) = A\sin\left(n\pi x^2\right),\ n = 1, 2, \ldots$$

or $$y(x) = A\left\{\sin\left[\left(\pm\tfrac{1}{3} + n\right)\pi x^2\right] \mp \sqrt{3}\cos\left[\left(\pm\tfrac{1}{3} + n\right)\pi x^2\right]\right\},\ n = 0, \pm 1, \ldots .$$

The eigenvalues are $\lambda = 4\omega^2 = 4n^2\pi^2$ ($n = 1, 2, \ldots$) and $\lambda = \frac{4}{9}(\pm 1 + 3n)^2\pi^2$ (for $n = 0, \pm 1, \ldots$); these eigenvalues are discrete, ordered and extend to (plus) infinity as $n \to \pm\infty$.

Finally, we apply the fundamental oscillation theorem in a very direct way, but now to two (independent) solutions of the same equation. Let $y_1(x)$ and $y_2(x)$ be the independent solutions of

$$\left[p(x)y'\right]' - k(x)y = 0 \ \left(p(x) > 0\right),$$

then the construction used in (11) yields

$$\left[p(x)\left(y_1'y_2 - y_1y_2'\right)\right]_{x_1}^{x_2} = 0.$$

Suppose that x_1 and x_2 are consecutive zeros of the solution $y_1(x)$ i.e. $y_1(x_1) = y_1(x_2) = 0$, then we obtain

$$p(x_2)y_1'(x_2)y_2(x_2) = p(x_1)y'(x_1)y_2(x_1);$$

but y_1' must have opposite signs at $x = x_1, x_2$ (as shown in the figure on p.18), and so y_2 must correspondingly have opposite signs at $x = x_1, x_2$. Thus $y_2(x)$ must be zero at least once between $x = x_1$ and $x = x_2$. We can apply exactly the same argument by reversing the rôles of y_1 and y_2; hence the zeros of $y_1(x)$ and $y_2(x)$ *interlace*.

Example 8

Demonstrate that the zeros of the two solutions of $y'' + y = 0$ interlace.

This is an elementary exercise: the two independent solutions are the familiar $y = \sin x$ and $y = \cos x$. The former has zeros at $x = n\pi$, and the latter at $x = \frac{1}{2}\pi + n\pi$ (each for $n = 0, \pm 1, \ldots$): the two sets of zeros do indeed interlace.

3.3 Orthogonality

A very significant and fundamental property of the eigenfunctions of a Sturm-Liouville problem, with far-reaching consequences, is that of *orthogonality*. (This word is used because, as we shall see, the property mimics the familiar orthogonality of vectors: $\mathbf{a} \cdot \mathbf{b} = 0$ implies that the vectors are mutually orthogonal (if they are both non-zero) and so linearly independent. Indeed, the connection goes far deeper when we consider how a set of linearly independent vectors spans a space; this is to be compared with a complete set of independent functions used to represent general functions as series – the Fourier series.)

The procedure follows very closely that which we adopted in the proof that the eigenvalues are real (§2.1). We consider the classical Sturm-Liouville problem

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0, \quad a \le x \le b,$$

with $$\alpha y(a) + \beta y'(a) = 0; \quad \gamma y(b) + \delta y'(b) = 0.$$

Let the solution $y_m(x)$ correspond to the eigenvalue λ_m, and similarly for $y_n(x)$ and λ_n (with $m \ne n$, so that we are working with two different solutions of the problem). Thus we have the pair of equations

$$(py_m')' + (q + \lambda_m r)y_m = 0; \quad (py_n')' + (q + \lambda_n r)y_n = 0,$$

and each solution satisfies the same boundary conditions. We follow the previous method, so we form $y_n \times \text{first} - y_m \times \text{second}$ and then integrate over the given domain (see §2.1), to give

$$\left[p(y_n y_m' - y_m y_n')\right]_a^b + (\lambda_m - \lambda_n)\int_a^b r y_m y_n \,\mathrm{d}x = 0.$$

The evaluations at $x = a$ and at $x = b$ are precisely as developed in §2.1, so these each contribute zero, leaving

$$(\lambda_m - \lambda_n)\int_a^b r(x)y_m(x)y_n(x)\,\mathrm{d}x = 0.$$

But we have $\lambda_m \neq \lambda_n$, so we see that

$$\int_a^b r(x)y_m(x)y_n(x)\,\mathrm{d}x = 0 \ \ (m \neq n);$$

the set of functions $y_n(x)$ are *orthogonal* on $[a,b]$ with respect to the *weight function* $r(x)$. We should note that, for $m = n$, we obtain

$$\int_a^b r(x)y_n^2(x)\,\mathrm{d}x > 0$$

for any non-zero solution (because $r(x) > 0$); it is fairly common practice to *normalise* the eigenfunctions by choosing the multiplicative arbitrary constant in each $y_n(x)$ so that

$$\int_a^b r(x)y_n^2(x)\,\mathrm{d}x = 1.$$

These two integral results can then be expressed in the compact form

$$\int_a^b r(x)y_m(x)y_n(x)\,\mathrm{d}x = \delta_{mn},$$

the *orthonormal* form, where δ_{mn} is the *Kronecker delta*. A set of functions that satisfies this integral condition, for all $n = 1, 2, \ldots$, is called an *orthonormal set*.

Example 9

Confirm by direct integration that the eigenfunctions of the Sturm-Liouville problem:

$$y'' + \lambda y = 0,\ 0 \le x \le 1, \text{ with } y(0) = y(1) = 0,$$

are orthogonal, and construct the corresponding set of orthonormal functions.

Set $\lambda = \omega^2$, then $y = A\sin(\omega x) + B\cos(\omega x)$ with $B = 0$ and $\sin\omega = 0$ ($A \neq 0$) i.e. $\omega = n\pi$; thus we may write

$$y_n(x) = A_n \sin(n\pi x),\ \lambda_n = n^2\pi^2\ \ (n = 1, 2, \ldots),$$

where the A_n are the arbitrary constants. (Note that $n = 0$ generates the zero solution.) First, we find

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,\mathrm{d}x = \frac{1}{2}\int_0^1 \left[\cos\{(m-n)\pi x\} - \cos\{(m+n)\pi x\}\right]\mathrm{d}x$$

$$= \frac{1}{2}\left[-\frac{\sin\{(m-n)\pi x\}}{(m-n)\pi} + \frac{\sin\{(m+n)\pi x\}}{(m+n)\pi}\right]_0^1 \quad (m \neq n)$$

$$= 0 \quad (\text{since } m \pm n \text{ are integers}).$$

Thus the eigenfunctions are certainly orthogonal. Now we consider

$$\int_0^1 A_n^2 \sin^2(n\pi x)\,\mathrm{d}x = \frac{1}{2}A_n^2\int_0^1 [1-\cos(2n\pi x]\,\mathrm{d}x$$
$$= \frac{1}{2}A_n^2\left[x - \tfrac{1}{2n}\sin(2n\pi x)\right]_0^1 = \frac{1}{2}A_n^2$$

which takes the value 1 if we choose $A_n = \sqrt{2}$ (the sign is irrelevant); thus the set of orthonormal functions is

$$y_n(x) = \sqrt{2}\sin(n\pi x),\ n = 1, 2, \ldots \ .$$

3.4 Eigenfunction expansions

The construction of an expansion of a given function as a power series in x is a very familiar (and useful) technique: the Maclaurin and Taylor expansions. Further, many readers will be almost equally familiar with the expansion of a function in a trigonometric series: the classical Fourier series. However, this latter series is simply one example of an expansion that is based on a set of orthogonal functions generated by a Sturm-Liouville problem. The statement of this property, and its construction, is readily presented.

Suppose that we are given a set of orthogonal functions , $y_n(x)$ ($n = 1, 2, \ldots$), of some Sturm-Liouville problem (defined for $a \le x \le b$). Further, suppose that we are given a function, $f(x)$, also defined for $a \le x \le b$; we assume that we may write

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

and then we address the question: how do we determine the coefficients c_n ? To accomplish this, we first multiply by $r(x)y_m(x)$ (where we must remember that the general Sturm-Liouville problem involves the weight function, $r(x)$), to give

$$f(x)r(x)y_m(x) = \sum_{n=1}^{\infty} c_n r(x) y_m(x) y_n(x) \ .$$

Now we integrate over the given domain, and in the process integrate each term in the series, to produce

$$\int_a^b f(x)r(x)y_m(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} c_n\left(\int_a^b r(x)y_n(x)y_m(x)\,\mathrm{d}x\right);$$

but $\int_a^b r(x) y_n(x) y_m(x)\, \mathrm{d}x = \delta_{mn}$ (since the y_n s form an orthonormal set), so we obtain

$$\int_a^b f(x) r(x) y_m(x)\, \mathrm{d}x = c_m .$$

This determines each coefficient c_m, $m = 1, 2, \dots$, and so we have demonstrated, formally at least, that the procedure is quite simple. Of course, this does not justify the correctness of the expansion itself – this issue concerns the existence, i.e. convergence, of the series representation for $f(x)$. The proof of convergence is altogether beyond the scope of this text (and beyond the assumed background of the reader); however, we can state the relevant theorem:

Let $y_n(x)$ be a set of orthonormal eigenfunctions of the Sturm-Liouville problem

$$(py')' + (q + \lambda r) y = 0, \; a \le x \le b,$$

with $\alpha y(a) + \beta y'(a) = 0$; $\gamma y(b) + \delta y'(b) = 0$.

Let $f(x)$ and $f'(x)$ be piecewise continuous functions on $[a,b]$; the series

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \; c_n = \int_a^b f(x) r(x) y_n(x)\, \mathrm{d}x, ,$$

then converges to $\frac{1}{2}\left[f(x^+) + f(x^-) \right]$ at every $x \in (a,b)$.

Comment: The notation x^+, x^-, means the approach to the value *x* from above and below, respectively. Thus, if $f(x)$ is continuous at $x = x_0$, the series converges to $f(x_0)$; if, however, $f(x)$ is discontinuous across $x = x_0$, then the series converges to the mid-point value.

Example 10

Find the solution of the zero-order Bessel equation

$$xy'' + y' + \lambda xy = 0, \; 0 < x \le 1,$$

which satisfies $y(1) = 0$ and is bounded as $x \to 0$. Hence use the resulting set of eigenfunctions to represent the general function $f(x)$.

We take $\lambda > 0$ and then write $y(x) = Y(x\sqrt{\lambda})$, to give

$$\lambda x Y'' + \sqrt{\lambda} Y' + \lambda x Y = 0 \text{ or } \left(x\sqrt{\lambda}\right) Y'' + Y' + \left(x\sqrt{\lambda}\right) Y = 0;$$

this equation has the general solution

$$y(x) = Y(x\sqrt{\lambda}) = A\mathrm{J}_0\left(x\sqrt{\lambda}\right) + B\mathrm{Y}_0\left(x\sqrt{\lambda}\right).$$

But $\mathrm{Y}_0 \to -\infty$ as $x \to 0$, so we set $B = 0$; then we require $y(1) = 0$ i.e. $\mathrm{J}_0(\sqrt{\lambda}) = 0$, and so we need to use the infinite set of discrete zeros of J_0: $x_1, x_2, \ldots$, say. Thus the eigenvalues are $\lambda_n = x_n^2$, $n = 1, 2, \ldots$. Further, the equation in Y can be written

$$\left(XY'\right)' + XY = 0 \text{ (where } X = x\sqrt{\lambda}\text{)},$$

and so we see that the weight function is $X = x\sqrt{\lambda}$; thus we have the orthogonality property

$$\int_0^1 x\mathrm{J}_0\left(x\sqrt{\lambda_n}\right)\mathrm{J}_0\left(x\sqrt{\lambda_m}\right)\mathrm{d}x = 0 \;\; (m \neq n).$$

Now we write

$$f(x) = \sum_{n=1}^{\infty} c_n \mathrm{J}_0\left(x\sqrt{\lambda_n}\right)$$

and then we form

$$\int_0^1 xf(x)\mathrm{J}_0\left(x\sqrt{\lambda_m}\right)\mathrm{d}x = c_m\int_0^1 x\mathrm{J}_0^2\left(x\sqrt{\lambda_m}\right)\mathrm{d}x\,;$$

note that we have not, here, introduced a normalised version of $\mathrm{J}_0\left(x\sqrt{\lambda_m}\right)$ – it is unnecessary. Thus the coefficients of the 'Fourier-Bessel' series are given by

$$c_n = \frac{\int_0^1 xf(x)\mathrm{J}_0\left(x\sqrt{\lambda_n}\right)\mathrm{d}x}{\int_0^1 x\mathrm{J}_0^2\left(x\sqrt{\lambda_n}\right)\mathrm{d}x}.$$

Comment: A Sturm-Liouville problem that does not satisfy the usual conditions prescribed for $p(x)$, $q(x)$ and $r(x)$ at the boundary points is called a *singular Sturm-Liouville problem*. In the example above, we have that $p(x) = x$ and $r(x) = x$, so the conditions $p > 0$ and $r > 0$ for $x \in [0,1]$ are not satisfied. The problem and its solution, as we have seen, turn out to be well-defined, although this explains why we were careful to prescribe the domain as the open interval $x \in (0,1]$, with a condition as $x \to 0$.

Exercises 3

1. Use the oscillation theorems to describe the nature of the solutions of
$$(1+x)y'' + (1+2x)y' + \left[2 + \lambda x^{-1}\left(2x^2 - 1\right)\right]y = 0,\ 1 \le x \le 2.$$

2. Use the substitution $y(x) = Y\left(\sqrt{1+x}\right)$ to find the eigenvalues and eigenfunctions of the Sturm-Liouville problem
$(1+x)y'' + \frac{1}{2}y' + \lambda y = 0$, $0 \le x \le 3$, with $y(0) = y(3) = 0$.

   Hence relate the nature of the eigenvalues, and the oscillatory solution, to the oscillation theorems.

3. Show by direct calculation that $\cos(n\pi x)$, $n = 1, 2, \ldots$ with $0 \le x \le 1$, form an orthogonal set; find the corresponding orthonormal functions.
4. Find the solution of the Sturm-Liouville problem $xy'' + y' + \lambda xy = 0$, $0 < x \le 1$, with $y'(1) = 0$ and which is bounded as $x \to 0$. Hence use the resulting set of eigenfunctions to represent the general function, $f(x)$.

4 Inhomogeneous equations

We now turn to the consideration of the equation which is, generally, of the Sturm-Liouville type but with a forcing term; see equation (5). Thus we will discuss

$$(py')' + (q + \lambda r)y = s(x)\,,\ a \le x \le b\,,$$

with the same homogeneous boundary conditions:

$$\alpha y(a) + \beta y'(a) = 0\,;\ \gamma y(b) + \delta y'(b) = 0\,.$$

All the same conditions (see §1.2) apply to the functions $p(x)$, $q(x)$ and $r(x)$; additionally, for this equation, we shall assume that $s(x)$ is a continuous function on $[a,b]$. The analysis of this problem, as we shall see, produces an intriguing and unexpected situation.

To initiate the discussion, we will make use of the eigenfunctions associated with the classical Sturm-Liouville problem i.e. obtained by setting $s(x) \equiv 0$; let $y_n(x)$ be the (orthogonal) eigenfunction associated with the eigenvalue λ_n. We now assume that the solution of the inhomogeneous equation can be represented by an expansion in these eigenfunctions, so

$$y(x) = \sum_{n=1}^{\infty} c_n y_n(x)\,.$$

The use of this form of solution in the original differential equation then gives

$$\sum_{n=1}^{\infty} c_n (py_n')' + (q + \lambda r)\sum_{n=1}^{\infty} c_n y_n = s(x)\,,$$

where we have differentiated, appropriately, each term in the series. But from the definition of $y_n(x)$, we have

$$(py_n')' + (q + \lambda_n r)y_n \quad \text{for each } n,$$

so we may use the identity

$$\sum_{n=1}^{\infty}\left\{c_n (py_n')' + qc_n y_n\right\} = -\sum_{n=1}^{\infty} \lambda_n c_n r y_n$$

to give

$$\lambda r\sum_{n=1}^{\infty}c_n y_n - \sum_{n=1}^{\infty}\lambda_n c_n r y_n = s(x)$$

or $$\lambda\sum_{n=1}^{\infty}c_n y_n - \sum_{n=1}^{\infty}\lambda_n c_n y_n = \frac{s(x)}{r(x)}.$$

To proceed, we now assume that we may write $s(x)/r(x)$ as an expansion in the same eigenfunctions (which would then be consistent with our earlier assumption), so

$$\frac{s(x)}{r(x)} = \sum_{n=1}^{\infty}\alpha_n y_n(x).$$

The coefficients, α_n, are defined in the conventional way:

$$\alpha_n = \int_a^b r(x)\frac{s(x)}{r(x)}y_n(x)\,\mathrm{d}x = \int_a^b s(x)y_n(x)\,\mathrm{d}x\,,$$

so the apparent (minor) complication in working with $s(x)/r(x)$, rather than $s(x)$, leads to an unlooked for simplification.

Thus we now have

$$\sum_{n=1}^{\infty}\left[(\lambda-\lambda_n)c_n-\alpha_n\right]y_n(x)=0,$$

which is a linear combination of linearly independent functions, so the only solution is

$$(\lambda-\lambda_n)c_n-\alpha_n=0 \text{ or } c_n=\frac{\alpha_n}{\lambda-\lambda_n} \quad (n=1,2,\ldots)$$

provided that $\lambda\neq\lambda_n$ for any *n*. We have therefore constructed, formally at least, a solution of the original inhomogeneous equation; it is

$$y(x)=\sum_{n=1}^{\infty}\frac{\alpha_n}{\lambda-\lambda_n}y_n(x)$$

and this automatically satisfies the homogeneous boundary conditions because each $y_n(x)$ does. (For the purposes of this discussion, we assume that this solution exists for $x\in[a,b]$, and this will be the case – according to the theorem stated in the previous section.) At first sight, all this appears quite satisfactory, but there is a complication lurking here.

In all the analysis so far, we have not made any detailed statements about λ, the parameter in the original equation. Certainly λ is a free parameter in the solution that we have obtained, and the solution is well-defined, provided that $\lambda\neq\lambda_n$ for any *n*. If it happens that λ has been chosen (or fixed by some additional constraint – perhaps a physical requirement) to be one of the eigenvalues, then it is immediately evident that we are presented with a difficulty. Let $\lambda=\lambda_m$, where *m* is one of the integers $n=1,2,\ldots$; then the equation defining c_m becomes

$$(\lambda_m-\lambda_m)c_m-\alpha_m=0\,.$$

There are thus two cases: either $\alpha_m\neq 0$ or $\alpha_m=0$. In the former case, we have an inconsistency, and so no solution exists. If $\alpha_m=0$, then we do have a solution, but not a unique solution because c_m is undetermined: it has become an arbitrary constant in the solution. Further, if $\alpha_m=0$, then we must have that

$$\alpha_m=\int_a^b s(x)y_m(x)\,\mathrm{d}x=0,$$

which implies that the forcing term, $s(x)$, must be orthogonal to the eigenfunction $y_m(x)$. Thus the existence, or otherwise, of the solution in the case $\lambda = \lambda_m$ hinges on the value of this integral: if it is non-zero, no solution exists; if it is zero, then a non-unique solution exists. (This situation, where a (non-unique) solution exists only if a quantity vanishes, is usually called the *Fredholm alternative*. It occurs in various branches of mathematics, and in various guises; for example, in matrix theory, in the form $|M| = 0$ which ensures that $MX = 0$ has a non-zero solution, and in the theory of integral equations, a field of analysis that was essentially founded by Fredholm.)

Example 11

Find the solutions, if they exist, of the problem:

$$y'' + \lambda y = \sin(3\pi x),\ 0 \le x \le 1,\ y(0) = y(1) = 0,$$

in the cases (a) $\lambda = 3\pi^2$; (b) $\lambda = 9\pi^2$.

The eigenfunctions are given by (see Example 3) $y_n(x) = \sin(n\pi x)$, $n = 1, 2, \ldots$; then $\sin(3\pi x)$, expanded as a Fourier series in terms of $\sin(n\pi x)$, is simply itself: $\sin(3\pi x)$. Now we seek a solution in the form

$$y(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x),$$

which gives

$$-\sum_{n=1}^{\infty} c_n n^2\pi^2 \sin(n\pi x) + \lambda \sum_{n=1}^{\infty} c_n \sin(n\pi x) = \sin(3\pi x)$$

which requires that $c_n = 0$ for $\forall n \ne 3$; otherwise $c_3\left(\lambda - 9\pi^2\right) = 1$..

Thus in the case $\lambda = 3\pi^2$ we have $c_3 = -1/6\pi^2$; the solution is therefore

$$y(x) = -\frac{1}{6\pi^2}\sin(3\pi x).$$

However, in the case $\lambda = 9\pi^2$, we see that no solution exists.

Exercise 4

Given $y'' + \lambda y = k + x$, $0 \le x \le 1$ with $y(0) = y(1) = 0$, determine the solutions (if they exist) in these two cases:

a) $\lambda = \pi^2$ (for all *k*? for any special value(s) of *k*?)
b) $\lambda = 4\pi^2$ (*ditto*).

Answers

Exercises 1

1. (a) $y = e^{-x}(A\sin 2x + B\cos 2x)$, $W = -2e^{-2x}$; (b) $y = A\sqrt{x} + Bx^{-1}$, $W = -\frac{3}{2}x^{-3/2}$; (c) $y = Ae^{-x} + Bxe^{-x}$, $W = e^{-2x}$.
2. (a) Yes; (b) No; (c) Yes; (d) Yes.
3. (a) $\left(e^{2x}y'\right)' + \left(xe^{2x}\right)y = 0$; (b) $\left(e^{x^2}y'\right)' + \left(\frac{3}{2}e^{x+x^2}\right)y = \frac{1}{2}xe^{x^2}$;

   (c) $(y')' + 4y = \sin x$; (d) $(y')' + \left(\dfrac{5x}{1+x}\right)y = \dfrac{x}{1+x}$
4. (a) Yes; (b) No; (c) Yes; (d) Yes.
5. $xv'' + (3x-1)v = 0$.

Exercises 2

1. $\lambda = -\omega^2 < 0$: $y = Ae^{\omega x} + Be^{-\omega x} \Rightarrow A = 0, B = 0$;
   $\lambda = 0$: $y = Ax + B \Rightarrow A = 0, B = 0$.
2. (a) $y = A\sin\left[\left(\frac{1}{2}+n\right)\pi x\right]$, $\lambda = \left(\frac{1}{2}+n\right)^2\pi^2$, $n = 0, 1, 2, \ldots$; (b) $y = A\cos\left[\left(\frac{1}{2}+n\right)\pi x\right]$,

   $\lambda = \left(\frac{1}{2}+n\right)^2\pi^2$, $n = 0, 1, 2, \ldots$; (c) $y = A\cos(n\pi x)$, $\lambda = n^2\pi^2$, $n = 0, 1, 2, \ldots$.
3. $y = Ax^{-1}\sin(n\pi\ln x)$, $\lambda = 1 + n^2\pi^2$, $n = 1, 2, \ldots$.
4. $y = A\left(x^{1+\sqrt{1-\lambda}} - x^{1-\sqrt{1-\lambda}}\right)$, $0 < \lambda < 1$; $y = Ax\ln x$, $\lambda = 1$.

Exercises 3

1. At least m zeros if $m - 1 \le \dfrac{1}{2\pi e}\sqrt{3(2+\lambda)}$.
2. $y = A\sin\left(n\pi\sqrt{1+x}\right)$, $\lambda = \frac{1}{4}n^2\pi^2$, at least m zeros, where $m \le 1 + \frac{3}{4}n$, $n = 1, 2, \ldots$.
3. Orthonormal functions is $\sqrt{2}\cos(n\pi x)$. .

4. $y = A\mathrm{J}_0\left(x\sqrt{\lambda_n}\right)$ where $\lambda_n = x_n^2$, x_n being the roots of $J_0'(x) = 0$, $n = 1, 2, \ldots$; in addition there is the solution $y = A$ for $\lambda_0 = 0$;

$$f(x) = \sum_{n=0}^{\infty} c_n \mathrm{J}_0\left(x\sqrt{\lambda_n}\right), \; c_n = \frac{\int_0^1 x f(x) \mathrm{J}_0\left(x\sqrt{\lambda_n}\right) \mathrm{d}x}{\int_0^1 x \mathrm{J}_0^2\left(x\sqrt{\lambda_n}\right) \mathrm{d}x}.$$

Exercise 4

a) Only for $k = -1/2$, then $y = \frac{1}{2\pi^2}\cos(\pi x) + \frac{1}{\pi^2}\left(x - \frac{1}{2}\right) + A\sin(\pi x)$;;
b) No solution for any *k*.

Part III

Integral transforms

Preface

This text is intended to provide an introduction to, and a discussion of, integral transforms. This topic, as a stand-alone course of study, may not appear in a particular degree programme, although the possibility of solving problems using this technique may be mentioned in connection with specific applications. For example, modules on complex analysis are likely to include a brief mention of the *Fourier Transform*, and a fairly comprehensive course on differential equations may mention the *Laplace Transform*. The material has been written to provide a general introduction to the relevant ideas (but it could provide the basis for a module based on these ideas). The intention is to present the material so that it can be used as an adjunct to a number of different modules – or simply to help the reader gain a broader experience of mathematical ideas. Necessarily, therefore, this material goes beyond the methods and techniques that are likely to be presented in any current programme of study, but any elementary ideas that might be mentioned are certainly included here.

It is assumed that the reader has a basic knowledge of, and practical experience in, various aspects of standard methods of integration, and is familiar with the more routine methods for solving ordinary and partial differential equations. Some use is also made of complex analysis (but this can be ignored without any detrimental effect on the basic ideas and their applications). This brief notebook does not attempt to include any 'applied mathematical' applications of these methods, although various differential (and integral) equations are solved, as examples of what can be done.

The approach adopted here is to present some general ideas, which might involve a definition, or a theorem, or an idea, but with the emphasis on methods and applications of the ideas, explained with a number of carefully worked examples – we present 35. A small number of exercises, with answers, are also offered, to reinforce the ideas and methods that are described here. In addition, a list of the transforms of the more common functions is provided at the end of the Notebook.

List of Problems

This is a list of the various problems, whose solutions are discussed in this Notebook. We have used the notation: LT = Laplace Transform, FT = Fourier Transform, HT = Hankel Transform, MT = Mellin Transform.

1 Introduction

A powerful technique – certainly for solving constant coefficient ordinary and partial differential equations – is the *integral transform*. The essential idea is to replace the unknown function (which may be the required solution of a differential equation) by a suitable integral of this function that also contains a parameter. When this is done, we can expect that ODEs are reduced to algebraic equations and that PDEs are replaced by differential equations of lower dimensionality. One of the simplest integral transforms is the *Laplace Transform*:

$$F(s) = \int_0^\infty f(x)\mathrm{e}^{-sx}\mathrm{d}x$$

(where $\mathrm{R}(s) > 0$), and one of the most commonly-used is the *Fourier Transform*:

$$F(k) = \int_{-\infty}^\infty f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$$

(for real k).

In this Notebook, we shall introduce four standard and popular integral transforms, describe some of their properties and apply them to a variety of problems (mainly differential equations, but not exclusively so). An important – and obvious – question to pose is simply: having obtained the transformed version of a problem, and generated a solution for the transform itself, (the 'F' above), how do we recover the solution (equivalently, the 'f above) to the original problem? Although the answer to this, in practice, is often very straightforward: look it up in a table, we shall comment on this important aspect. However, it is useful, first, to show how transforms appear fairly naturally within familiar contexts.

1.1 The appearance of an integral transform from a PDE

We shall start with the technique of separation of variables, and apply it to Laplace's equation in two dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \, .$$

We seek a solution $u(x, y) = X(x)Y(y)$, to give

$$X''Y + XY'' = 0$$

(where the prime denotes the derivative), and then we choose (for example)

$$X'' + \omega^2 X = 0\,;\; Y'' - \omega^2 Y = 0\,,$$

for real, positive ω. Thus a solution for $u(x,y)$ is

$$u(x,y) = \mathrm{e}^{\mathrm{i}\omega x - \omega y}$$

which, in this case, might be restricted to $y \ge 0$. (Note that this solution, as written, is complex valued; we could, however, elect to take the real or imaginary part, or construct a suitable linear combination with a complex conjugate, essentially as we do below.) The boundary conditions for a particular problem lead to an eigenvalue problem, and then we may write down a more general solution:

$$u(x,y) = \sum_n A_n \mathrm{e}^{\mathrm{i}\omega_n x - \omega_n y}\,, \tag{1}$$

where the A_ns are (complex) constants and the summation is over all the allowed values of ω_n (which are essentially the eigenvalues here). This solution, for suitable A_n, can be made real, as mentioned above. We have obtained the familiar structure of the separation of variables solution (although it is likely to have been expressed in terms of *sin* and/or *cos* functions, with real coefficients).

The presence of the summation in the solution (1) suggests that an equivalent formulation could be obtained by replacing $\sum$ by $\int$ i.e.

$$u(x,y)=\int_{-\infty}^{\infty}A(\omega)\,\mathrm{e}^{\mathrm{i}\omega x-|\omega|y}\mathrm{d}\omega\,, \tag{2}$$

where we have allowed all real ω (and so we have written ωy as $|\omega|y$ for $y\geq 0$); the original solution, (1), can be recovered by making a suitable choice of $A(\omega)$ e.g. as a sum of *delta functions*. Now is (2), albeit formally, a solution of the original PDE? In order to answer this, we first assume that $A(\omega)$ is such as to ensure that $u(x,y)$ exists – of course – and, indeed, that both u_{xx} and u_{yy} exist. When we employ the technique of differentiation under the integral sign, we find that

$$u_{xx}=\int_{-\infty}^{\infty}\left(-\omega^2\right)A(\omega)\,\mathrm{e}^{\mathrm{i}\omega x-|\omega|y}\mathrm{d}\omega \text{ and } u_{yy}=\int_{-\infty}^{\infty}|\omega|^2\,A(\omega)\,\mathrm{e}^{\mathrm{i}\omega x-|\omega|y}\mathrm{d}\omega\,;$$

thus $u_{xx}+u_{yy}=\int_{-\infty}^{\infty}\left(-\omega^2+\omega^2\right)A(\omega)\,\mathrm{e}^{\mathrm{i}\omega x-|\omega|y}\mathrm{d}\omega=0$

(because, of course, $|\omega|^2=\omega^2$). Thus (2) is indeed a solution of Laplace's equation.

This representation of the solution suggests that we might seek a solution in the form

$$u(x,y)=\int_{-\infty}^{\infty}f(y;k)\,\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k\,,$$

where we have used the more conventional *k*; this is the solution expressed as a *Fourier Transform*. In this formulation, we first solve for $f(y;k)$ (e.g. $f(y;k)=A(k)\mathrm{e}^{-|k|y}$) and then – presumably – we shall want to recover $u(x,y)$. An alternative approach is to construct an equation for $\int_{-\infty}^{\infty}u(x,y)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$ – and with the *k* used above, it turns out that we need $-k$ here – directly from the PDE. (The semi-colon notation here, $f(y;k)$, is the standard one used to separate variables from parameters in the arguments of a function.)

1.2 The appearance of an integral transform from an ODE

A second example is provided by the constant coefficient, second order, ordinary differential equation (with a right-hand side):

$$y''+ay'+by=r(x)\,,$$

where *a* and *b* are given (real) constants and $r(x)$ is a known function. There is a very familiar method for solving such equations: seek a solution for the complementary function (CF) by writing $y = e^{\lambda x}$, determine λ, write down the CF, and also find (by some appropriate means) a particular integral (PI). A slightly more formal approach, and one that is applicable for any $r(x)$, is to write $y(x) = e^{\lambda x} v(x)$ (where λ is chosen to be a value associated with the CF), solve for $v(x)$ and hence write down $y(x)$. (This is recognised as a version of the method of variation of parameters, resulting in a first-order equation for $v'(x)$.) We now demonstrate how this approach can be generalised and modified.

First, it is evident that it is more convenient to work with $v(x)$; second, we treat λ as a parameter – rather like the *k* and ω in the previous section – and then generate a new function (to replace $v(x)$) which depends only on λ; this is the integral transform. Thus, guided by these observations and given

$$y'' + ay' + by = r(x), \quad x \geq 0,$$

we multiply throughout by e^{-sx} (introducing the most commonly-used notation for the parameter, although some texts use *p* rather than *s*) and integrate over all *x*:

$$\int_0^\infty (y'' + ay' + by) e^{-sx} \mathrm{d}x = \int_0^\infty e^{-sx} r(x)\, \mathrm{d}x.$$

We use integration by parts, and suppose that we are given $y(0)=\alpha$ and $y'(0)=\beta$, then

$$\left[y'\mathrm{e}^{-sx}\right]_0^{\infty}+\int_0^{\infty}\left(sy'+ay'+by\right)\mathrm{e}^{-sx}\mathrm{d}x$$
$$=-\beta+\left[(s+a)y\mathrm{e}^{-sx}\right]_0^{\infty}+\int_0^{\infty}\left[\left(s^2+as\right)y+by\right]\mathrm{e}^{-sx}\mathrm{d}x$$
$$=-\beta-(s+a)\alpha+\left(s^2+as+b\right)\int_0^{\infty}\mathrm{e}^{-sx}y(x)\,\mathrm{d}x=\int_0^{\infty}\mathrm{e}^{-sx}r(x)\,\mathrm{d}x\,,$$

on the assumption that $\mathrm{e}^{-sx}y(x)$ and $\mathrm{e}^{-sx}y'(x)$ both tend to zero as $x\to\infty$. (This normally requires that the real part of *s*, R(*s*), is sufficiently large, in order to 'kill off' any exponential growth associated with the solution $y(x)$.) We express this equation in the form

$$-\beta-(s+a)\alpha+\left(s^2+as+b\right)Y(s)=R(s)\,,$$

where $Y(s)=\int_0^{\infty}\mathrm{e}^{-sx}y(x)\,\mathrm{d}x$ and $R(s)=\int_0^{\infty}\mathrm{e}^{-sx}r(x)\,\mathrm{d}x$
are the *Laplace Transforms* of $y(x)$ and $r(x)$, respectively. Thus we have

$$Y(s)=\frac{R(s)+(s+a)\alpha+\beta}{s^2+as+b}\,,$$

which represents the complete solution of the equation, and also satisfies the given boundary conditions. We have therefore reduced the calculation to an elementary algebraic exercise to find $Y(s)$; the solution for $y(x)$ can be recovered if we are able to map back from the known $Y(s)$ to $y(x)$.

We have introduced two of the most common integral transforms; in the following chapters, we shall discuss and examine these, and two others, in far more detail.

Exercise 1

A problem in heat conduction is described by

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with $u(x,0) = 0$ (in $x > 0$), $u(x,t) \to 0$ as $x \to \infty$ (for $t > 0$) and $u(0,t) = f(t)$ ($t > 0$), where k is a positive constant. Multiply the equation by e^{-st} and integrate $\int_0^\infty (.)\,\mathrm{d}t$ to form the Laplace Transform (in t). Introduce $U(x;s) = \int\limits_0^\infty \mathrm{e}^{-st}u(x,t)\,\mathrm{d}t$ and $F(s) = \int\limits_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$, show that $\dfrac{\mathrm{d}^2U}{\mathrm{d}x^2} = \dfrac{s}{k}U$ and hence that $U(x;s) = F(s)\,\mathrm{e}^{-x\sqrt{s/k}}$.

2 The Laplace Transform

The Laplace Transform (LT), of the function $f(x)$, is defined by

$$F(s)=\int_0^\infty \mathrm{e}^{-sx} f(x)\mathrm{d}x \,,$$

where *s* is, in general, a complex parameter but with R(*s*) large enough to ensure that $F(s)$ exists. (Of course, if such an *s* does not exist, then the LT cannot be defined; so, for example, the LT of $f(x)=\exp\left(x^2\right)$, $0\le x<\infty$, does not exist.) The LT, not surprisingly, was introduced by Pierre-Simon de Laplace (1749-1827), in 1782, but in the form of an integral equation:

$$f(x)=\int_0^\infty \mathrm{e}^{-xt} g(t)\mathrm{d}t \,.$$

He then posed the question: given $f(x)$, what is $g(t)$? (It was S.-D. Poisson who, in 1823, provided the general solution to this problem – but more of this later.) We start by obtaining a few LTs of simple functions; a list of the more common LTs can be found in Table 1 at the end of this Notebook.

2.1 LTs of some elementary functions

Here, given a suitable $f(x)$, we integrate directly to obtain

$$F(s)=\int_0^\infty \mathrm{e}^{-sx} f(x)\mathrm{d}x \,,$$

noting any restrictions on the choice of the parameter *s*; we shall present each as an individual example.

Example 1

Find the LT of $f(x)=1$, $0\le x<\infty$.

We write $F(s)=\int_0^\infty 1.\mathrm{e}^{-sx}\mathrm{d}x=\left[-\frac{1}{s}\mathrm{e}^{-sx}\right]_0^\infty=\frac{1}{s}$, provided that $\mathrm{R}(s)>0$, so that $\mathrm{e}^{-sx}\to 0$ as $x\to\infty$; this is therefore the required Laplace Transform.

The result is sometimes expressed as $\mathrm{L}[1;s] = s^{-1}$.

The extension to $f(x) = x^{\alpha}$ (on the same domain), at least for some α, is quite straightforward; the case $\alpha = 0$ should then recover the result of Example 1.

Example 2

Find the LT of $f(x) = x^{\alpha}$, $0 \le x < \infty$, for $\alpha > -1$.

We have $F(s) = \int_0^{\infty} \mathrm{e}^{-sx} x^{\alpha} \mathrm{d}x$, and it is convenient to introduce $u = sx$ to give

$$F(s) = \int_0^{s.\infty} \mathrm{e}^{-u} \left(\frac{u}{s}\right)^{\alpha} \frac{\mathrm{d}u}{s} = s^{-1-\alpha} \int_0^{s.\infty} \mathrm{e}^{-u} u^{\alpha} \mathrm{d}u$$

which, for $\alpha > -1$ (necessary for the integral to exist), can be expressed in terms of the *gamma function*:

$$F(s) = s^{-1-\alpha} \Gamma(1+\alpha).$$

We may note, for $\alpha = n$ (integer), that we have the familiar result $\Gamma(1+n) = n!$ (and then the special case $\alpha = n = 0$, in Example 1, is recovered). Thus we see that

$$\mathrm{L}[x^{\alpha};s] = s^{-1-\alpha} \Gamma(1+\alpha) \quad (\alpha > -1).$$

We note here that the upper limit in the *u*-integral has been written as $s.\infty$, because *s* is, in general, a complex parameter; if *s* were restricted to the reals, then we would simply write $\int_0^{\infty} .. \mathrm{d}x$. The inclusion of *s* makes clear that, in the complex-*u* plane, we go to infinity in a given direction (with *x* given as real, of course). This does not affect the definition of the Γ-function.

Example 3

Find the LT of $f(x) = \mathrm{e}^{\alpha x}$, $0 \le x < \infty$, where α is a complex constant.

This is particularly straightforward:

$$F(s) = \int_0^{\infty} \mathrm{e}^{-sx} \mathrm{e}^{\alpha x} \mathrm{d}x = \frac{1}{\alpha - s} \left[e^{(\alpha - s)x} \right]_0^{\infty} = \frac{1}{s - \alpha} \quad \text{for } \mathrm{R}(s) > \mathrm{R}(\alpha).$$

Thus we write $\mathrm{L}\left[\mathrm{e}^{\alpha x};s\right]=\dfrac{1}{s-\alpha}$ ($\mathrm{R}(s)>\mathrm{R}(\alpha)$).

We may use the preceding result in a number of ways, as we now demonstrate.

Example 4

Find the LTs of $\sinh(\alpha x)$ and $\cosh(\alpha x)$, both given on $[0,\infty)$, for α real.

Because integration is a linear operation, we may use linear combinations of, for example, exponential functions. Thus

$$\mathcal{L}[\sinh(\alpha x);s]=\tfrac{1}{2}\mathcal{L}\left[\mathrm{e}^{\alpha x};s\right]-\tfrac{1}{2}\mathcal{L}\left[\mathrm{e}^{-\alpha x};s\right]$$

$$=\frac{1}{2}\left(\frac{1}{s-\alpha}-\frac{1}{s+\alpha}\right)=\frac{\alpha}{s^2-\alpha^2}$$

$$(\mathcal{R}(s)>\alpha).$$

Similarly

$$\mathcal{L}[\cosh(\alpha x);s]=\frac{1}{2}\left(\frac{1}{s-\alpha}+\frac{1}{s+\alpha}\right)=\frac{s}{s^2-\alpha^2}\quad(\mathcal{R}(s)>\alpha).$$

Example 5

Find the LTs of $\sin(\alpha x)$ and $\cos(\alpha x)$, both given on $[0,\infty)$, for α real.

We use the result

$$\mathcal{L}\left[\mathrm{e}^{\mathrm{i}\alpha x};s\right]=\mathcal{L}\left[\cos(\alpha x)+\mathrm{i}\sin(\alpha x);s\right]=\mathcal{L}\left[\cos(\alpha x);s\right]+\mathrm{i}\,\mathcal{L}\left[\sin(\alpha x);s\right]$$

with $\mathcal{L}\left[\mathrm{e}^{\mathrm{i}\alpha x};s\right]=\displaystyle\int_0^\infty \mathrm{e}^{\mathrm{i}\alpha x}\mathrm{e}^{-sx}\mathrm{d}x=\frac{1}{s-\mathrm{i}\alpha}$ (from Example 3);

thus $\mathcal{L}\left[\mathrm{e}^{\mathrm{i}\alpha x};s\right]=\dfrac{s+\mathrm{i}\alpha}{s^2+\alpha^2}$.

So equating real and imaginary parts, we obtain the pair of results:

$$\mathcal{L}[\cos(\alpha x);s]=\frac{s}{s^2+\alpha^2}\,;\ \mathcal{L}[\sin(\alpha x);s]=\frac{\alpha}{s^2+\alpha^2}$$

for $\mathrm{R}(s)>\alpha$ in each case.

In the above, we have presented a few simple examples of LTs (and a few others are included in Table 1); we now examine some general properties of the LT, some of which lead to the construction of more LTs based on our earlier examples.

2.2 Some properties of the LT

We now consider a few algebraic, differential and integral properties of the Laplace Transform; many of these will prove to be useful when we turn to the problem of finding solutions to differential (and integral) equations.

(a) Algebraic

Here, given $f(x)$ and its LT (i.e. $F(s)$), we consider how we might obtain the LTs of the functions $\mathrm{e}^{-\alpha x} f(x)$, $f(\alpha x)$ and $\mathrm{H}(x-\alpha) f(x)$ (where α is a real constant and $\mathrm{H}(.)$ is the Heaviside step function). First we consider

$$\int_0^\infty e^{-sx} e^{-\alpha x} f(x)\,\mathrm{d}x = \int_0^\infty e^{-(s+\alpha)x} f(x)\,\mathrm{d}x = F(s+\alpha),$$

provided that the integral exists; thus the inclusion of the exponential factor is simply to shift the argument of the given LT. Now we construct, for α positive,

$$\int_0^\infty \mathrm{e}^{-sx} f(\alpha x)\,\mathrm{d}x = \frac{1}{\alpha}\int_0^\infty \mathrm{e}^{-(s/\alpha)u} f(u)\,\mathrm{d}u \quad (\text{where } u = \alpha x);$$

thus

$$\mathcal{L}[f(\alpha x); s] = \tfrac{1}{\alpha} F(s/\alpha).$$

Finally, we consider the function $\mathrm{H}(x-\alpha) f(x)$, again for positive α, where

$$\mathrm{H}(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

and then we form the resulting LT:

$$\int_0^\infty \mathrm{e}^{-sx} \mathrm{H}(x-\alpha) f(x)\,\mathrm{d}x = \int_\alpha^\infty \mathrm{e}^{-sx} f(x)\,\mathrm{d}x$$

$$= \int_0^\infty e^{-s(\alpha+u)} f(\alpha+u)\,\mathrm{d}u \quad (\text{where } x = \alpha + u)$$

$$= \mathrm{e}^{-\alpha s}\,\mathcal{L}[f(x+\alpha); s].$$

Example 6

Given that the LT of a function is $F(s) = \dfrac{s-2}{s^2-2s+5}$, use the results above (or from Table 1) to find $f(x)$.

First we write $F(s)=\dfrac{s-2}{(s-1)^2+4}=\dfrac{s-1}{(s-1)^2+4}-\dfrac{1}{(s-1)^2+4}$, and then the required function follows directly:

$$f(x)=\mathrm{e}^{x}(\cos 2x-\sin 2x),$$

because the two LTs that we have produced are shifted versions of those for the trigonometric functions.

Example 7

Find the LT of $f(x)=\mathrm{e}^{-x}\mathrm{H}(x-1)\sin x$.

This becomes $F(s)=\mathrm{e}^{-s}\,\mathrm{L}\left[\mathrm{e}^{-(x+1)}\sin(x+1);s\right]$

$$=\mathrm{e}^{-s}\mathcal{L}\left[\mathrm{e}^{-1}(\cos 1)\mathrm{e}^{-x}\sin x;s\right]+\mathrm{e}^{-s}\mathcal{L}\left[\mathrm{e}^{-1}(\sin 1)\mathrm{e}^{-x}\cos x;s\right]$$

$$=\mathrm{e}^{-(s+1)}\left\{\frac{(s+1)\cos 1}{(s+1)^2+1}+\frac{\sin 1}{(s+1)^2+1}\right\}.$$

(b) Differential

We start with a property that, at first sight, would appear to be purely algebraic (of the type encountered in the previous section): given the LT of $f(x)$, find that for $xf(x)$. So we require

$$\int_0^\infty \mathrm{e}^{-sx}xf(x)\,\mathrm{d}x,$$

given that $F(s)=\int_0^\infty \mathrm{e}^{-sx}f(x)\,\mathrm{d}x$; now this expression for $F(s)$, when we invoke differentiating under the integral sign, yields

$$\frac{\mathrm{d}F}{\mathrm{d}s}=\int_0^\infty(-x)\,\mathrm{e}^{-sx}f(x)\,\mathrm{d}x$$

(provided, of course, that this integral exists). Thus we immediately have the identity

$$\int_0^\infty \mathrm{e}^{-sx} x f(x)\,\mathrm{d}x = -F'(s)\,;$$

further, provided that all the relevant integrals also exist, we may generalise this to produce

$$\int_0^\infty \mathrm{e}^{-sx} x^n f(x)\,\mathrm{d}x = (-1)^n \frac{\mathrm{d}^n F}{\mathrm{d}s^n}\,.$$

However, the more natural LT to investigate (and arguably the more important) is that of $f'(x)$; by direct calculation, we have

$$\int_0^\infty \mathrm{e}^{-sx} f'(x)\,\mathrm{d}x = \left[\mathrm{e}^{-sx} f(x)\right]_0^\infty + s\int_0^\infty \mathrm{e}^{-sx} f(x)\,\mathrm{d}x$$

and so, provided that $f(0)$ exists and that $\mathrm{e}^{-sx} f(x) \to 0$ as $x \to \infty$ (which may require R(s) sufficiently large and positive), then

$$\mathcal{L}[f'(x);s] = sF(s) - f(0).$$

Similarly, we see that

$$\int_0^\infty \mathrm{e}^{-sx} f''(x)\,\mathrm{d}x = \left[\mathrm{e}^{-sx} f'(x)\right]_0^\infty + s\int_0^\infty \mathrm{e}^{-sx} f'(x)\,\mathrm{d}x$$

$$= -f'(0) + s\{-f(0) + sF(s)\} = -\{f'(0) + sf(0)\} + s^2 F(s),$$

which then generalises to give

$$\int_0^\infty \mathrm{e}^{-sx} f^{(n)}(x)\,\mathrm{d}x = s^n F(s) - \left\{s^{n-1} f(0) + s^{n-2} f'(0) + \ldots + f^{(n-1)}(0)\right\},$$

again provided that all the evaluations on $x = 0$ exist and that there are suitable decay conditions at infinity.

(c) Integral

The counterpart to the differential result above is

$$\int_0^\infty \mathrm{e}^{-sx}\left(\int_0^x f(x')\,\mathrm{d}x'\right)\mathrm{d}x$$

$$= \left[-\frac{1}{s}\mathrm{e}^{-sx}\int_0^x f(x')\,\mathrm{d}x'\right]_0^\infty + \frac{1}{s}\int_0^\infty \mathrm{e}^{-sx} f(x)\,\mathrm{d}x$$

and so, provided that $\mathrm{e}^{-sx}\int_0^x f(x')\,\mathrm{d}x' \to 0$ as $x \to \infty$ (at least, for suitable s), we obtain

$$\mathcal{L}\left[\int_0^x f(x')\,\mathrm{d}x';s\right] = \frac{1}{s}F(s).$$

Finally, another important integral for which we require the LT, takes the form

$$\int_0^x f(x-y)g(y)\,\mathrm{d}y \quad ,$$

which is usually denoted by $f*g(x)$; this is called the *convolution* (or *convolution integral*) of the two functions (sometimes the *Faltung integral*, which is the German for 'folding' or 'intertwining' i.e. convolution). In passing we may observe the symmetry property:

$$\int_0^x f(x-y)g(y)\,\mathrm{d}y = \int_0^x f(u)g(x-u)\,\mathrm{d}u \quad (x-y=u)$$

and so $f*g(x) = g*f(x)$. The LT of the convolution becomes

$$\int_0^\infty \mathrm{e}^{-sx}\left(\int_0^x f(x-y)g(y)\,\mathrm{d}y\right)\mathrm{d}x = \int_0^\infty\int_0^x \mathrm{e}^{-sx} f(x-y)g(y)\,\mathrm{d}y\mathrm{d}x \quad ,$$

where the region of integration is $0 \le y \le x$, $0 \le x < \infty$, or $y \le x < \infty$, $0 \le y < \infty$, and so we may write this double integral as

$$\int_0^\infty\int_y^\infty \mathrm{e}^{-sx} f(x-y)g(y)\,\mathrm{d}x\mathrm{d}y = \int_0^\infty g(y)\left(\int_y^\infty \mathrm{e}^{-sx} f(x-y)\,\mathrm{d}x\right)\mathrm{d}y \quad .$$

Now we introduce $x-y=u$ into the inner integral (at fixed *y*):

$$\int_0^\infty g(y)\left(\int_0^\infty \mathrm{e}^{-s(y+u)} f(u)\,\mathrm{d}u\right)\mathrm{d}y = \left(\int_0^\infty \mathrm{e}^{-sy} g(y)\mathrm{d}y\right)\left(\int_0^\infty \mathrm{e}^{-su} f(u)\mathrm{d}u\right) = G(s)F(s) \quad ,$$

where G and F are the LTs of $g(x)$ and $f(x)$, respectively.

Example 8

Find the LT of $\int_0^x y^\alpha \sin(x-y)\,\mathrm{d}y$ (for $\alpha > -1$, real).

From the result above, together with two of our earlier LTs, we see that the LT is simply

$$\frac{s^{-1-\alpha}\Gamma(1+\alpha)}{s^2+1},$$

being the product of the LTs for x^α and $\sin x$.

(d) Periodic functions

A fairly common type of function that is encountered (in various branches of applied mathematics, for example) is one that is periodic i.e. $f(x+X)=f(x)$ for all $x\in[0,\infty)$, where X (= constant) is the period. The LT of such a function can then be written

$$\int_0^\infty e^{-sx}f(x)\,dx=\sum_{n=0}^{\infty}\left(\int_{nX}^{(1+n)X}e^{-sx}f(x)\,dx\right)$$

$$=\sum_{n=0}^{\infty}\left(\int_0^X e^{-s(nX+u)}f(u+nX)\,du\right)\ (x=nX+u)$$

$$=\sum_{n=0}^{\infty}\left(\int_0^X e^{-s(nX+u)}f(u)\,du\right)=\left(\sum_{n=0}^{\infty}e^{-nsX}\right)\left(\int_0^X e^{-su}f(u)\,du\right);$$

but $\sum_{n=0}^{\infty}e^{-nsX}=\left(1-e^{-sX}\right)^{-1}$ (for $\left|e^{-sX}\right|<1$, which can be guaranteed for any given X by choosing s appropriately),

and so the required LT is

$$\frac{\hat{F}(s)}{1-e^{-sX}}$$

where $\hat{F}(s)$ is the LT defined over one period (e.g. the first period).

Example 9

Find the LT of $f(x)$, where $f(x+2)=f(x)$ (for $0\le x<\infty$) and

$$f(x)=\begin{cases}x, & 0\le x\le 1,\\ 2-x, & 1<x\le 2.\end{cases}$$

First we require

$$\hat{F}(s) = \int_0^2 \mathrm{e}^{-sx} f(x)\,\mathrm{d}x$$

$$= \int_0^1 x\mathrm{e}^{-sx}\mathrm{d}x + \int_1^2 (2-x)\mathrm{e}^{-sx}\mathrm{d}x$$

$$= \left[-\frac{1}{s}x\mathrm{e}^{-sx}\right]_0^1 + \frac{1}{s}\int_0^1 \mathrm{e}^{-sx}\mathrm{d}x + \left[-\frac{1}{s}(2-x)\mathrm{e}^{-sx}\right]_1^2 - \frac{1}{s}\int_1^2 \mathrm{e}^{-sx}\mathrm{d}x$$

$$= -\frac{\mathrm{e}^{-s}}{s} - \frac{(\mathrm{e}^{-s}-1)}{s^2} + \frac{\mathrm{e}^{-s}}{s} + \frac{(\mathrm{e}^{-2s}-\mathrm{e}^{-s})}{s^2} = \frac{(1-e^{-s})^2}{s^2}.$$

Thus the required LT is

$$\frac{\frac{1}{s^2}(1-\mathrm{e}^{-s})^2}{1-\mathrm{e}^{-2s}}=\frac{1}{s^2}\left(\frac{1-\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}}\right)\quad\left(=\frac{1}{s^2}\tanh(s/2)\right).$$

2.3 Inversion of the Laplace Transform

So far, we have discussed various aspects of the LT, from the viewpoint: given a function, the derivative of a function, and so on, what is the LT? However, we must now address the question of how we can recover the original function, knowing the LT of the function; we will examine some aspects of this issue. The conventional (and simple) approach is to have available a comprehensive list of functions and their LTs; a few are provided in our Table 1 (and most standard texts list many more). However, this assumes that a fundamental principle is at play here: there is a one-to-one correspondence between a function and its LT. This is the case, but only for continuous functions (which, fortunately, is the situation that we encounter most often). *Lerch's theorem* (1903) states that, if two functions have the same LT then these functions differ, at most, by a set of measure zero i.e. only at isolated points of discontinuity. If, on the other hand, the two functions are continuous, then the functions are identical; on this basis we may use a table to recover the functions from their LTs. [M. Lerch, 1860-1922, was a Czech mathematician who was taught by Weierstrass and Kroneker at Berlin University; he made contributions to analysis and number theory.] If the functions are not continuous, but their LTs are identical, we say that the functions are equal *almost everywhere*, usually written in mathematical shorthand as 'a.e.'. But how do we proceed if our LT is not on the list? This requires an inversion formula and, if possible, its evaluation, in order to obtain the original function.

On the basis of the Fourier Inversion Theorem (which we shall introduce in §3.3, and we shall mention there the connection with the Laplace Transform), it is reasonable to infer that the inverse Laplace Transform (ILT) takes the form

$$f(x)=\frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}F(s)\mathrm{e}^{sx}\mathrm{d}s,$$

where γ (real) is sufficiently large. We shall use this as our starting point, and then provide an outline proof that this is indeed the required ILT. First we need some fundamental properties of the LT

$$F(s)=\int_0^\infty f(x)\mathrm{e}^{-sx}\mathrm{d}x;$$

for this to exist, we require $f(x)=\mathrm{O}(\mathrm{e}^{\alpha x})$ as $x\to\infty$ (for some real α), and then we select $\gamma>\alpha$. We also assume that *f*(*x*) is a C^1 function i.e. it has continuous first derivatives. It then follows that *F*(*s*) is analytic in the complex half-plane R(*s*) > γ; this ensures that *F*(*s*) is sufficiently well-behaved for all that follows to be valid.

We form

$$\frac{1}{2\pi \mathrm{i}}\int_{\gamma-\mathrm{i}\tau}^{\gamma+\mathrm{i}\tau} F(s)\mathrm{e}^{sx}\mathrm{d}s = \frac{1}{2\pi \mathrm{i}}\int_{\gamma-\mathrm{i}\tau}^{\gamma+\mathrm{i}\tau}\left(\int_0^\infty f(x')\mathrm{e}^{-sx'}\mathrm{d}x'\right)\mathrm{e}^{sx}\mathrm{d}s,$$

where we are careful to relabel the dummy variable in the definition of *F*(*s*) and, for the moment, τ is finite. (We shall let $\tau \to \infty$ shortly.) Now we change the order of integration – rather simple in this case – thereby producing

$$\frac{1}{2\pi \mathrm{i}}\int_0^\infty \int_{\gamma-\mathrm{i}\tau}^{\gamma+\mathrm{i}\tau} f(x')\mathrm{e}^{s(x-x')}\mathrm{d}s\,\mathrm{d}x'$$

$$= \frac{1}{2\pi \mathrm{i}}\int_0^\infty f(x')\left(\int_{\gamma-\mathrm{i}\tau}^{\gamma+\mathrm{i}\tau}\mathrm{e}^{s(x-x')}\mathrm{d}s\right)\mathrm{d}x' = \frac{1}{2\pi \mathrm{i}}\int_0^\infty f(x')\left[\frac{\mathrm{e}^{s(x-x')}}{x-x'}\right]_{\gamma-\mathrm{i}\tau}^{\gamma+\mathrm{i}\tau}\mathrm{d}x'.$$

The evaluation here is

$$\frac{1}{x-x'}\left[\mathrm{e}^{(\gamma+\mathrm{i}\tau)(x-x')} - \mathrm{e}^{(\gamma-\mathrm{i}\tau)(x-x')}\right] = \frac{\mathrm{e}^{\gamma(x-x')}}{x-x'}2\mathrm{i}\sin\left[(x-x')\tau\right]$$

and so we obtain

$$\frac{1}{2\pi \mathrm{i}}\int_{\gamma-\mathrm{i}\tau}^{\gamma+\mathrm{i}\tau} F(s)\mathrm{e}^{sx}\mathrm{d}s = \frac{1}{\pi}\int_0^\infty f(x')\frac{\sin\left[(x-x')\tau\right]}{x-x'}\mathrm{e}^{\gamma(x-x')}\mathrm{d}x'$$

$$= \frac{1}{\pi}\int_{-x}^\infty f(x+u)\frac{\sin(u\tau)}{u}\mathrm{e}^{-\gamma u}\mathrm{d}u \quad (x'-x=u).$$

At this stage, we introduce the further change of variable: $u\tau = w$, to give

$$\frac{1}{\pi}\int_{-x\tau}^\infty f(x+\tfrac{w}{\tau})\frac{\sin w}{w}\mathrm{e}^{-\gamma w/\tau}\mathrm{d}w$$

which, for $x > 0$ and $\tau \to \infty$, becomes

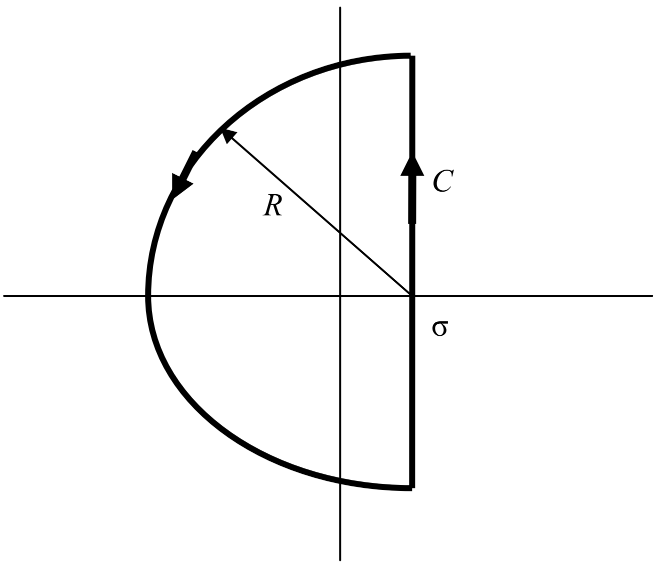
$$\frac{2}{\pi}f(x)\int_0^\infty \frac{\sin w}{w}\mathrm{d}w = f(x),$$

where we have used the property that the integrand is even, and incorporated a standard integral evaluation; thus we have demonstrated that

$$\frac{1}{2\pi \mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} F(s)\mathrm{e}^{sx}\mathrm{d}s$$

is the inverse Laplace transform (ILT).

This result, in conjunction with the integral theorems of complex analysis, enable the function to be recovered from the ILT. To accomplish this, we note that $F(s)$ is analytic in $\mathrm{R}(s) > \gamma$, and so we use the reversed-D-shaped contour (below) where $F(s)$ is analytic in $\gamma > \sigma$ and, indeed, everywhere outside C i.e. all the poles of $F(s)$ must lie inside C; the curved section of the contour is of radius R. (This contour is often called a *Bromwich contour*, after T.J.I'A. Bromwich (1875-1929), an English mathematician.) To proceed, we use Cauchy's residue theorem, followed by taking the limit $R \to \infty$ (and for all valid $F(s)$s, the integral along the semi-circular arc tends to zero as $R \to \infty$, enabling $f(x)$ to be recovered from $F(s)$). [A presentation of the relevant ideas, results and methods in complex analysis can be found, for example, in the volume of the Notebook Series entitled 'The integral theorems of complex analysis with applications to the evaluation of real integrals'.]



Example 10

Use complex integration to find the (continuous) function which has $\dfrac{1}{s-\alpha}$ as its LT (for real $\alpha > 0$).

We form $\dfrac{1}{2\pi \mathrm{i}}\oint\limits_C \dfrac{\mathrm{e}^{sx}}{s-\alpha}\mathrm{d}s$, with the contour shown above, where $\gamma > \alpha$ and $R > \gamma$. This function has a simple pole at $s = \alpha$, with a residue $\mathrm{e}^{\alpha x}$; thus the residue theorem (or simply Cauchy's Integral Formula) gives

$$\frac{1}{2\pi \mathrm{i}}\oint\limits_C \frac{\mathrm{e}^{sx}}{s-\alpha}\mathrm{d}s = \frac{1}{2\pi \mathrm{i}}2\pi \mathrm{i}\mathrm{e}^{\alpha x} = \mathrm{e}^{\alpha x};$$

now let $R \to \infty$, leaving the ILT of $1/(s-\alpha)$ as $\mathrm{e}^{\alpha x}$ (cf. Example 3).

We see that, to give the complete argument here, we use

$$\left|\frac{\mathrm{e}^{sx}}{s-\alpha}\right| \le \frac{\mathrm{e}^{\xi x}}{|s|-\alpha} \quad (s = \xi + \mathrm{i}\eta)$$

on the semi-circular arc ($s = \sigma + R\,\mathrm{e}^{i\theta}$), and this vanishes exponentially as the radius increases.

2.4 Applications to the solution of differential and integral equations

We conclude this chapter with a few simple applications of the Laplace Transform to various standard problems, notably differential equations (both ordinary and partial), and also to an integral equation. The essential idea is the same in all cases: take the LT of the given equation, find an expression for the LT of the solution that we seek, and then invert this.

Example 11

Find the solution, $y(x)$, of the equation

$$y'' + y = 3\mathrm{H}(x-1) \quad (x \geq 0),$$

satisfying $y(0) = y'(0) = 0$. [H(.) is the Heaviside step function; see §2.2(a).]

Let $\mathrm{L}[y(x);s] = Y(s)$, and then take the LT of the equation; this yields

$$s^2Y - sy(0) - y'(0) + Y = {}_{\mathrm{L}}[3\mathrm{H}(x-1);s] = \frac{3}{s}\mathrm{e}^{-s}$$

since the LT of 3 is $3\mathrm{e}^{-s}$, $s > 0$, and this is then shifted by 1 (where we have used results from Example 1, §2.2(a) and §2.2(b)). When we use the given initial values, we see that

$$Y(s) = \frac{3\mathrm{e}^{-s}}{s(1+s^2)} \quad (s > 0),$$

which we interpret as $3 \times \dfrac{\mathrm{e}^{-s}}{s} \times \dfrac{1}{1+s^2}$: the LT of a convolution (§2.2(c)). Now the inverse LTs of $\dfrac{\mathrm{e}^{-s}}{s}$ and $\dfrac{1}{1+s^2}$ are $\mathrm{H}(x-1)$ and $\sin x$, respectively (see Examples 1 and 5, with §2.2(a)), and so

$$y(x) = 3\int_0^x \mathrm{H}(u-1)\sin(x-u)\,\mathrm{d}u$$

$$= \begin{cases} 0, & 0 \leq x < 1, \\ 3\left[1-\cos(x-1)\right], & x \geq 1 \end{cases}$$

$$= 3[1-\cos(x-1)]\mathrm{H}(x-1).$$

If there is any doubt about the correctness of this answer – and there may be some concerns because of the appearance of the step function – it is left as an exercise to confirm by direct substitution that this is indeed the solution, defined for $x \geq 0$. (Of course, some care is required when differentiating the step function.) This example was a rather routine one in the context of ordinary differential equations and LTs; let us now consider a slightly more involved one.

Example 12

Find the solution, $y(x)$, of the equation

$$y'' + 2y' + y = x, \quad 0 \leq x \leq 1,$$

which satisfies $y(0) = 2$, $y(1) = 1$.

Let $\mathcal{L}[y(x); s] = Y(s)$, and then we take the LT of the equation, to give

$$s^2 Y - sy(0) - y'(0) + 2\left[sY - y(0)\right] + Y = \mathcal{L}[x; s] = \frac{1}{s^2};$$

we note that we may incorporate the conventional LT, defined for $x \in [0, \infty)$, even if we elect to use the solution only for $x \in [0, 1]$. We are given $y(0) = 2$, but we do not have $y'(0)$ (whose choice, presumably, controls the value of $y(1)$; cf. the 'shooting' method in the numerical solution of ODEs); let us write $y'(0) = a$, then we obtain

$$(s^2 + 2s + 1)Y = \frac{1}{s^2} + 2s + 4 + a$$

or $$Y(s) = \frac{1 + (4 + a)s^2 + 2s^3}{s^2(1 + s)^2}.$$

This we express in partial fractions:

$$Y(s) = -\frac{2}{s} + \frac{1}{s^2} + \frac{4}{s + 1} + \frac{3 + a}{(s + 1)^2}$$

and we recognise the LTs: $\frac{1}{s}$ (Ex.1), $\frac{1}{s^2}$ (Ex.2), $\frac{1}{s+1}$ (Ex.3), $\frac{1}{(s+1)^2}$ (Ex.2 with Ex.3). Thus we obtain the solution directly:

$$y(x) = -2 + x + 4\mathrm{e}^{-x} + (3 + a)x\mathrm{e}^{-x},$$

and we select *a* by imposing the condition $y(1)=1$, which requires $a=2\mathrm{e}-7$; the resulting solution is therefore

$$y(x)=x-2+\left[4+(2\mathrm{e}-4)x\right]\mathrm{e}^{-x} \quad (\text{for } 0\leq x\leq 1).$$

Again, it is left as an exercise to confirm (if it is thought necessary) that this function does indeed satisfy the ODE and the given boundary conditions. Perhaps a more interesting example is offered the problem of solving a partial differential equation – and we met one in Exercise 1 – and so we examine, carefully, a simple example of this type.

Example 13

Find the solution, $u(x,t)$, of the classical wave equation $u_{tt}-u_{xx}=0$, with $u(x,0)=f(x)$ and $u_t(x,0)=0$ (both for $-\infty<x<\infty$), and $u\to 0$ as $x\to\pm\infty$ (for suitable $f(x)$).

Because time (*t*) is defined so that $t\geq 0$, we first take the LT of the equation in *t*:

$$s^2U(x;s)-su(x,0)-u_t(x,0)-U''(x;s)=0,$$

where U is the LT of $u(x,t)$ (in t), at fixed x; the prime denotes the derivative with respect to x. The equation for U is therefore

$$U'' - s^2 U = -sf(x),$$

which has become an ODE for U as a function of x (with parameter s). The homogeneous equation here (i.e. the one with zero right-hand side) has the general solution

$$U = A\mathrm{e}^{sx} + B\mathrm{e}^{-sx}$$

(the complementary function), and then a particular integral is

$$\tfrac{1}{2}\mathrm{e}^{-sx}\int \mathrm{e}^{sx} f(x)\,\mathrm{d}x - \tfrac{1}{2}\mathrm{e}^{sx}\int \mathrm{e}^{-sx} f(x)\,\mathrm{d}x$$

(which can be obtained quite easily by seeking a solution $U = \mathrm{e}^{sx} v(x;s)$, for example, via the method of variation of parameters). Now, by virtue of the decay conditions as $|x| \to \infty$, we require $A = B = 0$; further, it is then convenient (for $\mathrm{R}(s) > 0$) to write the integrals correspondingly, and so give the solution in the form

$$U(x;s) = \tfrac{1}{2}\mathrm{e}^{-sx}\int_{-\infty}^{x} \mathrm{e}^{sx'} f(x')\,\mathrm{d}x' + \tfrac{1}{2}\mathrm{e}^{sx}\int_{x}^{\infty} \mathrm{e}^{-sx'} f(x')\,\mathrm{d}x'.$$

But the LT of, for example, $f(x-t)$ (in t) is

$$\int_{0}^{\infty} \mathrm{e}^{-st} f(x-t)\,\mathrm{d}t = \mathrm{e}^{-sx}\int_{-\infty}^{x} \mathrm{e}^{sx'} f(x')\,\mathrm{d}x' \quad (x - t = x');$$

thus the inversion of the LTs in U give directly the solution for $u(x,t)$:

$$u(x,t) = \tfrac{1}{2}[f(x-t) + f(x+t)],$$

which is the familiar result from d'Alembert's solution of the classical wave equation.

Finally, we show how the Laplace Transform can be used to solve problems that look very different and appear far more difficult: integral equations. A number of types of integral equation are encountered, particularly in branches of applied mathematics; we shall consider a simple – but important – class which are of *Volterra type* with a special form of the kernel:

$$\int_0^x f(y)k(x-y)\,\mathrm{d}y = g(x), \quad x > 0,$$

with $g(0)=0$ (which is necessary for a conventional solution to exist). [V. Volterra, 1860-1940, Italian mathematician, remembered for his work on functionals, PDEs and, of course, integral equations.] The problem here is: given $g(x)$ and the kernel $k(x-y)$, find the (continuous) function $f(x)$. We recognise, in the light of our work on LTs, that the left side of this integral equation is in convolution form, and so taking the transform immediately produces

$$F(s)K(s)=G(s),$$

where *F*, *K* and *G* are the LTs of $f(x)$, $k(x)$ and $g(x)$, respectively. The solution for *F*, and then for *f*, follows directly (at least, in principle).

Example 14

Find the solution of $\int_0^x \frac{f(y)}{(x-y)^\alpha}\mathrm{d}y = x$, $x>0$, where $0<\alpha<1$ is a real constant.

The transform of the integral equation gives

$$F(s)s^{\alpha-1}\Gamma(1-\alpha)=\frac{1}{s^2} \text{ and so } F(s)=\frac{s^{-1-\alpha}}{\Gamma(1-\alpha)};$$

this result uses Ex.2. Again, following this same example, we see that the required solution is immediately

$$f(x)=\frac{x^\alpha}{\Gamma(1+\alpha)\Gamma(1-\alpha)}.$$

This exercise demonstrates the ease with which some integral equations of this type – often regarded as technically more difficult to solve than differential equations – can be tackled.

Exercises 2

1. Find the Laplace transforms of these functions:
   (a) $2x^2+x-1$; (b) $\mathrm{e}^{-2x}\sin 5x$; (c) $x\cos(\alpha x)$; (d) $x\sinh(\alpha x)$;

   (e) $x^2\sin(\alpha x)$; (f) $\frac{\sin(\alpha x)}{x}$.

2. Find the solutions, $y(x)$, of these ODEs, subject to the conditions given, by using the Laplace transform:
   (a) $y'-2y=\mathrm{e}^{2x}$, $y(0)=1$; (b) $y''-3y'+2y=\mathrm{e}^{-x}$, $y(0)=y'(0)=0$;

   (c) $y''+4y=4[x-(x-1)\mathrm{H}(x-1)]$, $y(0)=1$, $y'(0)=0$.

3. Find the solution of the ODE $xy'' - y' - xy = 0$, which satisfies $y(0) = 0$, $y(1) = 1$.
   [The LT of $x^n I_n(x)$, where $I_n(x)$ is the modified Bessel function of the first kind

   (of order n), is $\dfrac{2^n \Gamma(n+\frac{1}{2})}{\sqrt{\pi}\left(s^2-1\right)^{n+\frac{1}{2}}}$.]

4. Find the LT of the solution of the heat conduction equation, $u_{xx} = u_t$, $0 \le x \le a$, $t > 0$, which satisfies $u(0,t) = f(t)$, $u(a,t) = 0$ and $u(x,0) = 0$ ($0 \le x \le a$).

5. Find the solution of Abel's integral equation: $\int_a^x \frac{y(t)}{\sqrt{x-t}}\,\mathrm{d}t = f(x)$, for suitable $f(x)$, where a is a real constant.

3 The Fourier Transform

We shall define the Fourier Transform (FT) of the function $f(x)$ as

$$F(k) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x,$$

provided, of course, that this integral exists. Other definitions are used by some authors, but the one we adopt is the most common; some replace *k* by $-k$, or use ξ (or $-\xi$) instead of *k*, or elect to work with the *symmetric form*:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x.$$

Also, corresponding to the notation introduced for the Laplace transform, we shall sometimes write the FT of $f(x)$ as $\mathcal{F}[f(x);k]$.

The Fourier transform was, not surprisingly, introduced by Fourier (in his paper on Fourier series published in 1811); however, the notion of it as a transform, in the modern sense, almost certainly did not occur to him, nor did he use it any way that we would recognise. Essentially, he allowed – without being too careful about the underlying procedures and necessary conditions – the summation in his Fourier series to go over to an integral and this, together with the integral representation of the coefficients of the Fourier series, constitutes a version of the FT and its inverse. It was Cauchy, in 1816, who produced a more mathematically robust and complete description, albeit in the context of a problem on water waves; he used this approach to construct a solution of that problem.

3.1 FTs of some elementary functions

We follow the procedure introduced in §2.1 by using direct integration to find the FTs of a few simple functions, presenting each as an individual example.

Example 15

Find the FT of $f(x) = \mathrm{e}^{-\alpha|x|}$ for $\alpha > 0$ (real).

We have immediately that $F(k) = \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha|x|-\mathrm{i}kx}\,\mathrm{d}x$

$$=\int_{-\infty}^{0} e^{(\alpha-ik)x}dx+\int_{0}^{\infty} e^{-(\alpha+ik)x}dx=\left[\frac{e^{(\alpha-ik)x}}{\alpha-ik}\right]_{-\infty}^{0}+\left[-\frac{e^{-(\alpha+ik)x}}{\alpha+ik}\right]_{0}^{\infty},$$

$$=\frac{1}{\alpha-ik}+\frac{1}{\alpha+ik}=\frac{2\alpha}{\alpha^2+k^2}.$$

because of the necessary decay at infinity (by virtue of the real $\alpha>0$).

Thus we may write $\mathcal{F}\left[e^{-\alpha|x|};k\right]=2\alpha/(\alpha^2+k^2)$.

A problem that, at first sight, seems to be very much more involved – perhaps impossible! – is the next one.

Example 16

Find the FT of $f(x)=e^{-\alpha^2x^2}$ for $\alpha\neq 0$ and real.

Here we have $F(k)=\int_{-\infty}^{\infty} e^{-\alpha^2x^2-ikx}dx$, and we note that

$$\alpha^2x^2+ikx=\left(\alpha x+\tfrac{1}{2}ik/\alpha\right)^2+\frac{k^2}{4\alpha^2},$$

which enables us to write $F(k)=e^{-k^2/4\alpha^2}\int_{-\infty}^{\infty} e^{-(\alpha x+ik/2\alpha)^2}dx$. Now we introduce a change of variable: $u=\alpha x+ik/2\alpha$, to give

$$F(k)=e^{-k^2/4\alpha^2}\int_{-\infty+\frac{ik}{2\alpha}}^{\infty+\frac{ik}{2\alpha}} e^{-u^2}\,\frac{du}{\alpha}$$

and, because this integral is along the line $u=ik/2\alpha$ (in the complex-*u* plane), from $-\infty$ to ∞ , the value of the integral is simply $\int_{-\infty}^{\infty} e^{-u^2}du=\sqrt{\pi}$; thus

$$F(k)=\frac{\sqrt{\pi}}{\alpha}e^{-k^2/4\alpha^2}\quad\text{i.e.}\quad\mathcal{F}\left[e^{-\alpha^2x^2};k\right]=\frac{\sqrt{\pi}}{\alpha}e^{-k^2/4\alpha^2}.$$

The next two examples demonstrate how we can find FTs of elementary functions defined piecewise (and, as we shall see later, these provide the basis for the evaluation of certain complicated real integrals).

Example 17

Find the FT of $f(x)=a=\text{constant}$, for $-b \le x \le b$, and zero elsewhere.

The FT becomes $F(k)=\int_{-b}^{b} a\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x=\left[-\frac{a}{\mathrm{i}k}\mathrm{e}^{-\mathrm{i}kx}\right]_{-b}^{b}$

$$=\frac{\mathrm{i}\,a}{k}(\mathrm{e}^{-\mathrm{i}kb}-\mathrm{e}^{\mathrm{i}kb})=\frac{2a}{k}\sin(bk).$$

Example 18

Find the FT of

$$f(x)=\begin{cases}\frac{a}{b}(x+b), & -b\le x\le 0\\ a, & 0<x\le c\\ \frac{a}{c-d}(x-d), & c<x\le d\\ 0 & \text{otherwise.}\end{cases}$$

In this case, we see that

$$F(k)=\int_{-b}^{0}\frac{a}{b}(x+b)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x+\int_{0}^{c}a\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x+\int_{c}^{d}\frac{a}{(c-d)}(x-d)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$$

$$=\left[-\frac{a}{\mathrm{i}bk}(x+b)\mathrm{e}^{-\mathrm{i}kx}\right]_{-b}^{0}+\frac{a}{\mathrm{i}kb}\int_{-b}^{0}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x-\left[\frac{a}{\mathrm{i}k}\mathrm{e}^{-\mathrm{i}kx}\right]_{0}^{c}$$

$$+\left[-\frac{a}{\mathrm{i}k}\left(\frac{d-x}{d-c}\right)\mathrm{e}^{-\mathrm{i}kx}\right]_{c}^{d}-\frac{a}{\mathrm{i}k}\frac{1}{d-c}\int_{c}^{d}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$$

$$= \mathrm{i}\frac{a}{k} + \frac{a}{bk^2}(1 - \mathrm{e}^{\mathrm{i}bk}) + \mathrm{i}\frac{a}{k}(e^{-\mathrm{i}bk} - 1) - \mathrm{i}\frac{a}{k}\mathrm{e}^{-\mathrm{i}ck} - \frac{a}{(d-c)k^2}(\mathrm{e}^{-\mathrm{i}dk} - \mathrm{e}^{-\mathrm{i}ck})$$

$$= \frac{a}{k^2}\left[\frac{1}{b}(1 - \mathrm{e}^{\mathrm{i}bk}) - \left(\frac{1}{d-c}\right)(\mathrm{e}^{-\mathrm{i}dk} - \mathrm{e}^{-\mathrm{i}ck})\right].$$

What is particularly useful here – and we will exploit this later – is that special choices can be made e.g. $c = 0$ and $d = b$, to give

$$F(k) = \frac{2a}{bk^2}\big(1 - \cos(bk)\big).$$

Finally, in this section, we consider the important example afforded by finding the FT of the – arguably – most useful *generalised function*: the *Dirac delta function*. This function is defined as

$$\delta(x)\begin{cases} \neq 0, & x = 0 \\ = 0 & \text{otherwise} \end{cases} \text{ and subject to } \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1,$$

although we will approach this exercise by working with a sequence of functions that, in the appropriate limit, recover the delta function. (This is the usual method that is adopted when introducing, and developing the properties of, the generalised functions.) Thus we write

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2},$$

which possesses the property

$$\int_{-\infty}^{\infty} \delta_n(x)\,dx = \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-n^2 x^2}\,dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\,du = 1,$$

for all n ($\neq 0$); as n increases, so we approach the definition of the delta function. Now, from Example 16, we see directly that

$$\mathcal{F}\left[\delta_n(x);k\right] = e^{-k^2/4n^2},$$

and this has a well-defined limit as $n \to \infty$, namely

$$\mathcal{F}[\delta(x);k] = 1.$$

Example 19

Find the FT of $\delta(x - x_0)$ (x_0 a real constant) directly from the definition of the delta function.

The FT is simply $\int_{-\infty}^{\infty} \delta\left(x - x_0\right) e^{-ikx} dx$, and then we introduce $u = x - x_0$ to give

$$F(k) = \int_{-\infty}^{\infty} \delta(u) e^{-ik(x+x_0)} dx = e^{-ikx_0} \int_{-\infty}^{\infty} \delta(u) e^{-iku} du$$

$$= e^{-ikx_0} \varepsilon \int_{-\infty}^{\infty} \delta(\varepsilon v) e^{-ik\varepsilon v} dv \to e^{-ikx_0} \varepsilon \int_{-\infty}^{\infty} \delta(\varepsilon v)\,dv$$

as $\varepsilon \to 0^+$ (where $u = \varepsilon v$). But $\varepsilon \int_{-\infty}^{\infty} \delta(\varepsilon v) dv = \int_{-\infty}^{\infty} \delta(u) du = 1$, and so we obtain

$$\mathcal{F}[\delta(x - x_0);k] = e^{-ikx_0},$$

which recovers our previous result when we select $x_0 = 0$.

The development just presented, we note, requires a little experience in working with generalised functions (but nothing, we suggest, too extreme). A short list of standard FTs can be found in Table 2, at the end of this Notebook. We now turn to a consideration of some important and general results associated with the Fourier Transform.

3.2 Some properties of the FT

Corresponding to the results that we presented for the Laplace transform, in §2.2, we now describe some standard properties of the Fourier transform, namely those that relate to algebraic and differential formulae.

(a) Algebraic

First, let us be given a function, $f(x)$, and its FT, $F(k)$; we form the FT of $f(ax)$ where $a > 0$ is a real constant. Thus we have

$$\mathcal{F}[f(ax);k] = \int_{-\infty}^{\infty} f(ax)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} f(u)\mathrm{e}^{-\mathrm{i}(k/a)u}\mathrm{d}u = \frac{1}{a}\mathcal{F}[f(x);k/a],$$

where we have used the substitution $u = ax$; it is clear, however, that this will be slightly adjusted if a is real and negative. In this case, we obtain

$$\mathcal{F}[f(ax);k] = \frac{1}{a}\int_{\infty}^{-\infty} f(u)\mathrm{e}^{-\mathrm{i}(k/a)u}\mathrm{d}u = -\frac{1}{a}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}(k/a)u}\mathrm{d}u$$

$$= -\frac{1}{a}\mathcal{F}[f(x);k/a],$$

which can be summarised, for any real $a \neq 0$, as

$$\mathcal{F}[f(ax);k] = \frac{1}{|a|}\mathcal{F}[f(x);k/a] = \frac{1}{|a|}F(k/a).$$

This has described a 'scaling' property (x is scaled and the resulting FT contains a scaled k); now let us perform a shift:

$$\mathcal{F}[f(x-a);k]=\int_{-\infty}^{\infty} f(x-a)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$$

$$=\int_{-\infty}^{\infty} f(u)\mathrm{e}^{-\mathrm{i}k(u+a)}\mathrm{d}u=\mathrm{e}^{-\mathrm{i}ka}\int_{-\infty}^{\infty} f(u)\mathrm{e}^{-\mathrm{i}ku}\mathrm{d}u \quad (u=x-a),$$

where a is a real constant. We see, therefore, that

$$\mathcal{F}[f(x-a);k]=\mathrm{e}^{-\mathrm{i}ka}\,\mathcal{F}[f(x);k]=\mathrm{e}^{-\mathrm{i}ka}F(k).$$

Further, it is possible to derive a converse of this result: let a be a real constant, then

$$\mathcal{F}\left[\mathrm{e}^{-\mathrm{i}ax}f(x);k\right]=\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}ax}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$$

$$=\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}(k+a)}\mathrm{d}x=\mathcal{F}[f(x);k+a]=F(k+a),$$

i.e. the multiplicative exponential factor generates a shift in k; cf. the previous result.

Example 20

Given that the FT of a function is $F(k)=\dfrac{e^{2ik}}{k^2+1}$, use the results above (or Table 2) to find $f(x)$.

We use the shift property, together with Ex. 15, and so write

$$\frac{e^{2ik}}{k^2+1}=e^{-ik(-2)}\frac{1}{2}\frac{2\times 1}{k^2+1^2};$$

this is the FT of $\frac{1}{2}e^{-1\times|x+2|}=\frac{1}{2}e^{-|x+2|}$.

(b) Differential

We now consider the problem of finding the FT of $\mathrm{d}f/\mathrm{d}x$, given the FT of $f(x)$ (and, of course, this is the type of calculation that we shall need to perform for the solution of differential equations). Thus we construct

$$\int_{-\infty}^{\infty} f'(x)e^{-ikx}\mathrm{d}x=\left[f(x)e^{-ikx}\right]_{-\infty}^{\infty}+ik\int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x$$

and, for the FT to exist, we certainly require $f(x)\to 0$ as $|x|\to\infty$; hence we obtain

$$\mathcal{F}[f'(x);k]=ik\int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x=ik\,\mathcal{F}[f(x);k].$$

This result is easily generalised to produce

$$\mathcal{F}\left[f^{(n)}(x);k\right]=(ik)^n\,\mathcal{F}[f(x);k],$$

Provided that the FT of each $f^{(m)}(x)$, $m=0,1,\ldots,n-1$, exists i.e. $f^{(m)}(x)\to 0$ as $|x|\to\infty$. (Some texts are keen to emphasise the operator identity: $\mathrm{d}^n/\mathrm{d}x^n$ maps to multiplication by $(ik)^n$, but such simplistic notions leave a lot to be desired.)

An important additional result, which we need on occasion, is the following. Let us suppose that $f(x)$ has a finite discontinuity at $x=x_0$, then we see that

$$\int_{-\infty}^{\infty} f'(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x = \int_{-\infty}^{x_0^-} f'(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x + \int_{x_0^+}^{\infty} f'(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x,$$

where we have indicated the evaluation from below (x_0^-) and from above (x_0^+). This gives

$$\left[f(x)\mathrm{e}^{-\mathrm{i}kx}\right]_{-\infty}^{x_0^-} + \mathrm{i}k\int_{-\infty}^{x_0^-} f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x + \left[f(x)\mathrm{e}^{-\mathrm{i}kx}\right]_{x_0^+}^{\infty} + \mathrm{i}k\int_{x_0^+}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$$

$$= f(x_0^-)\mathrm{e}^{-\mathrm{i}kx_0} - f(x_0^+)\mathrm{e}^{-\mathrm{i}kx_0} + \mathrm{i}k\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x,$$

with $f(x)\to 0$ as $|x|\to\infty$; thus we have

$$\mathcal{F}[f'(x);k] = \mathrm{i}k\,\mathcal{F}[f(x);k] - [f(x)]_{x_0}\,\mathrm{e}^{-\mathrm{i}kx_0},$$

where $[f(x)]_{x_0}$ denotes the jump in value at the discontinuity i.e. $= f(x_0^+) - f(x_0^-)$. (We note, as we would expect, that if *f*(*x*) is continuous everywhere, then $[f(x)]_{x_0} = 0$ and we recover the previous result.)

In our discussion of the LT, we observed that the use of the derivative can be extended to mean: differentiate the transform with respect to the parameter; the same type of results obtains with FTs. Let us be given *f*(*x*) and its FT *F*(*k*), so that we have

$$F(k) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x,$$

and then $F'(k) = \int_{-\infty}^{\infty} f(x)(-\mathrm{i}x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$,

provided that the FT of $xf(x)$ exists. In this case, we therefore have the identity

$$F'(k) = -\mathrm{i}\,\mathcal{F}[xf(x);k].$$

(c) Convolution integral

Although the Fourier transform of integrals, per se, are of no particular interest or relevance in standard applications of FTs, there is one of particular importance: the convolution integral. This (and we met a corresponding result in §2.2(c)) takes the form of the function defined (here) by

$$f \circ g(x) = \int_{-\infty}^{\infty} f(x-u)g(x)\,\mathrm{d}u$$
,

an integral over all u (cf. §2.2(c)); it is easily confirmed that $f \circ g = g \circ f$ (simply by changing the integration variable from u to $v = x - u$). Thus we have the FT

$$\mathcal{F}\left[f \circ g(x); k\right] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x-u)g(u)\,\mathrm{d}u \right) \mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x\,,$$

and we now transform this double integral according to $(x,u) \to (v,w)$, with $w = u$ and $v = x - w = x - u$; this produces

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)g(w)\mathrm{e}^{-\mathrm{i}k(v+w)}\mathrm{d}v\mathrm{d}w = \left(\int_{-\infty}^{\infty} f(v)\mathrm{e}^{-\mathrm{i}kv}\mathrm{d}v \right)\left(\int_{-\infty}^{\infty} g(w)\mathrm{e}^{-\mathrm{i}kw}\mathrm{d}w \right) = F(k)G(k)$$
,

where F and G are the FTs of $f(x)$ and $g(x)$, respectively.

Example 21

Given that the FT of a function, *f*(*x*), is $\mathrm{e}^{-k^2}\big/(1+k^2)$, use the results above (or Table 2) to find *f*(*x*).

We write this FT as $\dfrac{1}{2\sqrt{\pi}}\dfrac{\sqrt{\pi}}{(1/2)}\mathrm{e}^{-k^2/4(1/2)^2}\times\dfrac{1}{2}\dfrac{2\times 1}{k^2+1^2}$ (see Examples 15, 16), which is the product of the FTs of $f(x)=\dfrac{1}{2\sqrt{\pi}}\mathrm{e}^{-x^2/4}$ and $g(x)=\dfrac{1}{2}\mathrm{e}^{-1\times|x|}$, respectively. Thus the given FT is the transform of the convolution integral

$$\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi}}\mathrm{e}^{-(x-u)^2/4}\,\frac{1}{2}\mathrm{e}^{-|u|}\mathrm{d}u=\frac{1}{4\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-|u|-\frac{1}{4}(x-u)^2}\,\mathrm{d}u\,.$$

3.3 Inversion of the Fourier Transform

A complete and rigorous proof of the Fourier Integral Theorem (which provides the inversion of the FT) is well beyond the scope of the Notebook – it involves a number of functional-analytic ideas that we cannot assume of the reader (and cannot possibly develop here). Thus we will content ourselves with a proof in outline only, based on a complex version of the familiar Fourier series; this will therefore, in a small way, mimic what Fourier did in 1811. To accomplish this, we first express a (suitable) function *f*(*x*) in the form

$$f(x)=\sum_{n=-\infty}^{\infty}c_n\mathrm{e}^{\mathrm{i}n\pi/\ell}\,,$$

a function with period 2ℓ, where c_n is the set of complex-valued constants

$$c_n=\frac{1}{2\ell}\int_{-\ell}^{\ell}f(x)\mathrm{e}^{-\mathrm{i}n\pi x/\ell}\mathrm{d}x\,.$$

(This formulation is no more than the familiar Fourier series, containing both *sin* and *cos* terms, written in a complex form.) The important interpretation of this pair of definitions is afforded by considering the extension to any (suitable) general, non-periodic function; this is possible by allowing the period to increase indefinitely i.e. we take $\ell\to\infty$.

It is convenient, first, to write

$$2\ell c_n=\int_{-\ell}^{\ell}f(x)\mathrm{e}^{-\mathrm{i}n\pi x/\ell}\mathrm{d}x=F(n\pi/\ell)\,,$$

and then we have

$$f(x)=\sum_{n=-\infty}^{\infty}\frac{1}{2\ell}F(n\pi/\ell)\mathrm{e}^{\mathrm{i}n\pi x/\ell}=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{\pi}{\ell}F(n\pi/\ell)\mathrm{e}^{\mathrm{i}n\pi x/\ell}.$$

Now we set $\pi/\ell=\Delta k$, so that

$$F(n\Delta k)=\int_{-\pi/\Delta k}^{\pi/\Delta k} f(x)\mathrm{e}^{-\mathrm{i}n\Delta kx}\mathrm{d}x \text{ with } f(x)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}F(n\Delta k)\mathrm{e}^{\mathrm{i}n\Delta kx}\Delta k;$$

in this latter result, using the definition of the Riemann integral, we take the limit $\Delta k\to 0$, and so obtain

$$f(x)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}F(n\Delta k)\mathrm{e}^{\mathrm{i}n\Delta kx}\Delta k\to\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k.$$

Correspondingly, we have the definition

$$F(k)=\int_{-\infty}^{\infty}f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x,$$

which is the FT of $f(x)$; the expression for $f(x)$ (in terms of $F(k)$) then provides the inversion theorem:

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k.$$

As a commentary on, and a consequence of, this fundamental result, we choose to replace $f(x)$ by $\mathrm{e}^{-\gamma x}f(x)\mathrm{H}(x)$, where H(.) is the Heaviside step function and $\gamma>0$ is a real constant. Thus we may write

$$\mathrm{e}^{-\gamma x}f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_0^{\infty}\mathrm{e}^{-\gamma x'}f(x')\mathrm{e}^{-\mathrm{i}kx'}\mathrm{d}x']\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k,$$

and we introduce $\gamma+\mathrm{i}k=s$ inside the integral to produce

$$f(x)=\frac{\mathrm{e}^{\gamma x}}{2\pi}\int_{-\infty}^{\infty}[\int_0^{\infty}f(x')\mathrm{e}^{-sx'}\mathrm{d}x']\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k.$$

Finally, we define $\mathcal{L}[f(x);s]=\int_0^\infty f(x)\mathrm{e}^{-sx}\mathrm{d}x$, the Laplace Transform of $f(x)$; then, at fixed γ, we obtain

$$f(x)=\frac{\mathrm{e}^{\gamma x}}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}[\int_0^\infty f(x')\mathrm{e}^{-sx'}\mathrm{d}x']\mathrm{e}^{(s-\gamma)x}\mathrm{d}s$$

$$=\frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}[\int_0^\infty f(x')\mathrm{e}^{-sx'}\mathrm{d}x']\mathrm{e}^{sx}\mathrm{d}s \quad ,$$

the inverse Laplace Transform. These are the two identities introduced in §2.3.

Example 22

Given that the FT of a function is $\mathrm{e}^{-\alpha|k|}$, where $\alpha > 0$ is a real constant, use the inversion theorem to find the function.

We have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k = \frac{1}{2\pi}\int_{-\infty}^{0} \mathrm{e}^{\alpha k+\mathrm{i}kx}\mathrm{d}k + \frac{1}{2\pi}\int_{0}^{\infty} \mathrm{e}^{-\alpha k+\mathrm{i}kx}\mathrm{d}k$$

$$= \frac{1}{2\pi}\left\{\left[\frac{\mathrm{e}^{(\alpha+\mathrm{i}x)k}}{\alpha+\mathrm{i}x}\right]_{-\infty}^{0} + \left[\frac{\mathrm{e}^{(-\alpha+\mathrm{i}x)k}}{-\alpha+\mathrm{i}x}\right]_{0}^{\infty}\right\}$$

$$= \frac{1}{2\pi}\left(\frac{1}{\alpha+\mathrm{i}x} + \frac{1}{\alpha-\mathrm{i}x}\right) = \frac{\alpha/\pi}{\alpha^2+x^2}.$$

This result should be compared with that obtained in Example 15, demonstrating a symmetry between *f*(*x*) and *F*(*k*), in this case.

Example 23

Apply the inversion theorem for FTs to the general convolution integral (in *f* and *g*) and then evaluate on $x = 0$, with $f(u) = \bar{g}(-u)$; this yields an important identity.

Let *F*(*k*) and *G*(*k*) be the FTs of *f*(*x*) and *g*(*x*), respectively; the FT of $\int_{-\infty}^{\infty} f(x-u)g(u)\mathrm{d}u$ is then $F(k)G(k)$ (see §3.2(c)). Thus

$$\int_{-\infty}^{\infty} f(x-u)g(u)\,\mathrm{d}u = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)G(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k$$

and so on $x = 0$:

$$\int_{-\infty}^{\infty} f(-u)g(u)\,\mathrm{d}u = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)G(k)\,\mathrm{d}k.$$

Now let us take the special case $f(u) = \overline{g}(-u)$ (given), for *u* real, where the overbar denotes the complex conjugate (of course) since we may allow both *f* and *g* to be complex-valued; indeed, in this case, we see that

$$F(k) = \mathcal{F}[f(x);k] = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \int_{-\infty}^{\infty} \overline{g}(u)\mathrm{e}^{\mathrm{i}ku}\,\mathrm{d}u = \overline{G}(k)$$

for *k* (and $u = -k$) real, with $G(k) = \mathcal{F}[g(x);k]$. Thus, given the transform *G*(*k*) of *g*(*x*), so that of $f(-u) = \overline{g}(u)$ is $\overline{G}(k)$, we obtain

$$\int_{-\infty}^{\infty} g(u)\overline{g}(u)\,\mathrm{d}u = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(k)\overline{G}(k)\,\mathrm{d}k$$

or $$\int_{-\infty}^{\infty} |g(u)|^2\,\mathrm{d}u = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(k)|^2\,\mathrm{d}k\,.$$

This important identity is usually referred to as *Parseval's theorem*. [M.-A. Parseval, 1755-1836, a French mathematician who published only 5 papers; this was his second (1799), and was regarded by him as a self-evident property of integrals: he did not prove or derive it! Note that this pre-dates the introduction of the Fourier Transform, as we recognise it today.] This result is a quite powerful tool in analysis – and then it is usually expressed as an inner product – and it also constitutes a fundamental property in physics, namely, the conservation of energy (in particular, of a waveform over all wave number).

3.4 Applications to the solution of differential and integral equations

As we did for the Laplace Transform (§2.4), we shall conclude with a couple of applications: we solve some equations, but this can give little more than a hint of the usefulness of FTs in modern mathematics and physics. Our first example involves the solution of Laplace's equation.

Example 23

Find the solution of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ in $y > 0, -\infty < x < \infty$, given $u(x,0) = f(x)$, $-\infty < x < \infty$, with $f(x)$ and $f'(x)$ both tending to zero (suitably rapidly) as $|x| \to \infty$, and $u(x,y) \to 0$ as $x^2 + y^2 \to \infty$.

Because $x \in (-\infty, \infty)$, it is convenient (and natural) to take the FT of the equation (see §3.2(b)) in *x*:

$$(\mathrm{i}k)^2 U + U'' = 0 \text{ or } U'' - k^2 U = 0$$

where $U(y;k) = \mathcal{F}[u(x,y);k]$. Correspondingly, $U(0;k) = F(k) = \mathcal{F}[f(x);k]$ with $U \to 0$ as $y \to \infty$; thus we obtain the solution satisfying the equation (for *U*), and the boundary conditions, in the form

$$U(y;k) = F(k)\mathrm{e}^{-|k|y}.$$

This is in convolution form (§3.2(c)) and, with the result of Example 22, we therefore obtain

$$u(x,y) = \int_{-\infty}^{\infty} f(x-z)\frac{y/\pi}{z^2+y^2}\mathrm{d}z$$

$$= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(x-z)}{z^2+y^2}\mathrm{d}z \quad (\text{for } y > 0).$$

It is left as an exercise to confirm that $u \to f(x)$ as $y \to 0$.

We now turn to the problem of solving an integral equation. As we found for the LT (see Example 4), the most straightforward type of problem is one that involves a convolution – and the same applies here. Thus an integral equation of the form

$$\int_{-\infty}^{\infty} F(t)K(x-t)\,\mathrm{d}t = g(x) \quad (-\infty < x < \infty)$$

can be solved directly by employing the FT.

Example 24

Find the solution of $\int_{-\infty}^{\infty} \frac{y(t)}{|x-t|^{1/2}}\mathrm{d}t = g(x)$, where $g(x)$ is defined for all $-\infty < x < \infty$ and for which the FT exists.

We take the FT of the equation, to give

$$Y(k)\left(\frac{2\pi}{|k|}\right)^{1/2} = G(k),$$

where $\sqrt{2\pi}\,|k|^{-1/2}$ is the FT of $|x|^{-1/2}$ (see Exercise 3); thus we have

$$Y(k) = \left(\frac{|k|}{2\pi}\right)^{1/2} G(k).$$

Now $|x|^{-1/2}\operatorname{sgn}(x)$ is the inverse of $-\mathrm{i}\sqrt{2\pi}\,|k|^{-1/2}\operatorname{sgn}(k)$ (see Exercise 3) and so we write

$$Y(k) = \sqrt{2\pi}\,(\mathrm{i}k)G(k)\left(-\frac{\mathrm{i}|k|^{-1/2}}{\sqrt{2\pi}}\right)\operatorname{sgn}(k).\frac{1}{\sqrt{2\pi}};$$

Finally we use the derivative result (§3.2(b)) and, of course, split the inverse integral into $(-\infty, x)$ and (x, ∞). Thus we obtain

$$y(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\frac{g'(t)}{\sqrt{x-t}}\mathrm{d}t - \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}\frac{g'(t)}{\sqrt{t-x}}\mathrm{d}t,$$

which is the required solution (provided all the various integrals exist).

We conclude our brief set of problems that constitute some routine applications of the Ft by evaluating a (real) definite integral.

Example 25

Determine the integral $\int_{-\infty}^{\infty} \frac{(1-\cos(\alpha k))}{k^2} \mathrm{d}k$, where $\alpha\ (>0)$ is a real constant.

We know that the FT of $f(x)=\begin{cases} a(x+b)/b, & -b \le x \le 0 \\ a(b-x)/b, & 0 < x \le b \\ 0 & \text{otherwise} \end{cases}$ is $\frac{2a}{bk^2}(1-\cos(bk))$; see Example 18 (with $c=0, d=b$). Thus we may write, via the inverse transform,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2a}{bk^2}(1-\cos(bk))\, \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k$$;

we evaluate this on $x=0$:

$$f(0)=a=\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2a}{bk^2}(1-\cos(bk))\,\mathrm{d}k$$

or $$\frac{1}{b\pi}\int_{-\infty}^{\infty} \frac{1-\cos(bk)}{k^2}\mathrm{d}k=1$$

and so $$\int_{-\infty}^{\infty} \frac{1-\cos(\alpha k)}{k^2}\mathrm{d}k=\alpha\pi\,.$$

We have presented a few simple examples that embody the main (mathematical i.e. solving conventional problems) applications of the FT; we finish with some exercises.

Exercises 3

1. Find the Fourier Transforms of these functions:

   (a) $x^n f(x)$ ($n \geq 0$, integer) ; (b) $\dfrac{1}{1+x^2}$; (c) $|x|^{-1/2}$; (d) $|x|^{-1/2}\,\mathrm{sgn}(x)$;

   (e) $(a - \mathrm{i}x)^{-p}$ ($\mathscr{R}(a) > 0$, $\mathscr{R}(p) > 0$).

2. Find the Fourier Transform of these functions:

   (a) $f(x) = \begin{cases} a(b-x)/b, & 0 \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$; (b) $f(x) = \begin{cases} \sin x, & -\pi \leq x \leq \pi \\ 0 & \text{otherwise} \end{cases}$.

3. Use the Fourier Transform (in x) to find the solution of Laplace's equation, $u_{xx} + u_{yy} = 0$, $0 \leq y \leq a$, $-\infty < x < \infty$, subject to

   $$u(x,0) = f(x), u(x,a) = g(x), \; -\infty < x < \infty .$$

   [The Fourier Transform of $\dfrac{\sin(\pi y/a)}{\cosh(\pi x/a) + \cos(\pi y/a)}$ (in x) is $2a\dfrac{\sinh(ky)}{\sinh(ka)}$.]

4. Find the solution of the integral equation

   $$\int_{-\infty}^{\infty} y(t)\mathrm{e}^{-|x-t-a|}dt = x \quad (a \text{ real}).$$

   Find the Fourier Transform of the function $f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 1+x, & -1 \leq x < 0 \\ 0 & \text{otherwise} \end{cases}$, and hence evaluate

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{1 - \cos k + k \sin k}{k^2} \mathrm{d}k$; (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos k - \cos(3k) + 2k \sin k}{k^2} \mathrm{d}k$.

4 The Hankel Transform

In the final two chapters, we introduce and briefly describe two other fairly important integral transforms (and many others exist, and have been used in certain specific contexts). Here we present the main features of the *Hankel Transform* (HT). [These were introduced by H. Hankel (1839-1873), a German mathematician who made contributions to complex analysis and the theory of functions, as well as recasting Riemann's fundamental work on integration in terms of more modern measure theory. However, he is probably best remembered for his work on Bessel functions of the third kind: Hankel functions.]

In order to initiate our discussion, we note that any problem posed in cylindrical coordinates, together with the requirement to separate the variables, necessarily results in ordinary differential equations of Bessel type. Such problems are therefore likely to be neatly represented – and perhaps solved easily – by introducing an integral transform based on a Bessel function; this is the Hankel Transform. We define the HT, of order ν, as

$$F(k) = \mathcal{H}_\nu[f(x);k] = \int_0^\infty x f(x) \mathrm{J}_\nu(kx)\,\mathrm{d}x \quad (\nu > -\tfrac{1}{2}).$$

The corresponding inverse (which we do not derive here – it is considerably more involved than either of our earlier discussions) takes the form

$$\int_0^\infty kF(k)\mathrm{J}_\nu(kx)\,\mathrm{d}k = \frac{1}{2}[f(x_+) + f(x_-)],$$

which allows $f(x)$ to be discontinuous. We observe that *k* is therefore defined on $[0,\infty)$, and this applies throughout our discussions here. Of course, if $f(x)$ is continuous for all *x*, then the right-hand side is simply replaced by $f(x)$. These two results require that $\sqrt{x}f(x)$ is absolutely integrable on $(-\infty,\infty)$, with $\nu > -1/2$, where $\mathrm{J}_\nu(x)$ is the appropriate (i.e. bounded) solution of the ODE

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

It should already be evident that any use of the Hankel Transform – powerful though it is – necessarily involves some considerable knowledge of, and skills working with, the Bessel functions. Furthermore, many of the integrals that we encounter are not, in any sense, elementary; in consequence, in this chapter (and in the context of this Notebook), we shall proceed without giving all the details. Many of the main results that we mention can be found, described in detail, in a good text on transforms, read in conjunction with a standard work on the Bessel functions.

4.1 HTs of some elementary functions

The Hankel Transforms of many functions can be obtained from standard results in the theory of Bessel functions. As we have just implied, we shall not dwell upon this aspect of the derivations; we shall, however, outline the underlying structure in most cases. To start this process, we first consider the HT of the function

$$x^\nu(a^2 - x^2)^{\mu-\nu-1}\mathrm{H}(a-x),\ \mu > \nu \ge 0,\ a > 0,$$

where H(.) is the Heaviside step function; we therefore obtain the HT

$$\int_0^a x^{\nu+1}(a^2 - x^2)^{\mu-\nu-1}\mathrm{J}_\nu(xk)\,\mathrm{d}x.$$

The procedure is then slightly tedious (but routine): expand J_ν *via* its defining power series, integrate term-by-term and finally interpret the result (which turns out to be the power-series representation of another Bessel function). The result of all this is to generate

$$\mathcal{H}_\nu[x^\nu(a^2 - x^2)^{\mu-\nu-1}\mathrm{H}(x-a);k] = 2^{\mu-\nu-1}\Gamma(\mu-\nu)a^\mu k^{\nu-\mu}\mathrm{J}_\mu(ka).$$

The corresponding inversion theorem then gives

$$2^{\mu-\nu-1}\Gamma(\mu-\nu)a^\mu\int_0^\infty k^{\nu-\mu+1}\mathrm{J}_\mu(ka)\mathrm{J}_\nu(kx)\,\mathrm{d}k = x^\nu(a^2 - x^2)^{\mu-\nu-1}\mathrm{H}(a-x),$$

where we recognise that the integral on the left is, equivalently, the HT of $x^{\nu-\mu}\mathrm{J}_\mu(ax)$, and so we obtain the important transform result:

$$\mathcal{H}_\nu[x^{\nu-\mu}\mathrm{J}_\mu(ax);k] = \int_0^\infty x^{1+\nu-\mu}\mathrm{J}_\mu(ax)\mathrm{J}_\nu(kx)\,\mathrm{d}x$$

$$= \frac{k^\nu(a^2-k^2)^{\mu-\nu-1}}{2^{\mu-\nu-1}a^\mu\Gamma(\mu-\nu)}\mathrm{H}(a-k) .$$

Thus, with the choice $\mu = \nu + 1$, we obtain directly

$$\mathcal{H}_\nu[x^\nu \mathrm{H}(a-x);k] = \frac{a^{1+\nu}}{k}\mathrm{J}_{1+\nu}(ka)$$

and

$$\mathcal{H}_\nu[x^{-1}\mathrm{J}_{1+\nu}(ax);k] = \frac{k^\nu}{a^{1+\nu}}\mathrm{H}(a-k) .$$

Further, in this last result above, we now set $\nu = 0$ (for example) to give

$$\mathcal{H}_0[x^{-1}\mathrm{J}_1(ax);k] = a^{-1}\mathrm{H}(a-k) .$$

Correspondingly, with $\mu = \nu + \frac{1}{2}$, we obtain the pair

$$\mathcal{H}_\nu\left[\frac{x^\nu \mathrm{H}(a-x)}{\sqrt{a^2-x^2}};k\right] = \sqrt{\frac{\pi}{2k}}\,a^{\nu+\frac{1}{2}}\mathrm{J}_{\nu+\frac{1}{2}}(ak)$$

and

$$\mathcal{H}_\nu[x^{-1/2}\mathrm{J}_{\nu+\frac{1}{2}}(ax);k] = \sqrt{\frac{2}{\pi}}\,\frac{k^\nu}{a^{\nu+\frac{1}{2}}}\,\frac{\mathrm{H}(a-k)}{\sqrt{a^2-k^2}} ;$$

then, in this last result, with $\nu = 0$:

$$\mathcal{H}_0[x^{-1}\sin(ax);k] = \frac{\mathrm{H}(a-k)}{\sqrt{a^2-k^2}}.$$

We note, as we must expect, that we have used appropriate properties of the Bessel function (and its various integrals). Many other results follow from these, but we should mention another route that is useful in the construction of HTs, which is based on this observation:

$$\mathcal{H}_\nu[\mathrm{e}^{-sx}f(x);k] = \int_0^\infty x\mathrm{e}^{-sx}f(x)\mathrm{J}_\nu(kx)\,\mathrm{d}x;$$

this is the *Laplace Transform* of the function $xf(x)\mathrm{J}_\nu(kx)$ (with parameter *s*).

Example 26

Find the HT of $x^{\nu-1}\mathrm{e}^{-sx}$.

This is, directly, the integral $\int_0^\infty x^\nu \mathrm{e}^{-sx}\mathrm{J}_\nu(kx)\,\mathrm{d}x$,
which is a standard integral involving Bessel functions; the result is

$$\frac{2^\nu k^\nu \Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}(k^2+s^2)^{\nu+\frac{1}{2}}}.$$

The above example makes clear that a very good working knowledge of Bessel functions is required! Another rather similar example is

Example 27

Find the HT of e^{-sx}.

This is simply the integral $\int_0^\infty x\mathrm{e}^{-sx}\mathrm{J}_\nu(kx)\,\mathrm{d}x$, which is another (fairly) standard result:

$$\mathcal{H}_\nu[\mathrm{e}^{-sx};k] = \frac{s+\nu\sqrt{k^2+s^2}}{(k^2+s^2)^{3/2}}\left(\frac{k}{s+\sqrt{k^2+s^2}}\right)^\nu .$$

Note that these various results are greatly simplified if we elect to use, for example, $\nu = 0$, and in particular calculations we may do just this. So we have, from the last result above,

$$\mathcal{H}_0[\mathrm{e}^{-sx};k] = \frac{s}{(k^2+s^2)^{3/2}} .$$

4.2 Some properties of the HT

We shall discuss just two important properties of the Hankel Transform: a simple algebraic property and the HT of derivatives. These are – by far – the most useful, although others do exist (but are not derived in any routine or direct way).

(a) Algebraic

Given the HT

$$\mathcal{H}_\nu[f(x);k] = \int_0^\infty x f(x) \mathrm{J}_\nu(kx)\,\mathrm{d}x\,,$$

we form the HT of $f(ay)$ (where a is real and positive) to give

$$\mathcal{H}_\nu[f(ay);k] = \int_0^\infty y f(ay) \mathrm{J}_\nu(ky)\,\mathrm{d}y$$

$$= \frac{1}{a^2}\int_0^\infty z f(z) \mathrm{J}_\nu[(k/a)z]\,\mathrm{d}z \quad (ay = z).$$

Thus we see that we have a simple algebraic property:

$$\mathcal{H}_\nu[f(ax);k] = a^{-2}\mathcal{H}_\nu[f(x);k/a].$$

Example 28

Show that $\mathcal{H}_0[\mathrm{e}^{-sx};k]$ satisfies the above algebraic property.

We have that $\mathcal{H}_0[\mathrm{e}^{-x};k] = \dfrac{1}{(1+k^2)^{3/2}}$ (see Example 27)

and then the algebraic property yields

$$\mathcal{H}_0[\mathrm{e}^{-sx};k] = s^{-2}\mathcal{H}_0[e^{-x};k/s]$$

$$= \frac{1}{s^2}\frac{1}{(1+k^2/s^2)^{3/2}} = \frac{s}{(s^2+k^2)^{3/2}},$$

as required.

(b) Differential

The construction of the useful – and relevant – HTs of differentials requires a little care. First, consider the HT of $x^{\nu-1}(x^{1-\nu}f(x))'$ (where the prime denotes the derivative); this gives

$$\mathcal{H}_\nu[x^{\nu-1}(x^{1-\nu}f(x))';k]=\int_0^\infty x^\nu(x^{1-\nu}f)'\mathrm{J}_\nu(kx)\,\mathrm{d}x$$

$$=[xf(x)\mathrm{J}_\nu(kx)]_0^\infty-\int_0^\infty x^{1-\nu}f(x)[x^\nu\mathrm{J}_\nu(kx)]'\mathrm{d}x.$$

The first term certainly vanishes as $x\to\infty$, because this requires $x^{1/2}f(x)\to 0$ in this limit (which is a necessary requirement for the existence of the HT). If, in addition, $x^{1+\nu}f(x)\to 0$ as $x\to 0$, then the other evaluation of the first term is also zero; this is often the case, and we will invoke this condition here. Thus we obtain

$$\mathcal{H}_\nu[x^{\nu-1}(x^{1-\nu}f(x))';k]=-\int_0^\infty x^{1-\nu}f(x)[x^\nu\mathrm{J}_\nu(kx)]'\mathrm{d}x.$$

To proceed, we make use of a standard recurrence relation for Bessel functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^\nu J_\nu(kx)]=kx^\nu J_{\nu-1}(kx),$$

which gives

$$\mathcal{H}_\nu[x^{\nu-1}(x^{1-\nu}f(x))';k]=-k\int_0^\infty xf(x)\mathrm{J}_{\nu-1}(kx)\,\mathrm{d}x$$

$$=-k\,\mathcal{H}_{\nu-1}[f(x);k]$$

– our fundamental property. If we select $\nu=1$, then we obtain the particular identity

$$\mathcal{H}_1[f'(x);k]=-k\,\mathcal{H}_0[f(x);k],$$

where the removal of the contribution on $x=0$ now requires $x^2f(x)\to 0$ as $x\to 0$.

A corresponding result for the HT of $x^{-1-\nu}(x^{1+\nu}f(x))'$, when the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-\nu}\mathrm{J}_{\nu}(kx)] = -kx^{-\nu}\mathrm{J}_{1+\nu}(kx)$$

is used, is

$$\mathcal{H}_{\nu}[x^{-1-\nu}(x^{1+\nu}f(x))';k] = k\,\mathcal{H}_{\nu+1}[f(x);k]$$

which, in the case $\nu = 0$, yields

$$\mathcal{H}_{0}[x^{-1}(xf(x))';k] = k\,\mathcal{H}_{1}[f(x);k].$$

(The details of these calculations are left as an exercise; they follow precisely the pattern of the preceding one.)

Now, it is a familiar observation in the separation of variables applied to cylindrical coordinates – and therefore also for Bessel functions – that we encounter the differential operator

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\equiv x^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

Thus we are likely to want the HT of $x^{-1}(xf'(x))'$, which follows directly from our two previous identities:

$$\mathcal{H}_0[x^{-1}(xf')';k]=k\,\mathcal{H}_1[f'(x);k]=-k^2\,\mathcal{H}_0[f(x);k],$$

which enables us to carry out the Hankel transformation of, for example, PDEs expressed in cylindrical coordinates. We will use this important property in an application.

4.3 Application to the solution of a PDE

Partial differential equations, when written in cylindrical coordinates, contain the terms $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}$ (the form we mentioned above) and so the introduction of a suitable HT should enable us the construct a solution (albeit expressed as an integral involving Bessel functions).

Example 29

Find the solution of Laplace's equation, $u_{rr}+(1/r)u_r+u_{zz}=0$, in $r\geq 0$, $0\leq z\leq a$, which satisfies $u(r,0)=f(r)$, $u(r,a)=0$ (on the assumption that all relevant integrals exist).

We take the $\mathcal{H}_0$ transform of the equation, and employ the differential result above; this gives

$$\frac{\mathrm{d}^2U}{\mathrm{d}z^2}-k^2U=0 \text{ where } U(z;k)=\mathcal{H}_0[u(r,z);k].$$

The boundary conditions become $U(0;k)=F(k)=\mathcal{H}_0[f(r);k]$ and $U(a;k)=0$; thus the solution for U is simply

$$U(z;k)=F(k)\frac{\sinh[k(a-z)]}{\sinh(ka)}.$$

Taking the inverse, we then obtain the solution

$$u(r,z)=\int_0^\infty kF(k)\mathrm{J}_0(kr)\frac{\sinh[k(a-z)]}{\sinh(ka)}\,\mathrm{d}k,$$

where $F(k)=\int_0^\infty rf(r)\mathrm{J}_0(kr)\,\mathrm{d}r$.

It is possible to write this solution in alternative forms, mainly by re-expressing the double integral that is generated when the expression for $F(k)$ is inserted. Of course, simplifications can arise for appropriate, special choices for $f(r)$.

Exercises 4

1. Find the Hankel Transforms, $\mathcal{H}_0$, of these functions:
   (a) x^{-1}; (b) x; (c) $(x^2+a^2)^{-1/2}$ (a real and positive).

2. Given that $\mathcal{H}_0[f(x);k]=F(k)$, find an expression for $\mathcal{H}_0[-x^2f(x);k]$.

3. Show that the biharmonic operator $\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)$, under the Hankel Transform $\mathcal{H}_0$, becomes $(k^2)^2=k^4$. Hence find the solution of

$$\frac{\partial^2 u}{\partial t^2}+\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)^2 u=0\ (t\ge 0,\ r\ge 0)$$

which satisfies $u(r,0)=f(r)$ and $u_t(r,0)=0$.

5 The Mellin Transform

The Mellin Transform (MT) arises most naturally in problems that involve, for example, the solution of Laplace's equation in a wedge (an idea that we will not develop here, but this situation will arise in one of our examples). This type of problem then suggests the introduction of the integral

$$F(p) = \int_0^\infty x^{p-1} f(x)\mathrm{d}x = \mathcal{M}[f(x);p],$$

which requires that $x^{p-1}f(x)$ is absolutely integrable on $[0,\infty)$ for some $p>0$ (which we will take to be real here); the corresponding inverse (for continuous functions) is then

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-p} F(p)\mathrm{d}p$$

where c sits in the strip of analyticity of $F(p)$. [R.H. Mellin, 1854-1933, Finnish mathematician who worked mainly on the theory of functions; he introduced his transform, developed its theory and applied it to many and varied problems, from PDEs to number theory.]

5.1 MTs of some elementary functions

Mellin Transforms are readily obtained from some of our earlier transforms, so, for example, the Laplace Transform of x^{p-1} becomes the Mellin transform of e^{-sx} i.e.

$$\mathcal{L}[x^{p-1};s]=\int_0^\infty x^{p-1}\mathrm{e}^{-sx}\mathrm{d}x=\mathcal{M}[\mathrm{e}^{-sx};p] \quad (p>0),$$

and thus $\mathcal{M}[\mathrm{e}^{-sx};p]=s^{-p}\Gamma(p)$ (from Example 2).

Example 30

The evaluation of $\oint_C z^{p-1}\mathrm{e}^{-kz}\mathrm{d}z$, around a suitable contour, yields the result $\int_0^\infty y^{p-1}\mathrm{e}^{-\mathrm{i}ky}\mathrm{d}y=\mathrm{e}^{-\frac{1}{2}\mathrm{i}\pi p}k^{-p}\Gamma(p)$ ($0<p<1$, $k>0$); use this identity to find the MTs of $\sin x$ and $\cos x$.

We simply take real and imaginary parts, to give

$$\int_0^\infty y^{p-1}\cos(ky)\,\mathrm{d}y=k^{-p}\Gamma(p)\cos(\tfrac{1}{2}\pi p)$$

and $$\int_0^\infty y^{p-1}\sin(ky)\,\mathrm{d}y=k^{-p}\Gamma(p)\sin(\tfrac{1}{2}\pi p);$$

thus $\mathcal{M}[\sin x;p]=\Gamma(p)\sin(\tfrac{1}{2}\pi p)$ and $\mathcal{M}[\cos x;p]=\Gamma(p)\cos(\tfrac{1}{2}\pi p)$.

This example is based on the Fourier Transform; now let us construct an example that uses the Hankel Transform.

Example 31

Use the identity $\int_0^\infty x^{q-\nu-1}\mathrm{J}_\nu(x)\,\mathrm{d}x=\dfrac{2^{q-\nu-1}\Gamma(\frac{1}{2}q)}{\Gamma(\nu+1-\frac{1}{2}q)}$ ($0<q<1$, $\nu>-\frac{1}{2}$) to find the MT of $\mathrm{J}_\nu(x)$.

First, we set $q-\nu=p$ (so that $-\nu<p<\frac{3}{2}$), then we have

$$\mathcal{M}[\mathrm{J}_\nu(x);p]=\int_0^\infty x^{p-1}\mathrm{J}_\nu(x)\,\mathrm{d}x=\frac{2^{p-1}\Gamma(\frac{1}{2}\nu+\frac{1}{2}p)}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}p+1)},$$

which is the required MT.

5.2 Some properties of the MT

Corresponding to our development for the Hankel Transform, we show that there exist simple algebraic and differential properties that the MT satisfy; indeed, for this transform, we may also include the integral of a function.

Algebraic

Three results are readily obtained. First, given the MT of $f(x)$, we form the MT of $f(ax)$, where $a > 0$ is a real constant:

$$\int_0^\infty x^{p-1} f(ax)\,\mathrm{d}x = a^{-p}\int_0^\infty u^{p-1} f(u)\,\mathrm{d}u \quad (u = ax)$$

$$= a^{-p}\,\mathcal{M}[f(x);p].$$

Also, we may (for some real $a > 0$) form the MT of $x^a f(x)$:

$$\int_0^\infty x^{a+p-1} f(x)\,\mathrm{d}x$$

which simply replaces p by $a+p$, so that we obtain

$$\mathcal{M}[x^a f(x);p] = \mathcal{M}[f(x);p+a].$$

Perhaps rather surprisingly, we can find – for the same as – the MT of $f(x^a)$; in this case we obtain

$$\mathcal{M}[f(x^a);p] = \int_0^\infty x^{p-1} f(x^a)\,\mathrm{d}x$$

$$= \int_0^\infty u^{(p-1)/a} f(u).\frac{1}{a}u^{(1-a)/a}\mathrm{d}u \quad (u = x^a)$$

$$= \frac{1}{a}\int_0^\infty u^{(p/a)-1} f(u)\,\mathrm{d}u = \frac{1}{a}\mathcal{M}[f(x);p/a].$$

Example 32

Find the MT of e^{-x^2}.

From §5.1, we have $\mathcal{M}[\mathrm{e}^{-x};p] = \Gamma(p)$, and then the last result from those above gives:

$$\mathcal{M}[\mathrm{e}^{-x^2}; p] = \frac{1}{2}\mathcal{M}[e^{-x}; p/2] = \frac{1}{2}\Gamma(p/2)\ .$$

(b) Differential

We construct the MT of $f'(x)$ directly, as

$$\int_0^\infty x^{p-1} f'(x)\,\mathrm{d}x = [x^{p-1} f(x)]_0^\infty - (p-1)\int_0^\infty x^{p-2} f(x)\,\mathrm{d}x \quad ;$$

if $x^{p_1-1} f(x) \to 0$ as $x \to 0$, and $x^{p_2-1} f(x) \to 0$ as $x \to \infty$, then

$$\mathcal{M}[f'(x); p] = -(p-1)\,\mathcal{M}[f(x); p-1] \quad \text{(for } p_1 < p < p_2\text{)}.$$

This result is easily generalised to n derivatives, to produce

$$\mathcal{M}[f^{(n)}(x); p] = (-1)^n (p-1)(p-2)\ldots(p-n)\,\mathcal{M}[f(x); p-n]\,,$$

provided that the extension to higher derivatives of the conditions as $x \to 0$, and as $x \to \infty$, holds up to the $(n-1)$ th derivative.

Example 33

Find the MT of $xf'(x)$, from the results above.

First we have $\mathcal{M}[xf'(x); p] = \mathcal{M}[f'(x); p+1]$ and then

$$\mathcal{M}[f'(x); p+1] = -(p+1-1)\,\mathcal{M}[f(x); p+1-1]$$

i.e. $$\mathcal{M}[xf'(x); p] = -p\,\mathcal{M}[f(x); p].$$

This is a similarly neat result, which is readily generalised:

$$\mathcal{M}\left[x^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x); p\right] = (-1)^n p(p+1)\ldots(p+n-1)\,\mathcal{M}[f(x); p],$$

provided that all the appropriate decay conditions are met.

(c) Integral

The result we decsribe here can either be obtained by a direct calculation – which is left as an exercise – or derived from our differential identity. Consider the result from above (§5.2(b)):

$$\mathcal{M}[f'(x); p] = -(p-1)\,\mathcal{M}[f(x); p-1],$$

in which we elect to write $f(x) = \int_0^x g(u)\,\mathrm{d}u$ (so that $f'(x) = g(x)$); thus we obtain

$$\mathcal{M}[g(x); p] = -(p-1)\,\mathcal{M}\left[\int_0^x g(u)\,\mathrm{d}u; p-1\right].$$

Now replace $p-1$ by *p*:

$$\mathcal{M}\left[\int_0^x g(u)\,du; p\right] = -\frac{1}{p}\,\mathcal{M}[g(x); p+1],$$

which is the required identity.

Example 34

Find the MT of $\int_0^x \mathrm{e}^{-u^2}\,\mathrm{d}u$.

We have $\mathcal{M}\left[\int_0^x \mathrm{e}^{-u^2}\,\mathrm{d}u; p\right] = -\frac{1}{p}\mathcal{M}[\mathrm{e}^{-x^2}; p+1]$

$$= -\frac{1}{p}.\frac{1}{2}\Gamma(\tfrac{1}{2}p + \tfrac{1}{2})$$.

This example shows that the MT of a less-than-trivial function (essentially the error function, *erf*), which might have been expected to generate a complicated MT, presents us with a particularly routine exercise.

5.3 Applications to the solution of a PDE

Although the MT can be used to solve a class of integral equations (using the appropriate convolution theorem, much as we have seen for other transforms), arguably the most useful application for the applied mathematician is to the solution of certain types of PDE. However, we should mention that the MT also plays a rôle in the solution of functional equations and in certain problems in statistics. For us, a suitable problem involves Laplace's equation written in plane polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

which is conveniently expressed as

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$$.

We now observe that $\left(r\frac{\partial}{\partial r}\right)\left(r\frac{\partial}{\partial r}\right) \equiv r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r}$, and so we have one of the simple forms for the application of the MT i.e. $r^n\frac{\partial^n}{\partial r^n}$ (see after Example 33). The application of the MT therefore produces a fairly routine problem, although we may need to take care over where the MT exists; we rehearse all the salient points in our final example.

Example 35

Find the solution of $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$ in $r \geq 0$, $0 \leq \theta \leq \alpha$, with $u(r,0) = f(r)$ (given) and $u(r,\alpha) = 0$ (and $u \to 0$ as $r \to \infty$ for $\theta \in [0,\alpha]$).

Because the conditions as $r \to 0$, and $r \to \infty$, are not precise, we first apply the MT (in *r*) and perform the required integration by parts with some care:

$$\mathcal{M}[r(ru_r)_r; p] = [r^{p+1} u_r - p r^p u]_0^\infty + p^2 \mathcal{M}[u(r,\theta); p].$$

Let us suppose that $r^\mu u(r,\theta) \to \beta$ as $r \to 0$, and that $r^\nu u(r,\theta) \to \gamma$ as $r \to \infty$ (for β, γ non-zero constants), then for $\mu < p < \nu$ the evaluations at the endpoints vanish and so the MTs exist for these *p*s. Thus we obtain the equation

$$\frac{\mathrm{d}^2 U}{\mathrm{d}\theta^2} + p^2 U = 0 \text{ where } U(\theta; p) = \mathcal{M}[u(r,\theta); p]$$

with the boundary conditions

$$U(0; p) = \mathcal{M}[f(r); p] = F(p) \text{ and } U(\alpha; p) = 0.$$

The appropriate solution is the obtained directly:

$$U(\theta; p) = F(p) \frac{\sin[p(\alpha - \theta)]}{\sin(p\alpha)},$$

and then we may express the solution of the original problem as the integral

$$u(r,\theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-p} F(p) \frac{\sin[p(\alpha-\theta)]}{\sin(p\alpha)} \mathrm{d}p.$$

Of course, *c* must sit in the strip of analyticity of the integrand, which depends on *F*(*p*).

Alternative versions of this solution are possible, particularly when *F* turns out to be a simple function. This concludes our discussion of the Mellin Transform.

Exercises 5

1. Find the Mellin Transforms (parameter p) of these functions:
   (a) $\delta(x-a)$, $a>0$ constant; (b) $x^n \mathrm{H}(x-a)$, $a>0$ constant, $\mathrm{R}(p+n)>0$;

   (c) $\dfrac{1}{1+x}$ (valid for $0<\mathrm{R}(p)<1$).
2. Obtain the Mellin Transform of the Bessel equation
   $$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2-\nu^2)y=0$$

   in the form $F(p+2)=(\nu^2-p^2)F(p)$,

   assuming zero contributions from $x=0$ and $x\to\infty$.

   Show that this equation for $F(p)$ has a solution
   $$F(p)=k^p\Gamma\left[\tfrac{1}{2}(p+\nu)\right]\Gamma\left[\tfrac{1}{2}(p-\nu)\right]\cos(\tfrac{1}{2}p\pi)$$

   for a suitable choice of the constant k.

Tables of a few standard Integral Transforms

Table 1: Laplace Transforms

function	transform
$f(x)$	$F(s)=\int_0^\infty \mathrm{e}^{-sx} f(x)\,\mathrm{d}x$
1	s^{-1}
x^α	$s^{-1-\alpha}\Gamma(1+\alpha) \quad (\alpha>-1)$
$\mathrm{e}^{\alpha x}$	$\dfrac{1}{s-\alpha} \quad (\mathcal{R}(s)>\mathcal{R}(\alpha))$
$\sinh(\alpha x)$	$\dfrac{\alpha}{s^2-\alpha^2} \quad (\mathcal{R}(s)>\alpha)$
$\cosh(\alpha x)$	$\dfrac{s}{s^2-\alpha^2}$ (ditto)
$\sin(\alpha x)$	$\dfrac{\alpha}{s^2+\alpha^2}$ (ditto)
$\cos(\alpha x)$	$\dfrac{s}{s^2+\alpha^2}$ (ditto)
$\delta(x-a)$	$\mathrm{e}^{-as} \quad (a\ge 0)$
$\mathrm{erf}(x/2a)=\dfrac{2}{\sqrt{\pi}}\int_0^{x/2a}\mathrm{e}^{-u^2}\mathrm{d}u$	$s^{-1}\mathrm{e}^{a^2s^2}\mathrm{erfc}(as) \quad (\mathcal{R}(s)>0)$
$\mathrm{erf}(a\sqrt{x})$	$\dfrac{a}{s\sqrt{s+a^2}} \quad (s+a^2>0)$

Table 2: Fourier Transforms

function	transform
$f(x)$	$F(k)=\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$
$\mathrm{e}^{-\alpha\lvert x\rvert}$	$\dfrac{2\alpha}{k^2+\alpha^2}\quad(\alpha>0)$
$\mathrm{e}^{-\alpha^2x^2}$	$\dfrac{\sqrt{\pi}}{\alpha}\mathrm{e}^{-k^2/4\alpha^2}$
$\begin{cases} a, -b\le x\le b \\ 0, \text{otherwise}\end{cases}$	$\dfrac{2a}{k}\sin(bk)$
$\delta(x-a)$	$\mathrm{e}^{-\mathrm{i}ka}$
$\mathrm{sgn}(x)$	$-\dfrac{2\mathrm{i}}{k}$
1	$2\pi\delta(k)$
$\mathrm{e}^{\mathrm{i}\alpha x}$	$2\pi\delta(k-\alpha)$
x^n	$2\pi\mathrm{i}^n\delta^{(n)}(k)$
x^{-1}	$-\mathrm{i}\pi\,\mathrm{sgn}(k)$
x^{-n}	$-\mathrm{i}\pi\dfrac{(-\mathrm{i}k)^{n-1}}{(n-1)!}\mathrm{sgn}(k)$
$\dfrac{1}{\sqrt{\lvert x\rvert}}$	$\sqrt{\dfrac{2\pi}{\lvert k\rvert}}$
$\sin(\alpha x)$	$\mathrm{i}\pi[\delta(k+\alpha)-\delta(k-\alpha)]$
$\cos(\alpha x)$	$\pi[\delta(k+\alpha)+\delta(k-\alpha)]$

Table 3: Hankel Transforms

function	transform
$f(x)$	$F_\nu(k) = \int_0^\infty x f(x) \mathrm{J}_\nu(kx)\,\mathrm{d}x$
$x^{\nu-1}\mathrm{e}^{-sx}$	$\dfrac{2^\nu k^\nu \Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\left(k^2+s^2\right)^{\nu+\frac{1}{2}}}$
e^{-sx}	$\dfrac{s+\nu\sqrt{k^2+s^2}}{\left(k^2+s^2\right)^{3/2}}\left(\dfrac{k}{s+\sqrt{k^2+s^2}}\right)^\nu$
$f(x)$	$F_0(k) = \int_0^\infty x f(x) \mathrm{J}_0(kx)\,\mathrm{d}x$
$x^{-1}\mathrm{e}^{-sx}$	$(k^2+s^2)^{-1/2}$
e^{-sx}	$\dfrac{s}{\left(k^2+s^2\right)^{3/2}}$
x^{-1}	k^{-1}
$x^{-1}\mathrm{e}^{\mathrm{i}\alpha x}$	$\dfrac{\mathrm{i}}{\sqrt{\alpha^2 - k^2}}$ $(\alpha > 0,\ k < \alpha$
$\mathrm{e}^{-\frac{1}{2}\alpha^2 x^2}$	$\dfrac{1}{\alpha^2}\mathrm{e}^{-k^2/2\alpha^2}$
x^m (m odd)	$\dfrac{2^{m+1}}{k^{m+2}}\dfrac{\Gamma(1+\frac{1}{2}m)}{\Gamma(-\frac{1}{2}m)}$

Table 4: Mellin Transforms'

function	transform
$f(x)$	$F(p)=\int_0^\infty x^{p-1}f(x)\,\mathrm{d}x$
e^{-ax}	$a^{-p}\Gamma(p)$ $(\mathcal{R}(a)>0,\ \mathcal{R}(p)>0)$
e^{-x^2}	$\frac{1}{2}\Gamma(\frac{1}{2}p)$
$\sin x$	$\Gamma(p)\sin(\frac{1}{2}\pi p)$ $(-1<\mathcal{R}(p)<1)$
$\cos x$	$\Gamma(p)\cos(\frac{1}{2}\pi p)$ $(0<\mathcal{R}(p)<1)$
$\dfrac{1}{1+x}$	$\pi\mathrm{cosec}(\pi p)$ (ditto)
$\dfrac{1}{1+x^2}$	$\frac{1}{2}\pi\mathrm{cosec}(\frac{1}{2}\pi p)$ $(0<\mathcal{R}(p)<2)$
$\dfrac{1}{(1+x)^a}$	$\dfrac{\Gamma(a-p)\Gamma(p)}{\Gamma(a)}$ $(\mathcal{R}(a-p)>0,\ \mathcal{R}(p)>0)$
$\ln(1+x)$	$\dfrac{\pi}{p}\mathrm{cosec}(\pi p)$ $(-1<\mathcal{R}(p)<0)$
$\mathrm{erfc}(x)$	$\dfrac{1}{p\sqrt{\pi}}\Gamma(\frac{1}{2}+\frac{1}{2}p)$ $(\mathcal{R}(p)>0)$
$\delta(x-a)$	a^{p-1}
$\mathrm{H}(x-a)$	$-\dfrac{a^p}{p}$ $(a>0,\ p<0)$

Answers

Exercises 2

1. (a) $\dfrac{4}{s^3}+\dfrac{1}{s^2}-\dfrac{1}{s}$; (b) $\dfrac{5}{(s+2)^2+25}$; (c) $\dfrac{s^2-\alpha^2}{(s^2+\alpha^2)^2}$; (d) $\dfrac{2\alpha s}{(s^2-\alpha^2)^2}$ $(\mathrm{R}(s)>\alpha)$;

   (e) $\dfrac{6\alpha s^2-2\alpha^3}{(s^2+\alpha^2)^3}$; (f) $\operatorname{arccot}(s/\alpha)$.

2. (a) $Y(s)=\dfrac{1}{s-2}+\dfrac{1}{(s-2)^2}$ then $y(x)=1+x\mathrm{e}^{2x}$; (b) $Y(s)=\dfrac{1/6}{s+1}-\dfrac{1/2}{s-1}+\dfrac{1/3}{s-2}$ then

   $y(x)=\frac{1}{6}\mathrm{e}^{-x}-\frac{1}{2}\mathrm{e}^{x}+\frac{1}{3}\mathrm{e}^{2x}$; (c) $Y(s)=\dfrac{4(1-\mathrm{e}^{-s})+s^3}{s^2(s^2+4)}$ then

   $y(x)=x-\frac{1}{2}\sin 2x+\cos 2x+[\frac{1}{2}\sin(2x-2)-x+1]\mathrm{H}(x-1)$.

3. $Y(s)=\dfrac{A}{(s^2-1)^{3/2}}$ (arb. A) then $y(x)=x\mathrm{I}_1(x)/\mathrm{I}_1(1)$.

4. $U(x;s)=F(s)\dfrac{\sinh[(a-x)\sqrt{s}]}{\sinh(a\sqrt{s})}$.

5. $y(x)=\dfrac{1}{\pi}\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_a^x\dfrac{f(u)}{\sqrt{x-u}}\,\mathrm{d}u=\dfrac{f(a)}{\pi\sqrt{x-a}}+\dfrac{1}{\pi}\int_a^x\dfrac{f'(u)}{\sqrt{x-u}}\,\mathrm{d}u$.

Exercises 3

1. (a) $\mathrm{i}^n\dfrac{\mathrm{d}}{\mathrm{d}k}F(k)$; (b) $\pi\mathrm{e}^{-|k|}$; (c) $\sqrt{2\pi}\,|k|^{-1/2}$; (d) $-\mathrm{i}\sqrt{2\pi}\,|k|^{-1/2}\operatorname{sgn}k$;

   (e) $\begin{cases}2\pi k^{p-1}\,\mathrm{e}^{-ak}/\Gamma(k), & k>0\\ 0, & k<0\end{cases}$.

2. (a) $\dfrac{a}{bk^2}(1-\mathrm{e}^{-\mathrm{i}bk})-\dfrac{\mathrm{i}a}{k}$; (b) $\dfrac{2\mathrm{i}\sin(k\pi)}{k^2-1}$.

3. $U(y;k)=F(k)\dfrac{\sinh[k(a-y)]}{\sinh(ka)}+G(k)\dfrac{\sinh(ky)}{\sinh(ka)}$ then

   $u(x,y)=\dfrac{1}{2}\sin(\pi y)\displaystyle\int_{-\infty}^{\infty}\dfrac{f(t)}{\cosh[\pi(x-t)]-\cos(\pi y)}\,\mathrm{d}t$

$$+\frac{1}{2}\sin(\pi y)\int_{-\infty}^{\infty}\frac{g(t)}{\cosh[\pi(x-t)]+\cos(\pi y)}\,\mathrm{d}t\ .$$

4. $y(x)=\frac{1}{2}(x+a)$.
5. $F(k)=\dfrac{1}{k^2}(1-\mathrm{e}^{\mathrm{i}k})+\dfrac{\mathrm{i}}{k}\mathrm{e}^{-\mathrm{i}k}$ then (a) 2π ; (b) 4π .

Exercises 4

1. (a) $1/k$; (b) $-1/k^3$; (c) $\mathrm{e}^{-k|a|}/k$.
2. $\dfrac{\mathrm{d}^2F_0}{\mathrm{dk}^2}+\dfrac{1}{k}\dfrac{\mathrm{d}F_0}{\mathrm{d}k}$ where $F_0(k)=\mathrm{H}_0[f(x);k]$.
3. $U(t;k)=F_0(k)\cos(tk^2)$ then $u(r,t)=\dfrac{1}{2t}\displaystyle\int_0^{\infty} y\,f(y)\mathrm{J}_0(yr/2t)\sin[(y^2+r^2)/4t]\,\mathrm{d}y$.

Exercises 5

1. (a) a^{p-1} ; (b) $-\dfrac{a^{n+p}}{n+p}$; (c) $\pi\operatorname{cosec}(\pi p)$.
2. $k=\pm 2$.

Index